A BOUND FOR CANONICAL DIMENSION OF THE (SEMI-)SPINOR GROUPS

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Abstract. Using the theory of non-negative intersections, duality of the Schubert varieties, and Pieri type formula for the varieties of maximal totally isotropic subspaces, we get an upper bound for canonical dimension $\mathfrak{cd}(\mathrm{Spin}_n)$ of the spinor group Spin_n. A lower bound is given by the canonical 2-dimension $\mathfrak{cd}_2(\mathrm{Spin}_n)$, computed in [10]. If n or $n+1$ is a power of 2, no space is left between these two bounds; therefore the precise value of $\mathfrak{cd}(\mathrm{Spin}_n)$ is obtained for such n. We also produce an upper bound for canonical dimension of the semi-spinor group (giving the precise value of the canonical dimension in the case when the rank of the group is a power of 2) and show that spinor and semi-spinor groups are the only open cases of the question about canonical dimension of an arbitrary simple split group, possessing a unique torsion prime.

1. Introduction

Let F be an arbitrary field (of an arbitrary characteristic). Let X be a smooth algebraic variety over F. A field $L \supset F$ is called a *splitting field* (of X), if $X(L) \neq \emptyset$. A splitting field L is called *generic*, if for any splitting field L' there exists a place $L \to L'$. The *canonical dimension* $\mathrm{cd}(X)$ is defined as the minimum of transcendence degrees of generic splitting fields of X, cf. [10, §2] (since by lemma 4.1 in *loc.cit.* the function field of X is a generic splitting field of X, one has $\mathrm{cd}(X) \leq \dim X$).

Let G be an algebraic group over F. The *canonical dimension* $\mathfrak{cd}(G)$ of G, as introduced in [1], is the maximum of canonical dimensions of G-torsors, defined over field extensions of F (of course, $\mathfrak{cd}(G)$ is not the same as the canonical dimension $\mathrm{cd}(G)$ of the underlying variety of G, moreover, $\mathrm{cd}(G) = 0$ for any algebraic group G). The general question, raised in *loc.cit.*, is to determine $\mathfrak{cd}(G)$ for every split simple algebraic group G. For the spinor group, representing a particularly difficult case of the above general question, it is explained in lemma 12.1(b) of *loc.cit.*, that $\mathfrak{cd}(\mathrm{Spin}_{2n+1}) = \mathfrak{cd}(\mathrm{Spin}_{2n+2})$ (where n is a positive integer), so that we will discuss only $\mathfrak{cd}(\mathrm{Spin}_{2n+1})$ here.

Although the canonical dimension of, say, a smooth projective variety X can be expressed in terms of algebraic cycles on X (see [10, cor. 4.7]), there are no general recipes for computing $\mathrm{cd}(X)$ or $\mathfrak{cd}(G)$. A better situation occurs with the *canonical p-dimension* $\mathfrak{cd}_p$, a "p-local" version of $\mathfrak{cd}$ (p is a prime) introduced in §3 of *loc.cit.*: a recipe for computing $\mathfrak{cd}_p(G)$ of an arbitrary split simple G is obtained in *loc.cit.* In particular, one

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has

$$\mathfrak{cd}_2(\mathrm{Spin}_{2n+1}) = \frac{n(n+1)}{2} - 2^l + 1\ ,$$

where l is the minimal integer such that $2^l \geq n+1$ (and for any odd prime p, one has $\mathfrak{cd}_p(\mathrm{Spin}_{2n+1}) = 0$). Since $\mathfrak{cd}(G) \geq \mathfrak{cd}_p(G)$ for any G and p, we have a *lower bound* for the canonical dimension of the spinor group, given by its canonical 2-dimension.

In this note we establish the following *upper bound* for $\mathfrak{cd}(\mathrm{Spin}_{2n+1})$:

Theorem 1.1 (see §5)**.** *For any $n \geq 1$, one has $\mathfrak{cd}(\mathrm{Spin}_{2n+1}) \leq n(n-1)/2$.*

This result improves the previously known upper bound $n(n+1)/2$ for $\mathfrak{cd}(\mathrm{Spin}_{2n+1})$, established in [1] (see lemma 12.1(c) and proposition 12.3 in *loc.cit.*).

The proof of Theorem 1.1, given in section 5, makes use of the theory of non-negative intersections, of duality between Schubert varieties, and of the Pieri formula for varieties of maximal totally isotropic subspaces.

Note that the lower bound for $\mathfrak{cd}(\mathrm{Spin}_{2n+1})$, given by $\mathfrak{cd}_2(\mathrm{Spin}_{2n+1})$, coincides with our upper bound if (and only if) $n+1$ is a power of 2. Therefore, for such n, we get the precise value:

Corollary 1.2. *If $n+1$ is a power of 2, then*

$$\mathfrak{cd}(\mathrm{Spin}_{2n+1}) = \mathfrak{cd}(\mathrm{Spin}_{2n+2}) = \frac{n(n-1)}{2}\ . \qquad \square$$

Remark 1.3. For n up to 4, it is easy to see that $\mathfrak{cd}(\mathrm{Spin}_{2n+1}) = \mathfrak{cd}_2(\mathrm{Spin}_{2n+1})$ (see [1, example 12.2]), but for every $n \geq 5$ the bound of Theorem 1.1 is new.

