

CHOW RING OF $B\mathrm{SO}(2n)$ IN CHARACTERISTIC 2

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ABSTRACT. For $n \geq 1$, let $\mathrm{SO}(2n)$ be the special orthogonal group given by the standard split nondegenerate $2n$ -dimensional quadratic form over a field. The Chow ring $\mathrm{CH}(B\mathrm{SO}(2n))$ of its classifying space has been computed for the field of complex numbers in 2000 by R. Field. Arbitrary fields of characteristic $\neq 2$ have been treated, using a different method, in 2006 – by L. A. Molina Rojas and A. Vistoli. Using specialization from characteristic 0, we extend their computation to characteristic 2.

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0. INTRODUCTION

Let G be an affine algebraic group over a field F . The Chow ring $\mathrm{CH}(BG)$ of the classifying space of G , considered systematically for the first time in [17], is the graded ring of characteristic classes for G , where a degree i characteristic class is a functorial assignment for any G -torsor over a smooth F -variety X of a degree i element in the Chow ring $\mathrm{CH}(X)$ of X .

Example 0.1. Let G be the general linear group $\mathrm{GL}(d)$ for some $d \geq 1$. A G -torsor over a smooth variety X yields a rank d vector bundle E over X . For $i = 1, \dots, d$, its i th Chern class $c_i(E)$ is an element of $\mathrm{CH}^i(X)$ defining a characteristic class $c_i \in \mathrm{CH}^i(BG)$. By [17], c_1, \dots, c_d are independent generators of the ring $\mathrm{CH}(BG)$ identifying it with the polynomial ring $\mathbb{Z}[c_1, \dots, c_d]$.

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For arbitrary G , given a faithful representation $G \hookrightarrow \mathrm{GL}(d)$, the pull-back ring homomorphism $\mathrm{CH}(\mathcal{B}\mathrm{GL}(d)) \rightarrow \mathrm{CH}(\mathcal{B}G)$ transfers the Chern classes c_1, \dots, c_d to $\mathrm{CH}(\mathcal{B}G)$. Besides, evaluating the characteristic classes for G on the base $\mathrm{GL}(d)/G$ of the G -torsor $\mathrm{GL}(d)$, we get a ring homomorphism $\mathrm{CH}(\mathcal{B}G) \rightarrow \mathrm{CH}(\mathrm{GL}(d)/G)$.

Theorem 0.2 ([18, Theorem 5.1]). *For any $d \geq 1$ and any faithful G -representation $G \hookrightarrow \mathrm{GL}(d)$, the homomorphism $\mathrm{CH}(\mathcal{B}G) \rightarrow \mathrm{CH}(\mathrm{GL}(d)/G)$ is surjective; its kernel is the ideal generated by c_1, \dots, c_d .*

Remark 0.3. Theorem 0.2 is useful in both directions. First of all, it describes the Chow ring of the quotient variety $\mathrm{GL}(d)/G$ in terms of $\mathrm{CH}(\mathcal{B}G)$. On the other hand, any given generators of the ring $\mathrm{CH}(\mathrm{GL}(d)/G)$ can be lifted to $\mathrm{CH}(\mathcal{B}G)$; any such lifts together with the Chern classes c_1, \dots, c_d generate the Chow ring $\mathrm{CH}(\mathcal{B}G)$.

Example 0.4. For the orthogonal group $\mathrm{O}(d)$, given by the standard split nondegenerate d -dimensional quadratic form over a field (of arbitrary characteristic), and its standard representation $\mathrm{O}(d) \hookrightarrow \mathrm{GL}(d)$, the quotient $\mathrm{GL}(d)/\mathrm{O}(d)$ is an open subset in an affine space (see §1). It follows that $\mathrm{CH}(\mathrm{GL}(d)/\mathrm{O}(d)) = \mathbb{Z}$ and so the ring $\mathrm{CH}(\mathcal{B}\mathrm{O}(d))$ is generated by c_1, \dots, c_d . In characteristic $\neq 2$, the relations are: $2c_i = 0$ for every odd i , [17, §15]. In characteristic 2, the relations are: $c_i = 0$ for every odd i if d is even, and $2c_1 = 0$ and $c_i = 0$ for odd $i \geq 3$ if d is odd, [11, Appendix B]. For example, for d even, $d = 2n$, the vanishing of c_i for i odd follows from the fact that $\mathrm{O}(2n)$ is contained in the symplectic group $\mathrm{Sp}(2n)$ in characteristic 2.

