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Canonical Dimension

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Abstract

Canonical dimension is an integral-valued invariant of algebraic structures. We are mostly interested in understanding the canonical dimension of projective homogeneous varieties under semisimple affine algebraic groups over arbitrary fields. Known methods, results, applications, and open problems are reviewed, some new ones are provided.

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0. Introduction

A smooth projective variety X is *incompressible*, if any rational map $X \dashrightarrow X$ is dominant. *Canonical dimension* $\operatorname{cdim} X$, an invariant measuring the level of compressibility of X, is the minimum of the dimension of the image of a rational map $X \dashrightarrow X$. Formally introduced by G. Berhuy and Z. Reichstein only in 2005, [3], this invariant has been implicitly studied for a long time before. For instance, an old question of M. Knebusch, [19, Question 4.13], answered in Example 1.5, was about the canonical dimension of a quadric. Also the incompressibility of the Severi-Brauer variety of a primary division algebra – see Example 2.3 – has been known and intensively applied since 1995.

In this talk we look at the canonical dimension of a projective homogeneous variety X, mainly, through the motive of X. This approach is justified by Theorem 5.1.

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1. Definitions of Canonical Dimension

By *variety* we mean an *integral* separated scheme of finite type over a field.

Since we are mainly interested in canonical dimension of projective homogeneous varieties, we define it for smooth projective varieties only. We refer to [25] for the case of a more general variety.

Let X be a smooth projective variety over a field F.

Definition 1.1. *Canonical dimension* $\operatorname{cdim} X$ of X is the minimum of $\dim Y$, where Y runs over the closed subvarieties of X admitting a rational map $X \dashrightarrow Y$. Equivalently, Y runs over the closed subvarieties of X such that the scheme $Y_{F(X)}$ has a rational point.

Of course, $\operatorname{cdim} X = 0$ if X has a rational point. We are basically interested in varieties without rational points.

In general, $\operatorname{cdim} X$ is an integer satisfying

$$0 \le \operatorname{cdim} X \le \dim X.$$

Let p be a positive prime integer. We write Ch for the Chow group [7, §57] with coefficients in $\mathbb{F}_p$, the finite field of p elements. By a *correspondence* $X \rightsquigarrow Y$ we mean an element of the Chow group $\operatorname{Ch}_{\dim X}(X \times Y)$. The *multiplicity* $\operatorname{mult} \alpha \in \mathbb{F}_p$ of a correspondence $\alpha : X \rightsquigarrow Y$ (also called *degree* in the literature) is its image under the push-forward homomorphism

$$\operatorname{Ch}_{\dim X}(X \times Y) \to \operatorname{Ch}_{\dim X}(X) = \mathbb{F}_p$$

with respect to the projection $X \times Y \to X$. Finally, a 0-*cycle class* is an element of $\operatorname{Ch}_0(X)$, its *degree* is therefore an element of $\operatorname{Ch}_0(\operatorname{Spec} F) = \mathbb{F}_p$.

Our actual subject of study is the *canonical p-dimension*, a p-local version of the above notion, defined as follows:

Definition 1.2. *Canonical p-dimension* $\operatorname{cdim}_p X$ of X is the minimum of $\dim Y$, where Y runs over the closed subvarieties of X admitting a multiplicity 1 correspondence $X \rightsquigarrow Y$. Equivalently, Y runs over the closed subvarieties of X such that the scheme $Y_{F(X)}$ has a 0-cycle class of degree 1.

Of course, $\operatorname{cdim}_p X = 0$ if X has a 0-cycle class of degree 1. We are basically interested in varieties without 0-cycle classes of degree 1, that is, varieties where the degree of each closed point is divisible by p.

In general, $\operatorname{cdim}_p X$ is an integer satisfying

$$0 \le \operatorname{cdim}_p X \le \operatorname{cdim} X.$$

There are at least two more definitions of the canonical (p-)dimension looking quite differently. We refer to [25] for a proof that they are equivalent to the initial one. We start by the definition via the *essential dimension*. We refer to [25, §1.1] for the definition of the essential (p-)dimension of an arbitrary functor $\mathbf{Fields}_F \to \mathbf{Sets}$ of the category of the field extensions of F to the category of sets.

Definition 1.3. Let $\mathcal{F}_X : \mathbf{Fields}_F \to \mathbf{Sets}$ be the functor defined by the formulas $\mathcal{F}_X(L) = \emptyset$ if $X(L) = \emptyset$ and $\mathcal{F}_X(L) = \{L\}$ (a singleton) otherwise. We define $\operatorname{cdim} X$ as the essential dimension of the functor $\mathcal{F}_X$, and we define $\operatorname{cdim}_p X$ as its essential p-dimension.

We come to the last definition. It makes use of the notion of a *generic splitting field* of a variety. We say that a field L/F is a *splitting field* (or *isotropy field*) of X is $X(L) \neq \emptyset$. A splitting field E/F is *generic*, if for each splitting field L/F of X there exists an F-place $E \dashrightarrow L$. A splitting field E/F is *p-generic*, if for each splitting field L/F of X there exist a finite field extension L'/L of a p-prime degree and an F-place $E \dashrightarrow L'$. Of course, any generic splitting field is also p-generic (for any p); the function field $F(X)$ is a generic splitting field.

Definition 1.4. We define the canonical (p-)dimension of X as the minimum of the transcendence degree of a (p-)generic splitting field of X.

The last definition (as well as the previous one) naturally generalizes to the case of an arbitrary "algebraic structure" A in place of X as soon as we have a notion of a *splitting field* for A. We consider two examples of such a generalization. (However, one easily comes back to varieties in both examples.)