Now let us consider the semi-spinor group $\mathrm{Spin}^{\sim}_{2n+2}$ (where the positive integer n is odd). Our second main result is the following upper bound for $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2})$, obtained by the similar technique:

Theorem 1.4 (see §5)**.** *For any odd $n \geq 1$, one has $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) \leq n(n-1)/2 + 2^k - 1$, where $k = v_2(n+1)$ (the 2-adic order of $n+1$).*

The canonical 2-dimension of the semi-spinor group, computed in [10, §8.4], is equal to

$$\mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{2n+2}) = \frac{n(n+1)}{2} + 2^k - 2^l\ ,$$

where $k = v_2(n+1)$ and l is still the minimal integer such that $2^l \geq n+1$. Since the upper bound of Theorem 1.4 coincides with the lower bound for $\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2})$, given by $\mathfrak{cd}_2(\mathrm{Spin}^{\sim}_{2n+2})$, if (and only if) $n+1$ is a power of 2, we get the precise value of the canonical dimension in this case:

Corollary 1.5 (cf. Remark 5.5)**.** *If $n+1$ is a power of 2, then*

$$\mathfrak{cd}(\mathrm{Spin}^{\sim}_{2n+2}) = n(n+1)/2\ . \qquad \square$$

The importance of the spinor and semi-spinor groups in this context is explained by the fact that these groups represent the only difficult cases of the following general question: let G be a split simple algebraic group, having a unique torsion prime p (a prime p is a *torsion prime* of G if and only if $\mathfrak{cd}_p(G) \neq 0$, [10, rem. 6.10]); is it true that

$\mathfrak{cd}(G) = \mathfrak{cd}_p(G)$?[1] Below we provide the complete list of split simple groups (type by type), possessing a unique torsion prime, together with the corresponding references. Almost all cited results on canonical dimension are from [1] and on canonical p-dimension from [10]. In each subsection we assume without repeating it, that G is a split simple group of the type under consideration.

1.1. $\mathbf{A}_n$, $n \geq 1$. We have $G \simeq \mathrm{SL}_{n+1}/\mu_r$, where r is a positive integer, dividing $n+1$. The group G has a unique torsion prime p if and only if r is a positive power of p. In this case it is known that $\mathfrak{cd}(G) = p^{v_p(n+1)} - 1 = \mathfrak{cd}_p(G)$ ([1, cor. 11.5] and [10, §8.1]).

1.2. $\mathbf{C}_n$, $n \geq 2$. In the simply connected case, $G \simeq \mathrm{Sp}_{2n}$ has no torsion primes (and $\mathfrak{cd}(G) = 0$). Otherwise one has $G \simeq \mathrm{PGSp}_{2n}$, 2 is the unique torsion prime, and $\mathfrak{cd}(G) = 2^{v_2(n)+1} - 1 = \mathfrak{cd}_2(G)$ ([1, cor. 11.6] and [10, §8.3]).

1.3. $\mathbf{B}_n$, $n \geq 3$. The prime 2 is the unique torsion prime for all groups if this type. The group G is either isomorphic to Spin_{2n+1} or to SO_{2n+1}. In the latter case, it is known that $\mathfrak{cd}(G) = n(n+1)/2 = \mathfrak{cd}_2(G)$ (see [1, prop. 12.3] for the inequality $\mathfrak{cd}(G) \leq n(n+1)/2$ and [10, §8.2] for computation of $\mathfrak{cd}_2(G)$; the equality $\mathfrak{cd}(G) = n(n+1)/2$ is originally proved in [9]).

1.4. $\mathbf{D}_n$, $n \geq 4$. The prime 2 is the unique torsion prime for all groups of this type. The group G is either isomorphic to Spin_{2n}, or to SO_{2n}, or to PGO^+_{2n}, and if n is even, then one more possibility is added: $G \simeq \mathrm{Spin}^{\sim}_{2n}$ (the semi-spinor group). Since the canonical (2-)dimension of SO_{2n} coincides with that of SO_{2n-1} ([1, lemma 12.1(a)]), we only have to consider the case of $G = \mathrm{PGO}^+_{2n} = \mathrm{PGO}^+(\varphi)$, where $\varphi : F^{2n} \to F$ is a hyperbolic quadratic form. Let P be the stabilizer of a point for the standard action of G on the scheme of n-dimensional (maximal) totally isotropic subspaces of the vector space F^{2n}. By lemma 7.5(a) in *loc.cit.*, we have $\mathfrak{cd}(G) \leq \dim(G/P) + \mathfrak{cd}(P)$. The Levi subgroup of P (which has the same canonical dimension as P, see example 10.1 in *loc.cit.*) is isomorphic to GL_n/μ_2 for even n and to GL_n for odd n; therefore (see lemma 11.2 in *loc.cit.*) $\mathfrak{cd}(P) = 2^{v_2(n)} - 1$ (for any n). Since $\dim(G/P) = n(n-1)/2$, we get that $\mathfrak{cd}(G) \leq n(n-1)/2 + 2^{v_2(n)} - 1$ (see Remark 5.6). Since the obtained upper bound for $\mathfrak{cd}(G)$ coincides with $\mathfrak{cd}_2(G)$ ([10, §8.4]), the desired equality $\mathfrak{cd}(G) = \mathfrak{cd}_2(G)$ follows.