Now let us consider the special orthogonal group $\mathrm{SO}(d)$. For odd d , since

$$\mathrm{O}(d) = \mu_2 \times \mathrm{SO}(d),$$

the ring homomorphism $\mathrm{CH}(\mathcal{B}\mathrm{O}(d)) \rightarrow \mathrm{CH}(\mathcal{B}\mathrm{SO}(d))$, induced by the embedding of $\mathrm{SO}(d)$ into $\mathrm{O}(d)$, is surjective. Its kernel is generated by c_1 .

The case of even $d = 2n$ is much more difficult. The group $\mathrm{SO}(4)$ – the first nontrivial case – was done over the complex numbers in [14]. The group $\mathrm{SO}(2n)$ for arbitrary n – still over the complex numbers – has been treated in [5] (see also [6]). Over an arbitrary field of characteristic $\neq 2$, the (“same”) answer was obtained (by a different method) in [13]. Besides of the Chern classes, the answer, formulated below in Theorem 0.5, involves certain characteristic class $y \in \mathrm{CH}^n(\mathcal{B}\mathrm{SO}(2n))$ constructed by Edidin and Graham. When the base field is the complex numbers, the class y maps to 2^{n-1} times the Euler class in $H^{2n}(\mathcal{B}\mathrm{SO}(2n), \mathbb{Z})$.

Theorem 0.5 ([13]). *For $n \geq 1$, the group $\mathrm{SO}(2n)$, considered over a field of characteristic $\neq 2$, has the Chow ring $\mathrm{CH}(\mathcal{B}\mathrm{SO}(2n))$ generated by the Chern classes c_2, c_3, \dots, c_{2n} together with the Edidin-Graham characteristic class y . The generators are subject to the following relations:*

$$y^2 = (-1)^n 2^{2n-2} c_{2n} \quad \text{and} \quad 2c_i = 0 = c_i \cdot y \text{ for every odd } i.$$

We prove the analogue of Theorem 0.5 for characteristic 2. Any given field F of characteristic 2 is the residue field of some characteristic 0 discrete valuation field K , [2, Proposition 5 of §2.3 and Proposition 1 of §2.1 in Chapter IX]. We write $\mathrm{SO}(2n)_K$ and

$\mathrm{SO}(2n)_F$ for the special orthogonal group over the respective fields and consider the specialization ring homomorphism

$$(0.6) \quad \mathrm{CH}(B\mathrm{SO}(2n)_K) \rightarrow \mathrm{CH}(B\mathrm{SO}(2n)_F).$$

To explain the definition of (0.6), note that by [17, Theorem 1.3], the ring $\mathrm{CH}(BG)$ for an affine algebraic group G over any field is approximated by algebraic varieties over the field. By [17, Remark 1.4] (see also [11, Example 4.1]), in the case of $G = \mathrm{SO}(2n)$ it is enough to consider varieties obtained by base change from smooth schemes over the integers. For such varieties, the specialization homomorphism is a ring homomorphism defined in [8, Example 20.3.1] and discussed in §4.

Theorem 0.7. *The specialization homomorphism (0.6) is surjective; its kernel is generated by the odd Chern classes $c_3, c_5, \dots, c_{2n-1}$.*

The proof of Theorem 0.7 is given in the very end of the paper (see §5).