Example 1.5. Let φ be a finite-dimensional non-degenerate quadratic form over F. A field L/F is a splitting field (or isotropy field) of φ if the quadratic form φ_L has a non-trivial zero. This way we get the notion of the canonical (p-)dimension of φ. Let X be the projective quadric of φ. We have $\operatorname{cdim} \varphi = \operatorname{cdim} X$ and $\operatorname{cdim}_p \varphi = \operatorname{cdim}_p X$, because a splitting field of φ is the same as a splitting field of X. These invariants are computed. If $X(F) = \emptyset$, i.e., if the quadric X is *anisotropic*, then we have $\operatorname{cdim} \varphi = \operatorname{cdim}_2 \varphi = \dim X - \mathfrak{i}_1 + 1$, where $\mathfrak{i}_1$ is the *first Witt index* of φ, [7, Theorem 90.2]. (Of course, $\operatorname{cdim}_p \varphi = 0$ for $p \neq 2$.)

Example 1.6. Let A be a finite p-subgroup of the Brauer group $\operatorname{Br} F$ of F. A field L/F is a splitting field of A if $A_L = 0$, i.e., if A vanishes under the change of field homomorphism $\operatorname{Br} F \to \operatorname{Br} L$. We get the notion of the canonical (p-) dimension of A. Let $A_1, \dots, A_n$ be central simple F-algebras such that their classes are in A and generate A; let X be the direct product of the corresponding Severi-Brauer varieties. We have $\operatorname{cdim} A = \operatorname{cdim} X$ and $\operatorname{cdim}_p A = \operatorname{cdim}_p X$ (for any X obtained this way), because a splitting field of A is the same as a splitting field of X. These invariants are computed as $\operatorname{cdim} A = \operatorname{cdim}_p A = \min \dim X$, [18, §2]. (Of course, $\operatorname{cdim}_{p'} A = 0$ for $p' \neq p$.)

The result of Example 1.6 has numerous applications. Many of them compute the essential dimension of algebraic groups as the following series of papers on finite p-groups. It was initiated by M. Florence who used the case of *cyclic* A to compute the essential dimension of a finite *cyclic* p-group in [8]. Arbitrary finite constant p-groups have been treated later on in [18]. Finally, the case of an arbitrary finite p-group (as well as the case of an algebraic tori), still essentially using Example 1.6, has been recently done by R. Lötscher, M. Macdonald, A. Meyer, and Z. Reichstein, [22].

Here is an example of a class of projective homogeneous varieties for which the canonical p-dimension is computed in terms of their Chow groups. These are the *generically split* projective homogeneous varieties. A projective homogeneous variety X is *generically split*, if the $F(X)$-variety $X_{F(X)}$ is cellular.

Example 1.7 ([17, Theorem 5.8]). Let X be a generically split projective homogeneous variety and let $\bar{X} := X_{\bar{F}}$ with an algebraic closure $\bar{F}$ of F. The canonical p-dimension $\operatorname{cdim}_p X$ coincides with the minimal integer i such that the change of field homomorphism $\operatorname{Ch}_i(X) \to \operatorname{Ch}_i(\bar{X})$ is non-zero.

Let G be a *split* simple affine algebraic group, T a *generic* G-torsor, B a Borel subgroup of G. Using the result of Example 1.7, the canonical dimension of the (generically split) projective homogeneous variety T/B is determined: the case of a classical G is done in [17], the case of an exceptional G in [28].

Example 1.8. Let n be a positive integer and X be the variety of n-dimensional totally isotropic subspaces of a $2n+1$-dimensional non-degenerate quadratic form φ. The variety X is homogeneous and generically split. Its canonical (2-) dimension is the canonical (2-)dimension of φ if defining the splitting fields of φ we require that φ becomes completely split (i.e., almost hyperbolic). The canonical 2-dimension of X is known, [7, Theorem 90.3]; $\operatorname{cdim} X$, however, is not known in general. It is conjectured in [27, Conjecture 6.6] that $\operatorname{cdim} X = \operatorname{cdim}_2 X$.

2. Incompressible Varieties

A smooth projective variety X is *incompressible*, if $\operatorname{cdim} X = \dim X$; X is *p-incompressible*, if $\operatorname{cdim}_p X = \dim X$. Equivalently, X is incompressible if any rational map $X \dashrightarrow X$ is dominant, that is, no proper closed subset $Y \subset X$ admits a rational map $X \dashrightarrow Y$; X is p-incompressible, if no proper closed subset $Y \subset X$ admits a degree 1 correspondence $X \rightsquigarrow Y$.

Of course, any p-incompressible variety (for some p) is incompressible. An example of an incompressible and p-compressible (for any p) projective homogeneous variety is obtained in [21] with a help of the birational classification of geometrically rational surfaces:

Example 2.1. Let X_1 be the Severi-Brauer variety of a *quaternion* (i.e., *degree* 2 central) division algebra and let X_2 be the Severi-Brauer variety of a *degree* 3 central division algebra. The (projective homogeneous, 3-dimensional) variety $X := X_1 \times X_2$ is incompressible. However, $\operatorname{cdim}_2 X = \operatorname{cdim}_2 X_1 = \dim X_1 = 1$ and $\operatorname{cdim}_3 X = \operatorname{cdim}_2 X_2 = \dim X_2 = 2$ (and $\operatorname{cdim}_p X = 0$ for any other p).