1.5. $\mathbf{G}_2$. The prime 2 is the unique torsion prime here, and $\mathfrak{cd}(G) = 3 = \mathfrak{cd}_2(G)$ ([1, example 10.7] and [10, §8.5]).

1.6. $\mathbf{F}_4$ **and** $\mathbf{E}_n$, $n = 6, 7, 8$. We have multiple torsion primes here.

Remark 1.6. Since
$$\mathrm{cd}_p(G_1 \times G_2) = \mathrm{cd}_p(G_1) + \mathrm{cd}_p(G_2)$$
for any prime p and any split semisimple groups G_1 and G_2 ([10, rem. 7.4]), and at the same time
$$\mathrm{cd}(G_1 \times G_2) \leq \mathrm{cd}(G_1) + \mathrm{cd}(G_2)$$

[1]In this question we restrict ourself to the case of a single torsion prime because the case of multiple torsion primes (where $\mathfrak{cd}_p(G)$ should probably be replaced by $\max_p\{\mathfrak{cd}_p(G)\}$) is certainly of a higher level of difficulty; in fact, there does not exist any single G, having more than one torsion prime, for which $\mathfrak{cd}(G)$ is determined!

([1, lemma 7.5(d)]), it follows that $\mathrm{cd}(G) = \mathrm{cd}_p(G)$ for any split *semisimple* group G having a unique torsion prime p, provided that G is a direct product of split simple groups none of which is isomorphic to a (semi-)spinor group.

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2. Non-negativity

By *scheme* we mean a separated scheme of finite type over a field. A *variety* is an integral scheme.

Let X be a scheme. Following [5, ch. 12], an algebraic cycle $n_1Y_1 + \cdots + n_rY_r$ on X (where r is ≥ 0, Y_i are closed subvarieties of X, and n_i are integers) is called *non-negative*, if the coefficients $n_1, \ldots, n_r$ are non-negative. An element of the integral Chow group $\mathrm{CH}(X)$ is called *non-negative*, if it can be represented by a non-negative cycle.

I thank I. Panin for pointing me out the following as simple as useful fact:

Lemma 2.1 (see [7, prop. 2.1 of ch. I])**.** *Let L be a line vector bundle over a smooth variety X. The first Chern class $c_1(L) \in \mathrm{CH}^1(X)$ is non-negative if and only if L has a non-zero global section.*

Corollary 2.2. *Let X be a smooth absolutely irreducible variety over a field F, α an element of $\mathrm{CH}^1(X)$, E/F a field extension. If $\alpha_E \in \mathrm{CH}(X_E)$ is non-negative, then α itself is non-negative.* □

We are going to use the following

Theorem 2.3 ([5, §12.2])**.** *Let X be a smooth variety such that its tangent bundle is generated by the global sections. Then the product of non-negative elements in $\mathrm{CH}(X)$ is non-negative. Moreover, if $\alpha \in \mathrm{CH}(X)$ is represented by a non-negative cycle with support $A \subset X$, while $\beta \in \mathrm{CH}(X)$ is represented by a non-negative cycle with support $B \subset X$, then the product $\alpha\beta \in \mathrm{CH}(X)$ can be represented by a non-negative cycle with support on the intersection $A \cap B$.*

Remark 2.4. If X is a projective homogeneous variety under an action of an algebraic group, then the tangent bundle of X is generated by the global sections. Indeed, there exists a field extension E/F such that the variety X_E is isomorphic to the quotient G/P of a semisimple algebraic group G over E by a parabolic subgroup $P \subset G$. Therefore the tangent bundle of the variety X_E is generated by the global sections. Since the property of being generated by global sections is not changed under extension of the base field, the tangent bundle of the variety X is also generated by the global sections.

3. Dual Schubert varieties

Let G be a split semisimple algebraic group, B a Borel subgroup of G, containing a maximal split torus $\mathfrak{T}$. Let W be the Weyl group of G, and let $S \subset W$ be the set of reflections with respect to the simple roots. We fix a subset $\theta \subset S$, take the subgroup $W_\theta \subset W$, generated by θ, and consider the parabolic subgroup $P = P_\theta = BW_\theta B \subset G$.

Using the length function $l\colon W \to \mathbb{Z}_{\geq 0}$, induced by the set S of generators of the group W, we take in each coset of W/W_θ the unique minimal length element and write $W^\theta \subset W$ for the set of representatives thus obtained.