Theorems 0.5 and 0.7 together yield

Corollary 0.8. *For the special orthogonal group $\mathrm{SO}(2n)$, where $n \geq 1$, considered over a field of characteristic 2, the Chow ring $\mathrm{CH}(B\mathrm{SO}(2n))$ is generated by the even Chern classes c_2, c_4, \dots, c_{2n} together with the specialization $y \in \mathrm{CH}^n(B\mathrm{SO}(2n))$ of the Edidin-Graham characteristic class. These generators are subject to the unique relation*

$$y^2 = (-1)^n 2^{2n-2} c_{2n}.$$

Surjectivity of the specialization homomorphism is the most subtle part of Theorem 0.7. By Theorem 0.2, since the Chern classes specialize to “themselves” (see Corollary 4.5), it is equivalent to the surjectivity of specialization for the quotient variety $\tilde{X} := \mathrm{GL}(2n)/\mathrm{SO}(2n)$, investigated in §2 and §3. Note that a posteriori, the latter specialization homomorphism turns out to be an isomorphism.

We start in §1 with a study of the variety $X := \mathrm{GL}(2n)/\mathrm{O}(2n)$, which is much simpler although quite close to \tilde{X} .

Commutative rings we are considering are unital; ring homomorphisms preserve identities.

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1. $X := \mathrm{GL}(2n)/\mathrm{O}(2n)$

Let us consider the algebraic group schemes $\mathrm{GL}(2n)$ and $\mathrm{O}(2n)$ over the integers \mathbb{Z} . The quotient scheme $\mathrm{GL}(2n)/\mathrm{O}(2n)$ exists: it is the scheme of nondegenerate rank $2n$ quadratic forms. In more details, let us define a scheme X by the following condition: for any commutative ring R , the set $X(R)$ of R -points of X consists of all quadratic maps $q: R^{2n} \rightarrow R$ such that the determinant of the matrix of the associated symmetric bilinear form

$$b_q: R^{2n} \times R^{2n} \rightarrow R, \quad b_q(a, b) := q(a + b) - q(a) - q(b)$$

is invertible in R . Then X is an open affine subscheme in the affine space of all quadratic maps (including the degenerate ones) on which $\mathrm{GL}(2n)$ naturally acts. For any ring R

which is an algebraically closed field, the action of the abstract group $\mathrm{GL}(2n)(R)$ on the set $X(R)$ is transitive. By the very definition of $\mathrm{O}(2n)$, it is the stabilizer of the point in $X(\mathbb{Z})$ given by the standard split quadratic form

$$\mathbb{Z}^{2n} \rightarrow \mathbb{Z}, \quad (a_1, b_1, \dots, a_n, b_n) \mapsto (a_1 b_1 + \dots + a_n b_n).$$

Thus $X = \mathrm{GL}(2n)/\mathrm{O}(2n)$ over \mathbb{Z} by [3, Proposition 2.1 of Chapter III §3].

Let e_1, \dots, e_{2n} be the standard basis of the free R -module R^{2n} . We define a filtration by closed subschemes

$$(1.1) \quad X \supset X^1 \supset \dots \supset X^{2n-1} \supset X^{2n} = \emptyset,$$

where $X^1 \subset X$ is defined by the condition $q(e_1) = 0$ and for each $i = 2, \dots, 2n$ the condition defining X^i inside of X^{i-1} is $b_q(e_1, e_i) = 0$. Note that the scheme X^{2n} is empty – as claimed in (1.1) – because b_q is nondegenerate. The idea of considering this filtration comes from [7].

Lemma 1.2 (cf. [7, Lemma 5]). *For $n \geq 2$ and any $i = 2, \dots, 2n$, there is an isomorphism*

$$X^{i-1} \setminus X^i = \mathbb{G}_m \times \mathbb{A} \times X',$$

where \mathbb{G}_m is the multiplicative group scheme, \mathbb{A} is an affine space, and

$$X' := \mathrm{GL}(2n-2)/\mathrm{O}(2n-2) \text{ over } \mathbb{Z}.$$

Proof. For $U := X^{i-1} \setminus X^i$, a commutative ring R , and $q \in (U)(R)$, the value $b_q(e_1, e_i)$ is invertible in R . Therefore, since $q(e_1) = 0$, the restriction of q to the R -submodule