An important source of p-incompressible varieties is Proposition 2.2 below which is a consequence of the A. Merkurjev *degree formula* [24, Theorem 6.4], a generalization of the M. Rost degree formula.

For any sequence $R = (r_1, r_2, \dots)$ of non-negative almost all zero integers r_i, a homogeneous integral polynomial $T_R \in \mathbb{Z}[\sigma_1, \sigma_2, \dots]$ in variables $\sigma_1, \sigma_2, \dots$ of degree $|R| := \sum_{i\geq 1} r_i(p^i - 1)$ is defined in [24, §4], where for any $i \geq 1$ the degree of the variable σ_i is defined as i. (The polynomial T_R also depends on the prime p which we have fixed before.) For any smooth projective variety X of dimension $|R|$, the characteristic number $R(X)$ is defined as $R(X) := \deg c_R(-T_X) \in \mathbb{Z}$, where c_R is the characteristic class $c_R := T_R(c_1, c_2, \dots)$ (the polynomial T_R evaluated on the Chern classes $c_1, c_2, \dots$) and T_X (which has nothing to do with T_R) is the tangent bundle of X.

For any integer n, we write $v_p(n)$ for the value on n of the p-adic valuation. For any F-scheme X, we write $v_p(X)$ for the value of the p-adic valuation on the greatest common divisor of the degrees of the closed points on X.

Clearly, $v_p(R(X)) \geq v_p(X)$ for any R. A smooth projective variety X is *p-rigid*, if $v_p(R(X)) = v_p(X)$ for some R.

A smooth projective variety X is *strongly p-incompressible*, if for any projective variety Y with $v_p(Y) \geq v_p(X)$, $\dim Y \leq \dim X$, and a multiplicity 1 correspondence $X \rightsquigarrow Y$, one has: $\dim Y = \dim X$ (in particular, any strongly p-incompressible variety is p-incompressible) and there also exists a multiplicity 1 correspondence $Y \rightsquigarrow X$.

Proposition 2.2 ([24, Theorem 7.2]). *Assume that* $\operatorname{char} F \neq p$. *Then any p-rigid F-variety is strongly p-incompressible.*

For any projective scheme X and any positive integer $l \leq v_p(X)$, we define a homomorphism $\deg/p^l : \mathrm{Ch}_0(X) \to \mathbb{F}_p$ associating to the class $[x] \in \mathrm{Ch}_0(X)$ of a closed points $x \in X$ the class in $\mathbb{F}_p$ of the integer $(\deg x)/p^l$. Of course, $\deg/p^l = 0$ for $l < v_p(X)$. For any morphism $f : X \to Y$ of projective schemes X and Y and any $l \leq \min\{v_p(X), v_p(Y)\}$, the push-forward homomorphism $f_* : \mathrm{Ch}_0(X) \to \mathrm{Ch}_0(Y)$ satisfies $(\deg/p^l) \circ f_* = \deg/p^l$.

Since $\operatorname{char} F \neq p$, any sequence R as above determines certain degree $|R|$ homological operation S_R on the (modulo p) Chow group Ch, [24, §5]. This means that for any projective (not necessarily smooth) F-scheme Z, we are given a degree $-|R|$ homogeneous group homomorphism

$$S_R^Z : \mathrm{Ch}_*(Z) \to \mathrm{Ch}_{*-|R|}(Z)$$

commuting with the push-forward homomorphisms and such that

$$S_R^Z([Z]) = c_R(-T_Z) \mod p$$

if Z is smooth.

Proof of Proposition 2.2. Let X be a p-rigid variety and let R be a sequence such that $v_p(R(X)) = v_p(X)$. For checking the strong p-incompressibility of X, let us take a projective variety Y with $v_p(Y) \geq v_p(X)$, $\dim Y \leq \dim X$, and a multiplicity 1 correspondence $X \rightsquigarrow Y$. Then there exists a closed subvariety

$Z \subset X \times Y$ such that the degree $\deg pr_X \in \mathbb{F}_p$ of the projection $pr_X : Z \to X$ is non-zero. The proof plays with the following commutative diagram:

$$\begin{tikzcd}
& \mathrm{Ch}_{|R|}(Z) \arrow[dl, "(pr_X)_*"'] \arrow[dr, "(pr_Y)_*"] \arrow[dd, "S_R^Z"] & \\
\mathrm{Ch}_{|R|}(X) \arrow[dd, "S_R^X"'] & & \mathrm{Ch}_{|R|}(Y) \arrow[dd, "S_R^Y"] \\
& \mathrm{Ch}_0(Z) \arrow[dl, "(pr_X)_*"'] \arrow[dr, "(pr_Y)_*"] \arrow[dd, "\deg/p^l"] & \\
\mathrm{Ch}_0(X) \arrow[dr, "\deg/p^l"'] & & \mathrm{Ch}_0(Y) \arrow[dl, "\deg/p^l"] \\
& \mathbb{F}_p &
\end{tikzcd}$$

Since the operation S_R commutes with the push-forward $(pr_X)_*$ and $(pr_X)_*([Z]) = (\deg pr_X) \cdot [Z]$, we have $(\deg/p^l)\big(S_R^Z([Z])\big) = (\deg pr_X) \cdot (\deg/p^l)\big(S_R^X([X])\big) \neq 0$, where $l = v_p(X)$. Since the operation S_R also commutes with the push-forward with respect to the projection $pr_Y : Z \to Y$, we have $(\deg/p^l)\big(S_R^Z([Z])\big) = (\deg/p^l)\big(S_R^Y \circ (pr_Y)_*([Z])\big)$. It follows that $(pr_Y)_*([Z]) \neq 0$, that is, $\dim Y = \dim Z$ $(= \dim X)$ and $\deg pr_Y \neq 0$. Therefore the the class in $\mathrm{Ch}_{\dim Y}(Y \times X)$ of the transposition of Z is a required correspondence $Y \rightsquigarrow X$. □