The variety $X = G/P$ is cellular, the cells BwP/P are indexed by $w \in W^\theta$. We write X_w for the closure in G/P of the corresponding cell. The varieties X_w are called (generalized) Schubert varieties; their classes $[X_w] \in \mathrm{CH}(X)$, called (generalized) Schubert classes, form a basis of the group $\mathrm{CH}(X)$. Moreover, $\dim X_w = l(w)$.

Sometimes the upper indexation of the Schubert varieties, which respects their codimension, is more convenient. To define it, let us take the (unique) maximal length element w_0 of the whole group W and set $X^w = X_{w'}$, where w' is the minimal length element in the coset w_0wW_θ (so that $w' = w_0w''$, where w'' is the maximal length element of the coset wW_θ). Now we have $\operatorname{codim} X^w = \dim X_w = l(w)$.

Proposition 3.1 ([12, prop. 1.4])**.** *Let* $\deg\colon \mathrm{CH}(X) \to \mathbb{Z}$ *be the degree homomorphism. Then for any* $w, w' \in W^\theta$ *one has:*

$$\deg([X^w] \cdot [X_{w'}]) = \begin{cases} 1, & \text{if } w' = w; \\ 0, & \text{otherwise.} \end{cases}$$

Because of this property, we refer to the varieties X^w and X_w (as well as to their classes) as *mutually dual*.

Let us now specify the situation: take as G the special orthogonal group $\mathrm{SO}_{2n+1} = \mathrm{SO}(\psi)$ of a non-degenerate split quadratic form $\psi : F^{2n+1} \to F$. By saying *split*, we mean existence in F^{2n+1} of an n-dimensional totally isotropic subspace (and by saying *non-degenerate* in the characteristic 2 case we mean, as in [11], that the radical of the associated bilinear form is of dimension 1 and ψ is non-zero on the radical). Let us choose a complete flag

$$\mathcal{F} = (0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n)\ , \quad \dim \mathcal{F}_i = i$$

of totally isotropic subspaces of F^{2n+1} and a subspace D_i in each $\mathcal{F}_i$ such that $\mathcal{F}_i = \mathcal{F}_{i-1} \oplus D_i$. Then we take as $B \subset G$ the stabilizer of $\mathcal{F}$, as P the stabilizer of $\mathcal{F}_n$, and as $\mathfrak{T}$ the stabilizer of all D_i. The variety $X = G/P$ is therefore the variety of n-dimensional (maximal) totally isotropic subspaces of V.

Now the Schubert varieties on X are indexed by the strictly decreasing sequences $i_1 > i_2 > \cdots > i_s$ of positive integers, satisfying $n \geq i_1$ (or, if one prefers, by the subsets of the set $\{1, 2, \ldots, n\}$). The $(i_1 \ldots i_s)$-th Schubert variety $X^{i_1 \ldots i_s} \subset X$ is the closed subvariety of the subspaces W such that $\dim(W \cap \mathcal{F}_{n+1-i_t}) \geq t$ for $t = 1, 2, \ldots, s$; the variety $X^{i_1 \ldots i_s}$ has codimension $i_1 + \cdots + i_s$, and we write $e_{i_1 \ldots i_s}$ for its class in $\mathrm{CH}(X)$. The Schubert classes e_i $(i = 1, 2, \ldots, n)$ are called *special*.

As a specific case of Proposition 3.1 we get

Corollary 3.2. *For any* $e_{i_1 \ldots i_s}$ *there exists* $e_{i'_1 \ldots i'_{s'}}$ *such that for any* $e_{i''_1 \ldots i''_{s''}}$

$$\deg(e_{i''_1 \ldots i''_{s''}} \cdot e_{i'_1 \ldots i'_{s'}}) = \begin{cases} 1, & \text{if } e_{i''_1 \ldots i''_{s''}} = e_{i_1 \ldots i_s}; \\ 0, & \text{otherwise.} \end{cases}$$

□

Remark 3.3. Note that deducing Corollary 3.2 from Proposition 3.1, we do not need to know the precise relationship between the above two indexations of the Schubert varieties on $X = G/P$ (the indexation by the elements of the Weyl group and the indexation by the sequences of integers). The reader however might be interested to know how these indexations are related. It can be described even in a more general situation of an arbitrary parabolic subgroup P of the orthogonal group G. For $P = B$ (the Borel subgroup), where the Schubert varieties are indexed by *all* elements of W, the reference is [6, p.67]. Otherwise, in order to relate the Schubert classes on G/P with the Schubert classes on G/B, one makes use of the projection $\pi\colon G/B \to G/P$, which has the following property: for any $w \in W^\theta$, the pull-back homomorphism $\pi^* : \mathrm{CH}(G/P) \to \mathrm{CH}(G/B)$ maps the Schubert class $(G/P)^w$ to the Schubert class $(G/B)^w$ (see [12, lemma 1.2(b)] and [4, §3.3]); and (as follows from [12, lemma 1.2(c)]) for any $w \in W$, the push-forward homomorphism $\pi_*\colon \mathrm{CH}(G/B) \to \mathrm{CH}(G/P)$ maps $(G/B)_w$ to $(G/P)_w$ or to 0 depending on whether $w \in W^\theta$ or $w \notin W^\theta$.