$$Re_1 + Re_i \subset R^{2n},$$

generated by e_1 and e_i , is nondegenerate. The restriction q' of q to the orthogonal complement $(Re_1 + Re_i)^\perp$ with respect to b_q is also nondegenerate. Using the basis of the orthogonal complement, given by the orthogonal projections of $e_2, \dots, e_{i-1}, e_{i+1}, \dots, e_{2n}$, we get the morphism $U \rightarrow X'$, $q \mapsto q'$.

We define the morphism $U \rightarrow \mathbb{G}_m$ by associating to q the value $b_q(e_1, e_i)$. We finally map U to \mathbb{A} by associating to q the values $b_q(e_1, e_j)$ for $j = i+1, \dots, 2n$, the values $b_q(e_i, e_j)$, for $j = 2, \dots, i-1, i+1, \dots, 2n$, and the value $q(e_i)$.

We prove by constructing the inverse that the resulting morphism

$$U \rightarrow \mathbb{G}_m \times \mathbb{A} \times X'$$

is an isomorphism.

Given an R -point of $\mathbb{G}_m \times \mathbb{A} \times X'$, we first “reconstruct” q in the basis of R^{2n} , consisting of e_1, e_i and the “orthogonal projections” of the remaining standard basis elements. We also have all needed values to determine the matrix of the basis change. \square

2. $\tilde{X} := \mathrm{GL}(2n)/\mathrm{SO}(2n)$

For the algebraic group schemes $\mathrm{GL}(2n)$ and $\mathrm{SO}(2n)$ over the integers \mathbb{Z} , the quotient scheme $\tilde{X} := \mathrm{GL}(2n)/\mathrm{SO}(2n)$ exists as well: for any commutative ring R , the set $\tilde{X}(R)$ of R -points of \tilde{X} consists of the pairs (q, ε) with $q \in X(R)$ and ε being an isomorphism of R -algebras $Z(C_0(q)) \rightarrow R \times R$, where $Z(C_0(q))$ is the center of the even Clifford algebra $C_0(q)$ of q . Our reference for Clifford algebras of quadratic forms over general rings is [1, Chapter II].

Remark 2.1. Composing an isomorphism $\varepsilon: Z(C_0(q)) \rightarrow R \times R$ with the first projection $R \times R \rightarrow R$, we get a homomorphism of R -algebras $f: Z(C_0(q)) \rightarrow R$. Conversely, for any homomorphism of R -algebras $f: Z(C_0(q)) \rightarrow R$, the formula $a \mapsto (f(a), f(\sigma(a)))$, where σ is the canonical involution on $Z(C_0(q))$, defines an isomorphism $\varepsilon: Z(C_0(q)) \rightarrow R \times R$. The two maps $\varepsilon \mapsto f$ and $f \mapsto \varepsilon$ are mutually inverse bijections.

Remark 2.2 (A. Merkurjev). Let A be the commutative ring representing X : $X(R) = \text{Hom}(A, R)$ for any R . Let $q_A \in X(A)$ be the quadratic form corresponding to $\text{id}_A \in \text{Hom}(A, A)$. Then the commutative ring $\tilde{A} := Z(C_0(q_A))$ represents \tilde{X} . Indeed, an element of $\text{Hom}(\tilde{A}, R)$ is the same as an element q of $X(R) = \text{Hom}(A, R)$ together with a homomorphism of R -algebras $f: Z(C_0(q)) \rightarrow R$. So, $\tilde{X}(R) = \text{Hom}(\tilde{A}, R)$ by Remark 2.1. In particular, \tilde{X} is an affine scheme.