Certainly, the strength of the above approach to the p-incompressibility is in the fact that it gives a stronger property – the strong p-incompressibility. Moreover, if X is a p-rigid variety, then for any field extension L/F, any twisted form X'/L of X with $v_p(X') = v_p(X)$ is also p-rigid. Therefore we get the p-incompressibility not only for X, but also for any such X'. Sometimes, however, this is too much, becoming a weakness of the approach: it cannot possibly succeed for a variety possessing a p-compressible twisted form with the same v_p. Besides that, the approach does not exist in characteristic p at all because a construction of the operations on the Chow group modulo p is not available in characteristic p.

Example 2.3. Let n be a positive integer and let D be a central division F-algebra of degree p^n. The Severi-Brauer variety X of D is p-rigid, [24, §7.2]. Therefore, if $\operatorname{char} F \neq p$, the variety X is strongly p-incompressible. Consequently, X is p-incompressible. (This is the particular case of Example 1.6 with cyclic A and $\operatorname{char} F \neq p$.) For F with $\operatorname{char} F = p$, it is not known whether X is strongly p-incompressible. The general case of Example 1.6 (even with the characteristic p excluded) cannot be done by the degree formula method. For instance, the product of two non-isomorphic anisotropic conics possessing a common quadratic splitting field is 2-incompressible but not 2-rigid (and even not strongly 2-incompressible). In general, a product $X = X_1 \times \cdots \times X_n$ of arbitrary smooth projective varieties $X_1, \ldots, X_n$ can be p-rigid only if $v_p(X) = v_p(X_1) + \cdots + v_p(X_n)$.

Example 2.4 ([13]). Here is another proof of the p-incompressibility for the variety X of Example 2.3, which works for F of arbitrary characteristic and which also works in the general case of Example 1.6. Using the computation of $K_0(X)$ and the relationship between $K_0(X)$ and $\mathrm{Ch}(X)$, one shows that the image of $\mathrm{Ch}(X) \to \mathrm{Ch}(\bar{X})$ is generated by the class of X. Since the variety X is projective homogeneous and generically split, it follows by Example 1.7 that X is p-incompressible.

Example 2.5. An immediate consequence of the above result concerns an orthogonal involution σ on a central division F-algebra D. An F-linear involution – a self-inverse anti-automorphism of the algebra D – is *orthogonal*, if the induced involution on the split algebra $D_{F(X)} \simeq \mathrm{End}(V)$ is adjoint to a non-alternating bilinear form b on the vector space V. Possessing an involution, D has to be 2-primary, so that we have the incompressibility statement which implies that b is anisotropic, or, equivalently, that $\sigma_{F(X)}$ is anisotropic. Indeed, otherwise the proper closed subvariety $Y \subset X$ of the *isotropic* ideals in D (i.e., ideals $I \subset D$ with $\sigma(I) \cdot I = 0$) would have an $F(X)$-point. Note that in contrast to the original paper [15], containing this observation, we do not exclude the case of characteristic 2 here. Moreover, we can replace the involution by a *quadratic pair*, [20, Definition 5.4]; the conclusion obtained this way differs from the previous one in characteristic 2 (and coincides with it in characteristic $\neq 2$).

Example 2.6 (cf. Example 1.5). Let X be an anisotropic smooth projective quadric of the first Witt index 1. Then X is strongly 2-incompressible, [7, Theorem 76.1]. The degree formula approach works only if $\dim X + 1$ is a 2-power: otherwise, X has a 2-compressible twisted form X' (another quadric) with $v_2(X') = v_2(X) = 1$ so that the degree formula approach cannot possibly work.

We terminate this Section by a criterion of p-incompressibility in terms of the correspondence multiplicities:

Lemma 2.7. *A projective homogeneous variety X is p-incompressible if and only if* $\operatorname{mult} \alpha = \operatorname{mult} \alpha^t$ *for any correspondence $\alpha : X \rightsquigarrow X$, where α^t is the transposition of α.*

Proof. If X is p-compressible, there exists a multiplicity 1 correspondence $\alpha : X \rightsquigarrow Y$ to a proper closed subvariety $Y \subset X$. Considering α as a correspondence $X \rightsquigarrow X$, we have $\operatorname{mult} \alpha = 1$ and $\operatorname{mult} \alpha^t = 0$. Therefore the "only if" part of Lemma 2.7 holds for an arbitrary X, not only for a homogeneous one.