4. Pieri formula

The classical Pieri formula deals with grassmannians and expresses the product of a special Schubert class by an arbitrary Schubert class as a linear combination of Schubert classes. We are going to use an analogous formula for the variety of maximal totally isotropic subspaces, which is due to Hiller and Boe, [8] (a way of a simpler proof is suggested in [15]):

Theorem 4.1. *Let n be a positive integer and let X be the variety of n-dimensional totally isotropic subspaces of a non-degenerate split $(2n+1)$-dimensional quadratic form. The following multiplication formula holds for the Schubert classes in $\mathrm{CH}(X)$: for any strictly decreasing sequence of positive integers $x = (x_1, \dots, x_k)$, satisfying $x_1 \le n$, and any positive integer $p \le n$, one has*

$$e_p \cdot e_x = \sum_y 2^{m_y^{x,p}} e_y \ ,$$

where the sum runs over all strictly decreasing sequences of integers $y = (y_1, \dots, y_{k+1})$, satisfying

$$n \ge y_1 \ge x_1 \ge y_2 \ge \dots \ge y_k \ge x_k \ge y_{k+1} \ge 0$$

and $y_1 + \dots + y_{k+1} = p + x_1 + \dots + x_k$ (in the case of $y_{k+1} = 0$, we define e_y as $e_{y_1 \dots y_k}$). The exponent $m_y^{x,p}$ of the coefficient of e_y is determined as follows:

$$m_y^{x,p} = \begin{cases} \text{the number of } i \in [1,\ k] \text{ such that } y_i > x_i > y_{i+1}, & \text{if } y_{k+1} \ne 0; \\ \text{the above number minus } 1, & \text{if } y_{k+1} = 0. \end{cases}$$

Remark 4.2. Theorem 4.1 is proved in [8] under the assumption that the base field is algebraically closed; the base field in [15] is $\mathbb{C}$. However, as shown in [4], the multiplication table for the Schubert classes in $\mathrm{CH}(G/B)$, where G is a split semisimple algebraic group and B is its Borel subgroup, depends only on the type of G and does not depend on the base field. Now if $P \subset G$ is a parabolic subgroup, containing B, then, as already notices in Remark 3.3, the pull-back with respect to the projection $G/B \to G/P$ is an injective ring homomorphism, mapping each Schubert class $[X^w] \in \mathrm{CH}(G/P)$ (the notation is introduced in §3) to the "same" Schubert class in $\mathrm{CH}(G/B)$. Therefore the multiplication table for the Schubert classes in $\mathrm{CH}(G/P)$ depends only on the type of the pair (G, P) and does not depend on the base field either.

Corollary 4.3. *Under condition of Theorem 4.1, one has $e_1^n = e_n + \dots$, where dots stand for a linear combination of Schubert classes different from e_n.*

Proof. Let x and y be two strictly decreasing sequences of positive integers $\leq n$. We say that y is an *deformation* of x (and write $x \rightsquigarrow y$), if e_y appears in the formula for $e_1 \cdot e_x$, given by Theorem 4.1, in which case we refer to the number $m_y^{x,1}$ as the *exponent* of the deformation.

There is a unique chain of deformations, transforming (1) to (n), namely, the chain $(1) \rightsquigarrow (2) \rightsquigarrow \dots \rightsquigarrow (n)$. Since the exponent of each deformation in the chain is 0, the statement follows. □

5. Proofs of Theorems 1.1 and 1.4

Proposition 5.1. *For a positive integer n, let us fix a non-degenerate split quadratic form $F^{2n+1} \to F$. Let G be a split simple algebraic group over F and let P be its parabolic subgroup such that the quotient G/P is isomorphic to the variety of n-dimensional (maximal) totally isotropic subspaces of F^{2n+1}. Let T be a G-torsor and let L be a splitting field of T.*

If for the quotient variety $X = T/P$ the restriction homomorphism $\mathrm{CH}^1(X) \to \mathrm{CH}^1(X_L)$ is surjective, then X contains a closed subvariety Y of dimension $n(n-1)/2$ with $Y(L) \neq \emptyset$.

Proof. For any F-scheme Z, let us write $\bar{Z}$ for Z_L. Since the variety $\bar{X}$ is isomorphic to $\overline{G/P}$ (which is isomorphic to the variety of n-dimensional (maximal) totally isotropic subspaces of L^{2n+1}), we may speak about the Schubert classes $e_{i_1 \dots i_s}$ in $\mathrm{CH}(\bar{X})$ (see §3). We write $\bar{\mathrm{CH}}(X)$ for the image of the restriction homomorphism $\mathrm{CH}(X) \to \mathrm{CH}(\bar{X})$.