Recall that the discriminant of $q \in X(R)$ is defined as the isomorphism class of the separable quadratic R -algebra $Z(C_0(q))$. The set of all separable quadratic R -algebras form an exponent 2 abelian group whose 0 is the class of the split quadratic separable algebra $R \times R$. In particular, the discriminant of q is trivial for every $(q, \varepsilon) \in \tilde{X}(R)$. The group of quadratic separable R -algebras is canonically isomorphic to the multiplicative group of R modulo the squares for every characteristic $\neq 2$ field R (see [4, Example 98.2]); for every characteristic 2 field R (where the discriminant is also called Arf invariant), it is isomorphic to the additive group of R modulo the image of the Artin-Schreier map $R \rightarrow R, r \mapsto r^2 + r$ (see [4, Example 98.3]).

To verify that the scheme \tilde{X} is indeed the above quotient, one can use [3, Proposition 2.1 of Chapter III §3] as we did for X in §1: the natural action of $\text{GL}(2n)$ on \tilde{X} has the required properties. In particular, by its very definition, the algebraic group scheme $\text{SO}(2n)$ is the stabilizer of the point in $\tilde{X}(\mathbb{Z})$ given by the standard split quadratic form q together with an identification $Z(C_0(q)) = \mathbb{Z} \times \mathbb{Z}$, c.f. [12, §23A].

Since $\text{SO}(2n) \subset \text{O}(2n)$, the homogeneous space \tilde{X} maps to X . For any R , the map $\tilde{X}(R) \rightarrow X(R)$ is the forgetting map $(q, \varepsilon) \mapsto q$.

We pull-back along the morphism $\tilde{X} \rightarrow X$ the filtration (1.1) and write

$$\tilde{X} \supset \tilde{X}^1 \supset \dots \supset \tilde{X}^{2n-1} \supset \tilde{X}^{2n} = \emptyset$$

for the resulting filtration on \tilde{X} . Given any $i = 2, 3, \dots, 2n$, we write – as in Lemma 1.2 – U for the difference $X^{i-1} \setminus X^i$. We consider the morphism $U \rightarrow X' := \text{GL}(2n-2)/\text{O}(2n-2)$ over \mathbb{Z} of Lemma 1.2, write \tilde{X}' for the homogeneous space $\text{GL}(2n-2)/\text{SO}(2n-2)$, and write \tilde{U} for the difference $\tilde{X}^{i-1} \setminus \tilde{X}^i$.

In the construction of the morphism $U \rightarrow X'$ in the proof of Lemma 1.2, given some R -point $q \in U(R)$ of U , we split off from the quadratic form q its binary subform living on $Re_1 + Re_i$ and associate to q its restriction q' to $(Re_1 + Re_i)^\perp$. Since $q(e_1) = 0$, this binary subform is hyperbolic so that we have an isomorphism of the centers of the even Clifford algebras $Z(C_0(q)) = Z(C_0(q'))$. Thus we get a morphism $\tilde{U} \rightarrow \tilde{X}'$ making the

square

$$(2.3) \quad \begin{array}{ccc} \tilde{U} & \longrightarrow & \tilde{X}' \\ \downarrow & & \downarrow \\ U & \longrightarrow & X' \end{array}$$

commutative. The following property is immediate:

Lemma 2.4. *The square (2.3) is cartesian.* \square

Corollary 2.5. *For any field F , the pull-back homomorphism $\mathrm{CH}(\tilde{X}'_F) \rightarrow \mathrm{CH}(\tilde{U}_F)$ is surjective.*

Proof. Since by Lemma 1.2, $U \rightarrow X'$ is a trivial fibration with the fiber $\mathbb{G}_m \times \mathbb{A}$, the morphism $\tilde{U} \rightarrow \tilde{X}'$ is also a trivial fibration with the same fiber. Homotopy invariance and localization property of Chow groups (see [8, Theorem 3.3(a)] and Proposition 1.8] or [4, Theorem 57.13 and Proposition 57.9]) yield the surjectivity. \square

3. $\tilde{X} \setminus \tilde{X}^1$

We turn attention to the difference $Y := \tilde{X} \setminus \tilde{X}^1$ which has not been treated so far. We introduce the filtration

$$(3.1) \quad Y := Y^1 \supset Y^2 \supset \dots \supset Y^{2n},$$

where for each $i = 2, 3, \dots, 2n$ the condition defining Y^i inside Y^{i-1} is $b_q(e_1, e_i) = 0$. Note that $Y_F^{2n} = \emptyset$ for any field F of characteristic 2.