The other way round, suppose that we are given a correspondence $\alpha : X \rightsquigarrow X$ with $\operatorname{mult} \alpha \neq \operatorname{mult} \alpha^t$. Adding a multiple of the diagonal class and multiplying by an element of $\mathbb{F}_p$, we may achieve that $\operatorname{mult} \alpha = 1$ and $\operatorname{mult} \alpha^t = 0$. In this case the pull-back of α with respect to the morphism $X_{F(X)} \to X \times X$ induced by the generic point of the second factor of the product $X \times X$, is a 0-cycle class of degree 0. Since X is homogeneous, the degree homomorphism $\operatorname{Ch}_0(X_{F(X)}) \to \mathbb{F}_p$ is an isomorphism. Therefore the pull-back of α is 0. By the continuity property of Chow groups [7, Proposition 52.9], there exists a non-empty open subset $U \subset X$ such that the pull-back of α to $X \times U$ is already 0. By the localization sequence [7, Proposition 57.9], it follows that α is the push-forward of some correspondence $\beta : X \rightsquigarrow Y \in \operatorname{Ch}_{\dim X}(X \times Y)$, where Y is the proper closed subset $Y = X \setminus U$ of X. Since $\operatorname{mult} \beta = \operatorname{mult} \alpha = 1$, the variety X is p-compressible. □

3. Motives

The classical Grothendieck Chow motives [7, Chapter XII] we are going to use are simply a convenient language to work with the correspondences. Since our correspondences live in the Chow groups with coefficients in $\mathbb{F}_p$, our motives also have coefficients in $\mathbb{F}_p$. Thus, a motive is a direct sum of triples (X, π, i), where X is a smooth projective variety, $\pi : X \rightsquigarrow X$ a projector, and i an integer. Given two such triples (X_1, π_1, i_1) and (X_2, π_2, i_2), one defines

$$\operatorname{Hom}\big((X_1, \pi_1, i_1), (X_2, \pi_2, i_2)\big) := \pi_2 \circ \operatorname{Ch}_{\dim X_1 + i_1 - i_2}(X_1 \times X_2) \circ \pi_1.$$

For any smooth projective X, the motive $M(X)$ of X is the triple $(X, \mathrm{id}_X, 0)$. For any integer j, the shift functor $M \mapsto M(j)$ is identity on the homomorphisms, additive, and takes (X, π, i) to $(X, \pi, i + j)$. The motive $M(\operatorname{Spec} F)$ is denoted by $\mathbb{F}_p$; any its shift $\mathbb{F}_p(j)$ is called a *Tate* motive.

The *Krull-Schmidt principle* holds for the motives of projective homogeneous varieties: any direct summand of the motive of a projective homogeneous variety decomposes – and in a unique way – into a direct sum of indecomposable motives, see [6] or [12].

The *nilpotence principle*, initially discovered in the case of quadrics by M. Rost, holds for the motives of projective homogeneous varieties, [5, Theorem 8.2]. In particular, a motivic summand of a projective homogeneous variety becoming 0 over an extension of F is 0. However, in contrast to the Krull-Schmidt principle, the nilpotence principle is not really required for our purposes. It allows us to work with the usual Chow motives with coefficients in $\mathbb{F}_p$ (which is probably more interesting from the view point of the theory of motives itself). Alternatively, we could have constructed our motives out of the *reduced* Chow groups $\bar{\operatorname{Ch}}$ which are defined as Ch modulo everything vanishing over an extension of the base field. In this "simplified" motivic category, the nilpotence principle vanishes as well.

Let X be a projective homogeneous variety. The motive $\bar{M}(X)$ (which is $M(X)$ over an algebraic closure of F) is a sum of Tate motives $\mathbb{F}_p(j)$, with j varying between 0 and $\dim X$; moreover, there is precisely one summand with $j = 0$ (as well as with $j = \dim X$). Therefore, there is one and unique (up to an isomorphism) indecomposable summand $U(X)$ of $M(X)$ such that the Tate motive $\mathbb{F}_p$ is a summand of $\bar{U}(X)$. We call this $U(X)$ the *upper indecomposable motivic summand* of X or simply the *upper motive* of X. (The *lower* motive of X is defined in the same way by taking the Tate motive $\mathbb{F}_p(\dim X)$ in place of $\mathbb{F}_p$.)

Upper motives are easy to handle. For instance, $U(X) \simeq U(Y)$ for two projective homogeneous varieties X and Y if and only if $v_p(Y_{F(X)}) = 0 = v_p(X_{F(Y)})$, [12].

Upper motives are important: any indecomposable summand of the motive of a projective homogeneous variety under an algebraic group of inner type is the upper motive of some (other) projective homogeneous variety. A more precise statement is given in [12]. A generalization including the outer type case is given in [16, Theorem 1.1].

A projector $\pi : X \rightsquigarrow X$ determines an upper summand of $M(X)$ if and only if $\operatorname{mult} \pi = 1$; π determines a lower summand if and only if $\operatorname{mult} \pi^t = 1$ (see [12]). Since moreover, an appropriate power of any correspondence $X \rightsquigarrow X$ is a projector (see [12]), Lemma 2.7 can be reformulated as follows:

Lemma 3.1. *A projective homogeneous variety X is p-incompressible if and only if its upper motive is lower.* □

A simple but extremely useful tool for proving p-incompressibility is the following lemma. For any direct summand M of the motive of a projective homogeneous variety X, the *rank* $\operatorname{rk} M$ of M is the number of summands in the complete decomposition of $\bar{M}$.

Lemma 3.2 ([12]). $v_p(\operatorname{rk} M) \geq v_p(X)$.

Proof. Let π be a lifting of the projector on X defining M to the Chow group with coefficients in $\mathbb{Z}/p^l\mathbb{Z}$, where $l = v_p(X)$. Some power of the correspondence π is a projector and its pull-back with respect to the diagonal morphism $X \to X \times X$ is a (modulo p^l) 0-cycle class on X of degree $\operatorname{rk} M \mod p^l$. □

Example 3.3. Let X be the Severi-Brauer variety of a p-primary central division F-algebra D. Lemma 3.2 shows that the motive of X is indecomposable. Indeed, if $\deg D = p^n$, where $\deg D := \sqrt{\dim_F D} \in \mathbb{Z}$, then $v_p(X) = n$ and it follows that the rank of any non-zero summand of $M(X)$ is at least $p^n = \operatorname{rk} M(X)$. After the proofs of Examples 2.3 and 2.4, this is the third proof of the p-incompressibility of X.