Let us take the Schubert class $e_1 \in \mathrm{CH}^1(\bar{X})$. By our assumption on $\mathrm{CH}^1(X)$, we have $\bar{\mathrm{CH}}^1(X) = \mathrm{CH}^1(\bar{X})$; therefore $e_1 \in \bar{\mathrm{CH}}^1(X)$. Moreover, by Corollary 2.2, e_1 is non-negative element of $\bar{\mathrm{CH}}(X)$. It follows by Theorem 2.3 and Remark 2.4 that the n-th power of e_1 is also a non-negative element of $\bar{\mathrm{CH}}(X)$, so that we can write

$$e_1^n = n_1[Y_1] + \dots + n_r[Y_r] \in \bar{\mathrm{CH}}(X)$$

with some non-negative integers n_i and some closed subvarieties $Y_i \subset X$. Note that $\dim Y_i = \dim X - n = n(n-1)/2$ for all i.

By Corollary 4.3, we have $e_1^n = e_n + \dots \in \mathrm{CH}(\bar{X})$, where dots stand for a linear combination of Schubert classes different from e_n. Using Corollary 3.2, we find the Schubert class $e \in \mathrm{CH}(\bar{X})$ dual to e_n, that is, such that $\deg(e_n e) = 1$ and $\deg(e' e) = 0$ for any Schubert class e' different from e_n. For this e we have $\deg(e_1^n e) = 1$. Since for any i the product $[\bar{Y}_i \cdot e]$ is non-negative, it follows that $\deg([\bar{Y}_i] \cdot e) \geq 0$ for any i and therefore $\deg([\bar{Y}_i] \cdot e) = 1$ (and $n_i = 1$) for some $i \in [1,\ r]$.

Let Z be the Schubert variety, representing e. Since the product $[\bar{Y}_i] \cdot [Z]$ is a 0-cycle class of degree 1 and can be represented by a non-negative cycle on the intersection $\bar{Y}_i \cap Z$ (see Theorem 2.3 and Remark 2.4), the scheme $\bar{Y}_i$ has a rational point, that is, $Y_i(L) \neq \emptyset$. □

Remark 5.2. If for some n, at least one of the Schubert classes would appear with coefficient 1 in the decomposition of e_1^{n+1}, we could improve (by 1) the bounds of Theorem 1.1 and (if n is odd) also the bound of Theorem 1.4 for this n. Unfortunately, for any n, none of the Schubert classes appears with coefficient 1 in the decomposition of e_1^{n+1} (of course, "unfortunately" does not concern such n for which $n+1$ is a 2-power, because in that case our bounds on $\mathfrak{cd}(\mathrm{Spin}_{2n+1})$ and on $\mathfrak{cd}(\mathrm{Spin}_{2n+2})$ are equal to the canonical dimension).

Proof of Theorem 1.1. For $G = \mathrm{Spin}_{2n+1} = \mathrm{Spin}(\psi)$, where $\psi : F^{2n+1} \to F$ is a non-degenerate split quadratic form, let T be a G-torsor (over F). In order to prove Theorem 1.1, it suffices to check that $\mathrm{cd}(T) \le n(n-1)/2$.

Let P be the stabilizer of a point for the standard action of G on the variety of n-dimensional totally isotropic subspaces of F^{2n+1}. Since P is special, that is, has no non-trivial torsors over any extension of the base field ([10, §8.2]), we have $\mathrm{cd}(T) = \mathrm{cd}(X)$, where $X = T/P$ (lemma 6.5 in *loc.cit.*). By corollary 4.7 of *loc.cit.*, the canonical dimension $\mathrm{cd}(X)$ is the minimum of $\dim Y$, where Y runs over all closed subvarieties of X such that $Y(F(X)) \neq \emptyset$. To finish the proof, we apply Proposition 5.1 to the group G, the parabolic subgroup P, the torsor T, and the splitting field $L = F(X)$ of T. Note that the homomorphism $\mathrm{CH}^1(X) \to \mathrm{CH}^1(\bar{X})$ is surjective (see [10, §8.2] or [16, proof of lemma 3.1] or the proof of Theorem 1.4). □

Before starting the proof of Theorem 1.4, we establish one general bound on canonical dimension of torsors.

Let G be a split semisimple algebraic group over a field F, P a parabolic subgroup of G, P' a special parabolic subgroup of G (for example, a Borel subgroup) sitting inside of P. For any G-torsor T, let us write $\mathrm{cd}'(T/P)$ for $\min_x\{\dim x\}$, where x runs over all points of the variety T/P admitting an F-embedding of the residue field $F(x)$ into the function field $F(T/P')$. Note that replacing P' by P in the definition of $\mathrm{cd}'(T/P)$, we get $\mathrm{cd}(T/P)$ because by [10, cor. 4.6], for any smooth projective F-variety X (and in particular for $X = T/P$), $\mathrm{cd}(X) = \min_x\{\dim x\}$, where x runs over all points of X having residue field F-embeddable into the function field of X (a slightly different formulation of this principle can be seen in the proof of Theorem 1.1). The projection $T/P' \to T/P$ induces an F-embedding $F(T/P) \hookrightarrow F(T/P')$, showing that $\mathrm{cd}'(T/P) \le \mathrm{cd}(T/P)$.