Lemma 3.2. *For $n \geq 2$ and for every $i = 2, \dots, 2n$, there is an open subscheme*

$$U \subset Y^{i-1} \setminus Y^i$$

and a morphism

$$U \rightarrow X' := \mathrm{GL}(2n-2)/\mathrm{O}(2n-2)$$

over \mathbb{Z} such that for any field F of characteristic 2 one has $U_F = (Y^{i-1} \setminus Y^i)_F$ and the morphism $U_F \rightarrow X'_F$ is flat with the fibers $\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A}$ for some affine space \mathbb{A} .

Proof. To define the subscheme $U \subset Y^{i-1} \setminus Y^i$, we define for every commutative ring R the subset $U(R) \subset (Y^{i-1} \setminus Y^i)(R)$ as the set of $(q, \varepsilon) \in (Y^{i-1} \setminus Y^i)(R)$ such that the restriction of the bilinear form b_q to the submodule $Re_1 + Re_i \subset R^{2n}$ is nondegenerate. We can define then the morphism $U \rightarrow X'$ “as usual” – by mapping (q, ε) to the restriction q' of q to the orthogonal complement $(Re_1 + Re_i)^\perp$. As in the proof of Lemma 1.2, we use the basis of the orthogonal complement given by the orthogonal projections of $e_2, \dots, e_{i-1}, e_{i+1}, \dots, e_{2n}$.

Let F be a field of characteristic 2. For any F -algebra R and any

$$(q, \varepsilon) \in (Y^{i-1} \setminus Y^i)(R),$$

the value $b_q(e_1, e_1) = 2q(e_1)$ is zero and the value $b_q(e_1, e_i)$ is invertible in R implying that the restriction of b_q to $Re_1 + Re_i$ is nondegenerate. It follows that $U_F = (Y^{i-1} \setminus Y^i)_F$.

Let \mathbb{A} be the affine space of dimension $4n - i - 1$. To finish the proof, it suffices to show that for any *local* F -algebra R and any $q' \in X'(R)$, the fiber of the morphism $U_F \rightarrow X'_F$ over q' is the product $(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A})_R$ (cf. [4, Lemma 52.12]).

For any commutative R -algebra S and any $(q, \varepsilon) \in U(S)$, lying over the image of $q' \in X'(R)$ in $X'(S)$, the values $q(e_1)$ and $b_q(e_1, e_i)$ are invertible in S . We use them to define the morphism of the fiber to $(\mathbb{G}_m \times \mathbb{G}_m)_R$. The morphism of the fiber to \mathbb{A}_R is defined using the values $b_q(e_1, e_j)$ for $j = i + 1, \dots, 2n$, the values $b_q(e_i, e_j)$, for $j = 2, \dots, i - 1, i + 1, \dots, 2n$, and one more value which we define in the next paragraph.

Let us choose a basis $f_2, g_2, \dots, f_n, g_n$ of the free R -module R^{2n-2} , on which the quadratic form q' is defined, such that $b_q(f_j, g_j) = 1$ for all j and $b_q(f_j, f_k) = b_q(f_j, g_k) = b_q(g_j, g_k) = 0$ for $j \neq k$. We can find such a basis by [1, Proposition 3.4 of Chapter I] because the ring R is local. We identify R^{2n-2} with the submodule $(Re_1 + Re_i)^\perp$ of R^{2n} . By [1, Proposition 4.4 of Chapter II], the center of the even Clifford algebra $C_0(q)$ is generated by the single element

$$t := e_1 e_i / b_q(e_1, e_i) + f_2 g_2 + \dots + f_n g_n$$

subject to the single relation $t^2 + t + d = 0$, where $d \in R$ is defined by

$$d := q(e_1)q(e_i)/b_q(e_1, e_i)^2 + q(f_2)q(g_2) + \dots + q(f_n)q(g_n).$$

By Remark 2.1, ε determines an element $s \in S$ satisfying $s^2 + s = d$. We use this s as the last value in the definition of the morphism of the fiber to \mathbb{A}_R .