Let A be a central simple F-algebra. For any integer i with $0 \leq i \leq \deg A$ we write $\mathrm{SB}_i(A)$ for the following generalized Severi-Brauer variety of A: the variety of the right ideals in A of the *reduced dimension* i (that is, of the F-dimension $i \cdot \deg A$). For instance, $\mathrm{SB}_1(A)$ is the usual Severi-Brauer variety $\mathrm{SB}(A)$.

For $p = 2$, the opposite to the Severi-Brauer case has been considered by B. Mathews:

Example 3.4 ([23]). Let D be a non-trivial 2-primary central division F-algebra. Then the variety $X := \mathrm{SB}_{(\deg D)/2}(D)$ is 2-incompressible. Indeed, according to [4] or [5] or [14], the motive $M(X)_{F(X)}$ is a sum of one $\mathbb{F}_2$, one $\mathbb{F}_2(\dim X)$, and of shifts of $M(Y)$, where Y runs over some projective homogeneous $F(X)$-varieties with $v_2(Y) > 0$. It follows that $U(X)_{F(X)}$ contains the summand $\mathbb{F}_2(\dim X)$. Therefore X is 2-incompressible by Lemma 3.1. (This proof differs from the original one.) In contrast to Example 3.3, the motive of X is *decomposable* as far as $v_2(\deg D) > 2$: this is a special case of motivic decompositions found by M. Zhykhovich in [29].

Although the rank of $U(X)$ in Example 3.4 is not determined, one can show that $v_2 \operatorname{rk} U(X) = 1$, [12]. Together with the incompressibility of X, this is a basement for the following result concerning isotropy of an orthogonal involution on an arbitrary (not necessarily division) central simple algebra:

Theorem 3.5 ([11]). *Assume that* $\operatorname{char} F \neq 2$. *Any orthogonal involution* σ *on a central simple* F*-algebra* A *becoming isotropic over the function field of* $\mathrm{SB}(A)$, *also becomes isotropic over a finite odd degree field extension of* F.

An F-linear involution on a central simple F-algebra A is *hyperbolic*, if A possesses a σ-isotropic ideal of the reduced dimension $(\deg A)/2$.

The following non-hyperbolicity result is an immediate consequence of Theorem 3.5 and [1, Proposition 1.2]:

Theorem 3.6 ([9]). *Assume that* $\operatorname{char} F \neq 2$. *Any non-hyperbolic orthogonal involution* σ *on a central simple* F*-algebra* A *remains non-hyperbolic over the function field of* $\mathrm{SB}(A)$.

The symplectic version of Theorem 3.6 has been obtained by J.-P. Tignol:

Theorem 3.7 ([26]). *Assume that* $\operatorname{char} F \neq 2$. *Any non-hyperbolic* symplectic *(i.e., non-orthogonal) involution* σ *on a central simple* F*-algebra* A *remains non-hyperbolic over the function field of* $\mathrm{SB}_2(A)$.

Tensor products of F-linear involutions on quaternion F-algebras are called *Pfister involutions*. This is a generalization of the classical *Pfister forms*. Any isotropic Pfister form is hyperbolic. An over 30 years old conjecture saying that any isotropic Pfister involution on a central simple algebra A is hyperbolic, has been proved for algebras A of index ≤ 2 by K. Becher 3 years ago, [2]. Theorems 3.6 and 3.7 give the general case:

Theorem 3.8. *Any isotropic Pfister involution (over a field of characteristic* $\neq 2$*) is hyperbolic.*

Proof. Let σ be an isotropic Pfister involution on a central simple F-algebra A. If σ is orthogonal, $\sigma_{F(X)}$ with $X := \mathrm{SB}(A)$ is hyperbolic by [2, Theorem 1]; therefore σ is hyperbolic by Theorem 3.6. If σ is symplectic, $\sigma_{F(X)}$ with $X := \mathrm{SB}_2(A)$ is hyperbolic by [2, Corollary]; therefore σ is hyperbolic by Theorem 3.7. $\square$

4. General Generalized Severi-Brauer Varieties

The following result generalizes Examples 3.3 and 3.4:

Theorem 4.1 ([12]). *Let n be a positive integer and let D be a central division F-algebra of degree p^n. For any integer i with $0 \le i < n$, the generalized Severi-Brauer variety $\mathrm{SB}_{p^i}(D)$ is p-incompressible.*

The proof is based on the properties of upper motives formulated in Section 3. It makes use of a double induction on n and i with a simultaneous computation of the p-adic valuation of the rank of the upper motive of $\mathrm{SB}_{p^i}(D)$ which turns out to be

$$v_p \operatorname{rk} U\big(\mathrm{SB}_{p^i}(D)\big) = v_p \operatorname{rk} M\big(\mathrm{SB}_{p^i}(D)\big) = n - i.$$

Theorem 4.1 actually computes the canonical p-dimension of an arbitrary generalized Severi-Brauer variety:

Corollary 4.2 ([12]). *Let A be a central simple F-algebra, i any integer with $0 \le i \le \deg A$. Then*

$$\operatorname{cdim}_p \mathrm{SB}_i(A) = \dim \mathrm{SB}_{p^{v_p(i)}}(D_p) = p^{v_p(i)}(p^{v_p(\operatorname{ind} A)} - p^{v_p(i)}),$$

where D_p is the p-primary part of a central division algebra Brauer-equivalent to A.