Let S be the fiber of the projection $T \to T/P$ over a point $x \in T/P$. Note that S is a $P_{F(x)}$-torsor and the fiber of the projection $T/P' \to T/P$ over x is the projective $P_{F(x)}$-homogeneous variety $S/P'_{F(x)}$. To simplify notation, let us agree to write S/P' instead of $S/P'_{F(x)}$.

Lemma 5.3. *In the above notation, one has*

$$\mathrm{cd}(T) \le \mathrm{cd}'(T/P) + \max_Y \mathrm{cd}(Y) ,$$

where Y runs over all fibers of the projection $T/P' \to T/P$.

Proof. Let x be a point of T/P with the residue field $K = F(x)$ being F-embeddable into $F(T/P')$ and such that $\dim x = \mathrm{cd}'(T/P)$. Let S be the fiber of the projection $T \to T/P$ over the point x. Then S is a P_K-torsor, and the fiber of the projection $T/P' \to T/P$

over the point x is given by the projective P_K-homogeneous variety S/P'. Note that according to [2, th. 4.13], for any field $L \supset F$, the abstract group $\operatorname{Aut} T_L$ of G_L-equivariant automorphisms of T_L acts transitively on the set $(T/P)(L)$ of L-points of T/P and therefore the fibers of $T \to T/P$ over any two L-points are P_L-equivariantly isomorphic. In particular, the generic fiber with the scalars extended to the field $F(T/P') \supset F(T/P)$ is equivariantly isomorphic to S with the scalars extended to $F(T/P') \supset K$. Consequently, the generic fiber of the projection $T/P' \to T/P$ with the scalars extended to the field $F(T/P')$ is isomorphic to $(S/P')_{F(T/P')}$.

Let y be a point of S/P' with residue field K-embeddable into $K(S/P')$ and such that $\dim y = \operatorname{cd}(S/P')$. As a point of T/P', y has dimension

$$\dim_{T/P'} y = \dim x + \dim_{S/P'} y = \operatorname{cd}'(T/P) + \operatorname{cd}(S/P')\ .$$

Since $K(S/P') \subset F(T/P')(S/P') \simeq F\big((T/P')^2\big)$ (where $(T/P')^2 = (T/P') \times_{T/P} (T/P')$) by the established above isomorphism of fibers, the residue field of y is embeddable into the field $F\big((T/P')^2\big)$. The F-variety $(T/P')^2$ is smooth and has an $F(T/P')$-point (the generic point of the diagonal) so that there exists an F-place $F\big((T/P')^2\big) \dashrightarrow F(T/P')$ (in fact, the field extension $F\big((T/P')^2\big)/F(T/P')$ is purely transcendental); composing it with an F-embedding $F(y) \hookrightarrow F\big((T/P')^2\big)$, we get an F-place $F(y) \dashrightarrow F(T/P')$. Therefore $Y(F(T/P')) \neq \emptyset$ for the closure Y of the point y in T/P', and we get that

$$\operatorname{cd}(T) = \operatorname{cd}(T/P') \leq \dim Y = \dim_{T/P'} y. \qquad \square$$

Remark 5.4. We would not see the *square* of the variety T/P' in the proof of Lemma 5.3, if we would replace cd' by ordinary cd in the statement of Lemma 5.3. However the inequality with $\operatorname{cd}(T/P)$ instead of $\operatorname{cd}'(T/P)$ would be weaker and not enough for the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\varphi : F^{2n+2} \to F$ be a hyperbolic quadratic form such that the restriction $\varphi|_{F^{2n+1}}$ is non-degenerate. Let us consider the standard action of $G = \operatorname{Spin}^{\sim}_{2n+2} = \operatorname{Spin}^{\sim}(\varphi)$ on the scheme of $(n+1)$-dimensional (maximal) totally isotropic subspaces of the vector space F^{2n+2}. This scheme has two components (each isomorphic to the variety of n-dimensional totally isotropic subspaces of $F^{2n+1} \subset F^{2n+2}$ with the isomorphism induced by intersection with F^{2n+1}); for an appropriately chosen component, the stabilizer P of any point on the component has the following property: for any field extension K/F and any G_K-torsor T, the relative Brauer group

$$\operatorname{Br}(F(X)/F) = \operatorname{Ker}\big(\operatorname{Br}(F) \to \operatorname{Br}(F(X))\big)$$

of the function field of the variety $X = T/P$ is trivial, [13, appendix] ($\operatorname{char} F \neq 2$ case) and [14, rem 9.2] ($\operatorname{char} F = 2$ case). So, we fix a parabolic subgroup $P = \operatorname{Stab}(x)$ with this property. This parabolic subgroup is not special yet, a special parabolic subgroup $P' \subset P$ can be obtained as follows: we consider the standard action of G on the scheme of flags "a 1-dimensional subspace of F^{2n+2} sitting inside of an $(n+1)$-dimensional totally isotropic subspace" and define $P' = \operatorname{Stab}(x')$, where x' is a point lying over the point x (see [10, §8.4] for a proof that P' is special).