We prove that the resulting morphism of the fiber to $(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A})_R$ is an isomorphism by constructing its inverse. Given an S -point of $(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{A})_R$, using all its components aside from the last component in \mathbb{A} , we almost “reconstruct” the matrix of q in the usual way. It only remains to determine the value $q(e_i)$. This value is uniquely determined by the condition $s^2 + s = d$. \square

Corollary 3.3. *The pull-back homomorphism $\text{CH}(X'_F) \rightarrow \text{CH}(U_F)$ is surjective.*

Proof. Since $U_F \rightarrow X'_F$ is a flat morphism such that each fiber is an open subset in an affine space, the statement follows with a help of the spectral sequence [16, Corollary 8.2] (see also [10, §3]). Another possibility to prove the result is to adopt the proof of [4, Proposition 52.10] to the situation. \square

4. SPECIALIZATION

Let Λ be a discrete valuation ring, let K be its field of fractions, and let F be the residue field of Λ . For a separated Λ -scheme X of finite type, we are going to use the specialization homomorphism $\text{CH}(X_K) \rightarrow \text{CH}(X_F)$ defined in [8, §20.3]. This is a homomorphism of graded groups, where the grading on the Chow groups is defined by dimension of cycles. It has a very simple description on the level of algebraic cycles: for a reduced closed subvariety in X_K , its closure in X intersected with X_F is a closed subscheme in X_F yielding an algebraic cycle on X_F .

The specialization homomorphism commutes with proper push-forwards and flat pull-backs, [8, Proposition 20.3]. If X is smooth over Λ , then the specialization is a homomorphism of graded rings (with the grading by codimension of cycles), [8, Example 20.3.1], commuting with arbitrary pull-backs.

Lemma 4.1. *For any $n \geq 0$, the specialization homomorphism*

$$\mathrm{CH}(\mathbb{P}_K^n) \rightarrow \mathrm{CH}(\mathbb{P}_F^n)$$

maps the hyperplane class to “itself”.

Proof. Since the Λ -scheme \mathbb{P}_Λ^{n-1} is smooth, the specialization homomorphism

$$\mathrm{CH}(\mathbb{P}_K^{n-1}) \rightarrow \mathrm{CH}(\mathbb{P}_F^{n-1}),$$

being a unital ring homomorphism, maps $1 = [\mathbb{P}_K^{n-1}]$ to $1 = [\mathbb{P}_F^{n-1}]$. Since the specialization homomorphism commutes with proper push-forward, the square

$$\begin{array}{ccc} \mathrm{CH}(\mathbb{P}_K^{n-1}) & \longrightarrow & \mathrm{CH}(\mathbb{P}_K^n) \\ \downarrow & & \downarrow \\ \mathrm{CH}(\mathbb{P}_F^{n-1}) & \longrightarrow & \mathrm{CH}(\mathbb{P}_F^n) \end{array}$$

is commutative. □

Similarly, one proves

Lemma 4.2. *For X a product of projective spaces, the specialization homomorphism*

$$\mathrm{CH}(X_K) \rightarrow \mathrm{CH}(X_F)$$

is “identical”. □

Since for any field L , the Chow ring $\mathrm{CH}(BT_L)$ for a split \mathbb{Z} -torus T is approximated by products of projective spaces, we get

Corollary 4.3. *For a split \mathbb{Z} -torus T , the specialization homomorphism*

$$\mathrm{CH}(BT_K) \rightarrow \mathrm{CH}(BT_F)$$

is “identical”. □

Corollary 4.4. *For any $d \geq 1$, the specialization homomorphism*

$$\mathrm{CH}(\mathcal{B}\mathrm{GL}(d)_K) \rightarrow \mathrm{CH}(\mathcal{B}\mathrm{GL}(d)_F)$$

maps the generators c_1, \dots, c_d to “themselves”.