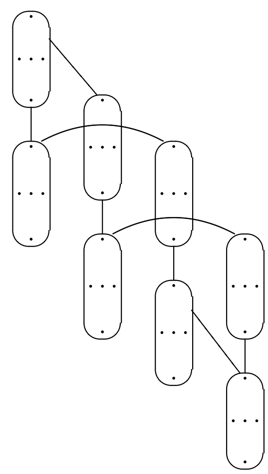
Example 4.3 (J.-P. Tignol, [26]). The particular case of Theorem 4.1 with $p = 2$ and $i = 1$ has the following application to a symplectic involution σ on a central division F-algebra D: $\sigma_{F(X)}$ is anisotropic, where $X = \mathrm{SB}_2(D)$. Indeed, otherwise the proper closed subvariety $Y \subset X$ of the *isotropic* ideals in D would have an $F(X)$-point. (This proof differs from the original one.) Note that the characteristic 2 case is included here. We do not get the same result for $X = \mathrm{SB}_1(D)$ because $Y = X$ for such X.

We have already spoken in Example 1.6 about the incompressibility of some products of Severi-Brauer varieties. There is one more related class of incompressible projective homogeneous varieties. It is useful in study of *unitary* involutions.

Given a finite separable field extension L/F, we write $\mathcal{R}_{L/F}X$ for the *Weil transfer* of an L-variety X.

Theorem 4.4 ([10]). *Let F be a field, L/F a quadratic separable field extension, n a non-negative integer, and D a central division L-algebra of degree 2^n such that the norm algebra $N_{L/F}D$ is trivial. For any integer $i \in [0,\ n]$, the variety $X := \mathcal{R}_{L/F}\,\mathrm{SB}_{2^i}(D)$ is 2-incompressible.*

The proof, using induction on n, considers some indecomposable motivic summands – the upper one, the lower one, and some of their shifts – of the variety X_{EL}, where $E = F\big(\mathcal{R}_{L/F}\,\mathrm{SB}_{2^{n-1}}(D)\big)$. "Connections" between these summands existing over E (known by induction) and over L are represented in the diagram below, where the ovals represent the summands. Since the upper and the lower summand are connected (by a chain of connections), the variety X is 2-incompressible.



Example 4.5 (J.-P. Tignol, [26]). Theorem 4.4 with $i = 0$ has the following application to a *unitary* involution σ on a 2-primary central division L-algebra D (an F-linear involution σ on D is unitary if it acts on L by the non-trivial F-automorphism): $\sigma_{F(X)}$ is anisotropic, where $X = \mathcal{R}_{L/F}\,\mathrm{SB}_1(D)$. Indeed, otherwise the proper closed subvariety $Y \subset X$ of the *isotropic* ideals in D would have an $F(X)$-point. (This proof differs from the original one.) Characteristic 2 case is included here. Unlike [26], we do not need to assume that the exponent of D is 2.

5. Dimension of Upper Motive

Let X be a projective homogeneous variety. In this final section we will show that $\mathrm{cdim}_p(X)$ is determined by the upper motive $U(X)$. Since $\mathrm{cdim}_p(X)$ is not changed under field extensions of p-prime degrees, [25, Proposition 1.5], we may assume that the semisimple affine algebraic group G acting on X has the following property: G becomes of inner type over some p-primary field extension of F.

Dimension $\dim U(X)$ of $U(X)$ is the biggest integer i such that the Tate motive $\mathbb{F}_p(i)$ is a summand of $\bar{U}(X)$. More generally, *dimension* of a summand M of the motive of a projective homogeneous variety is the maximum of $i-j$, where i and j run over the integers such that $\mathbb{F}_p(i)$ and $\mathbb{F}_p(j)$ are summands of $\bar{M}$.

Theorem 5.1. *$\dim U(X) = \operatorname{cdim}_p X$.*

For a motive M, M^* is its dual. The cofunctor $M \mapsto M^*$ transposes the homomorphisms, is additive, and takes (Y, π, i) to $(Y, \pi^t, -\dim Y - i)$ for any smooth projective variety Y, where π^t is the transposition of the projector π. In particular, $M(Y)^* = M(Y)(-\dim Y)$.

Proposition 5.2. *$U(X)^* \simeq U(X)\big(-\dim U(X)\big)$. In other words, the lower indecomposable motivic summand of X, that is, $U(X)^*(\dim X)$, is isomorphic to*

$$U(X)\big(\dim X - \dim U(X)\big).$$

Remark 5.3. Note that $\mathrm{Ch}_i \bar{U}(X) = 0 = \mathrm{Ch}^i \bar{U}(X)$ for any integer $i > \dim U(X)$ by the very definition of $\dim U(X)$. Proposition 5.2 shows that actually

$$\mathrm{Ch}_i U(X) = 0 = \mathrm{Ch}^i U(X)$$

for i as above. Indeed, for $d := \dim U(X)$, we have:

$$\mathrm{Ch}^i U(X) = \mathrm{Ch}_{-i} U(X)^* \simeq \mathrm{Ch}_{-i} U(X)(-d) = \mathrm{Ch}_{d-i} U(X) \subset \mathrm{Ch}_{d-i} X = 0$$

and $\mathrm{Ch}_i U(X) = \mathrm{Ch}^{-i} U(X)^* \simeq \mathrm{Ch}^{d-i} U(X) \subset \mathrm{Ch}^{d-i} X = 0$. (Of course, since $U(X)$ is a summand of the motive of a variety, we also have $\mathrm{Ch}_i(U) = 0 = \mathrm{Ch}^i(U)$ for any $i < 0$.)