Let T be a G-torsor. To prove Theorem 1.4 it suffices to show that

$$\operatorname{cd}(T) \leq \frac{n(n-1)}{2} + 2^k - 1\ .$$

By Lemma 5.3, applied to the subgroups P' and P introduced above, we have

$$\operatorname{cd}(T) \leq \operatorname{cd}'(T/P) + \max \operatorname{cd}(Y) ,$$

where Y runs over the fibers of the projection $T/P' \to T/P$. Any fiber of the projection $G/P' \to G/P$ is an n-dimensional projective space. Therefore the fiber over any point of the projection $T/P' \to T/P$ is an n-dimensional Severi-Brauer variety; moreover, this Severi-Brauer variety splits over any splitting field of T, in particular, it splits over a 2-primary extension of the base field (because 2 is the unique torsion prime of G). It follows that such a Severi-Brauer variety corresponds to a $(\operatorname{SL}_{n+1}/\mu_{2^k})$-torsor and therefore the canonical dimension of the fiber is bounded by $\operatorname{cd}(\operatorname{SL}_{n+1}/\mu_{2^k}) = 2^k - 1$ (we recall that $k = v_2(n+1)$).

It remains to check that $\operatorname{cd}'(T/P) \leq n(n-1)/2$. We get this inequality applying Proposition 5.1 to our G, P and T, taking as the splitting field L of the torsor T the function field $F(T/P')$. We only need to check that the homomorphism $\operatorname{CH}^1(X) \to \operatorname{CH}^1(X_L)$, where $X = T/P$, is surjective. Note that the group $\operatorname{CH}(X_L)$ is canonically isomorphic to the group $\operatorname{CH}(X_{\bar F})$, where $\bar F$ is a separable closure of F: the isomorphism is obtained as the composition of the isomorphisms

$$\operatorname{CH}(X_L) \xrightarrow{\operatorname{res}_{\bar L/L}} \operatorname{CH}(X_{\bar L}) \xrightarrow{\operatorname{res}^{-1}_{\bar L/\bar F}} \operatorname{CH}(X_{\bar F}) ;$$

in particular, the homomorphism $\operatorname{CH}^1(X) \to \operatorname{CH}^1(X_L)$ is surjective if the homomorphism $\operatorname{CH}^1(X) \to \operatorname{CH}^1(X_{\bar F})$ is surjective. The absolute Galois group of F acts trivially on $\operatorname{CH}^1(X_{\bar F}) = \mathbb{Z}\cdot e_1$. Consequently, the cokernel of the homomorphism $\operatorname{CH}^1(X) \to \operatorname{CH}^1(X_{\bar F})$ is identified with the relative Brauer group $\operatorname{Br}(F(X)/F)$ (see, e.g., [3, proof of th. 3.1]), which is trivial by our choice of P. □

Remark 5.5. Corollary 1.5 of Theorem 1.4 has a short direct proof. Let $P \subset G = \operatorname{Spin}^{\sim}_{2n+2}$ be the stabilizer of a point of the scheme of totally isotropic subspaces of F^{2n+2}, lying on the other component than the component used in the proof of Theorem 1.4. Then P is special (the Levi subgroup of P is isomorphic to GL_{n+1}) and therefore

$$\mathfrak{cd}(G) \leq \dim(G/P) = n(n+1)/2 .$$

Since $n(n+1)/2 = \mathfrak{cd}_2(G)$ if $n+1$ is a 2-power, Corollary 1.5 follows (however, for any n such that $n+1$ is not a 2-power, the bound given in Theorem 1.4 is better than the bound $n(n+1)/2$).

Remark 5.6. Following the spirit of the proof of Theorem 1.4, one can give an alternative proof of the inequality $\mathfrak{cd}(\operatorname{PGO}^+_{2n}) \leq n(n-1)/2+2^{v_2(n)}-1$ (see §1.4), avoiding computation of Levi subgroups. Consider the standard action of $G = \operatorname{PGO}^+_{2n}$ on the scheme of n-dimensional totally isotropic subspaces of F^{2n}, and let P be the stabilizer of a point x. Also consider the standard action of G on the scheme of flags "a 1-dimensional subspace contained in an n-dimensional totally isotropic subspace", and let P' be the stabilizer of a point x' lying over x. The parabolic subgroup P' is special and $P' \subset P$. Let T be a G-torsor. Any fiber Y of the projection $T/P' \to T/P$ (in fact, we are interested only in the generic fiber here) is an $(n-1)$-dimensional Severi-Brauer variety, which can be split by a 2-primary field extension; therefore $\operatorname{cd}(Y) \leq 2^{v_2(n)} - 1$. Since $\operatorname{cd}(T) = \operatorname{cd}(T/P') \leq$

$\mathrm{cd}(Y) + \dim(G/P)$ (this inequality formally follows from Lemma 5.3, but is in fact an easier statement) and $\dim(G/P) = n(n-1)/2$, we get the desired bound on $\mathfrak{cd}(G)$.

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