Proof of Proposition 5.2. For G as above, let $r = r(X)$ be the rank of the semisimple anisotropic kernel of $G_{F(X)}$. We induct on r.

The motive $U(X)^*(d)$, where now $d := \dim X$, is an indecomposable summand of $M(X)$. Therefore, by [16, Theorem 1.1] and according to the assumption on G made in the beginning of this Section, there exists a finite separable field extension L/F, a projective G_L-homogeneous L-variety Y, and an integer n such that $U(X)^*(d) \simeq U(Y)(n)$ and the Tits index of $G_{L(Y)}$ contains the Tits index of $G_{F(X)}$. Here we consider the upper motive of Y, which originally lives over L, as a motive over F (strictly speaking, we apply to the L-motive $U(Y)$ the functor $\mathrm{cor}_{L/F}$ of [16, §3]).

Since $\mathrm{Ch}_d \bar{U}(X)^*(d) = \mathrm{Ch}_0 \bar{U}(X)^* = \mathrm{Ch}^0 \bar{U}(X) = \mathbb{F}_p$ and $\dim_{\mathbb{F}_p} \mathrm{Ch}_d \bar{U}(Y)(n)$ is a multiple of $[L : F]$, it follows that $L = F$. Besides,

$$n = \min\{i \mid \ \mathrm{Ch}^i \bar{U}(Y)(n) \neq 0\}$$

and $\min\{i \mid \operatorname{Ch}^i \bar{U}(X)^*(d) \neq 0\} = d - \dim U(X)$, therefore $n = d - \dim U(X)$, and we have $U(X)^* \simeq U(Y)\big(-\dim U(X)\big)$.

If the Tits index of $G_{F(Y)}$ coincides with the Tits index of $G_{F(X)}$, the motives $U(X)$ and $U(Y)$ are isomorphic, and we are done in this case. Otherwise, the rank of the semisimple anisotropic kernel of $G_{F(Y)}$ is smaller than r, and, by the induction hypothesis, we have $U(Y)^* \simeq U(Y)\big(-\dim U(Y)\big)$. Dualizing and substituting, we see that

$$U(X) \simeq U(Y)\big(\dim U(X) - \dim U(Y)\big).$$

It follows that $\dim U(X) = \dim U(Y)$ and $U(X) \simeq U(Y)$. □

Proof of Theorem 5.1. We start by proving the easier inequality

$$\dim U(X) \leq \operatorname{cdim}_p X.$$

We can find a closed subvariety $Y \subset X$ with $\dim Y = \operatorname{cdim}_p X$ and with a multiplicity 1 correspondence $\pi : X \rightsquigarrow Y$. Considering π as a correspondence $X \rightsquigarrow X$, we can find an integer $m \geq 1$ such that $\pi^{\circ m}$ is a projector. Let $M = (X, \pi^{\circ m})$. Since $\operatorname{mult} \pi^{\circ m} = \operatorname{mult} \pi = 1$, the motivic summand M of X is upper and so, $\dim U(X) \leq \dim M$. Since $\operatorname{Ch}_i \bar{M} \subset \operatorname{Im}(\operatorname{Ch}_i \bar{Y} \to \operatorname{Ch}_i \bar{X})$ for any integer i, and $\operatorname{Ch}_i \bar{Y} = 0$ for $i > \dim Y$, we get the inequality $\dim M \leq \dim Y$ proving that $\dim U(X) \leq \operatorname{cdim}_p X$.

The opposite inequality $\dim U(X) \geq \operatorname{cdim}_p X$ requires Proposition 5.2. We set $n := \dim X - \dim U(X)$. Since $U(X)(n)$ is a motivic summand of X, shifting, we have morphisms

$$U(X) \xrightarrow{\ f\ } M(X)(-n) \xrightarrow{\ g\ } U(X)$$

with $g \circ f = \mathrm{id}$. Since $U(X)$ is an upper summand of $M(X)$, the subgroup $\operatorname{Ch}^0 U(X)$ of $\operatorname{Ch}^0 X$ coincides with $\operatorname{Ch}^0 X$ and, in particular, the class $[X] \in \operatorname{Ch}^0 X$ belongs to $\operatorname{Ch}^0 U(X)$. Applying $f_* : \operatorname{Ch}^0 U(X) \to \operatorname{Ch}^0 M(X)(-n) = \operatorname{Ch}^n X$, we get an element $\alpha := f_*([X]) \in \operatorname{Ch}^n X$ such that $g_*(\alpha) = [X]$. Therefore, there exists a closed subvariety $Y \subset X$ of codimension n such that $g_*([Y]) \neq 0$. We claim that $Y_{F(X)}$ has a closed point of a p-prime degree, and this claim proves Theorem 5.1.

To prove the claim, it suffices to notice that the relation $g_*([Y]) \neq 0 \in \operatorname{Ch}^0(X)$ implies that $\xi^* g_*([Y]) \neq 0 \in \operatorname{Ch}^0 \operatorname{Spec} F(X) = \mathbb{F}_p$, where $\xi : \operatorname{Spec} F(X) \to X$ is the generic point. In the same time, the modulo p integer $\xi^* g_*([Y]) \in \mathbb{F}_p$ is the degree of the 0-cycle class $[Y_{F(X)}] \cdot (\mathrm{id}_X \times \xi)^*(g)$ which is represented by a 0-cycle on $Y_{F(X)}$. □

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