Proof. Let $T \subset \mathrm{GL}(d)$ be standard split \mathbb{Z} -torus of rank d . Since the specialization homomorphism commutes with pull-backs, the square

$$\begin{array}{ccc} \mathrm{CH}(\mathcal{B}\mathrm{GL}(d)_K) & \longrightarrow & \mathrm{CH}(BT_K) \\ \downarrow & & \downarrow \\ \mathrm{CH}(\mathcal{B}\mathrm{GL}(d)_F) & \longrightarrow & \mathrm{CH}(BT_F) \end{array}$$

is commutative. Since the horizontal maps are injective, the statement follows. □

Let G be an algebraic group scheme over \mathbb{Z} with a faithful representation $G \hookrightarrow \mathrm{GL}(d)$. As in §0, for any field L , we continue to write c_1, \dots, c_d for the images of c_1, \dots, c_d under the restriction homomorphism $\mathrm{CH}(\mathcal{B}\mathrm{GL}(d)_L) \rightarrow \mathrm{CH}(BG_L)$.

Corollary 4.5. *The specialization homomorphism*

$$\mathrm{CH}(BG_K) \rightarrow \mathrm{CH}(BG_F)$$

maps the elements c_1, \dots, c_d to “themselves”. \square

5. SURJECTIVITY

In this section, K is a discrete valuation field of characteristic 0 with a residue field F of characteristic 2.

Proposition 5.1. *The specialization homomorphisms*

$$\mathrm{CH}(\tilde{X}_K) \rightarrow \mathrm{CH}(\tilde{X}_F) \quad \text{and} \quad \mathrm{CH}(B\mathrm{SO}(2n)_K) \rightarrow \mathrm{CH}(B\mathrm{SO}(2n)_F)$$

are surjective, where $\tilde{X} := \mathrm{GL}(2n)/\mathrm{SO}(2n)$.

Proof. We prove the statement using an induction on $n \geq 1$. As explained in §0, surjectivity of specialization for \tilde{X} is equivalent to the same property of $B\mathrm{SO}(2n)$. For $n = 1$, the group $\mathrm{SO}(2n) = \mathrm{SO}(2) = \mathbb{G}_m$ has this property (see Corollary 4.3). Below we are assuming that $n \geq 2$ and prove the statement for \tilde{X} .

Using descending induction on $i \leq 2n$, we first prove that for any $i = 2n, 2n-1, \dots, 2, 1$, the specialization homomorphism for \tilde{X}^i is surjective. We start with $i = 2n$, where $\tilde{X}^{2n} = \emptyset$.

For $i < 2n$, we have the commutative diagram

$$\begin{array}{ccccccc} \mathrm{CH}(\tilde{X}^{i+1})_K & \longrightarrow & \mathrm{CH}(\tilde{X}^i)_K & \longrightarrow & \mathrm{CH}(\tilde{X}^i \setminus \tilde{X}^{i+1})_K & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{CH}(\tilde{X}^{i+1})_F & \longrightarrow & \mathrm{CH}(\tilde{X}^i)_F & \longrightarrow & \mathrm{CH}(\tilde{X}^i \setminus \tilde{X}^{i+1})_F & \longrightarrow & 0 \end{array}$$

where the rows are exact localization sequences (see [8, Proposition 1.8] or [4, Proposition 57.9]) and the vertical arrows are the specialization maps. The left specialization map is surjective by induction hypothesis on $i + 1$. The right specialization map is surjective by Corollary 2.5 and the induction hypothesis on $n - 1$. Therefore the specialization map in the middle is surjective as well.

We proved that the specialization map for \tilde{X}^1 is surjective. To prove the same statement for \tilde{X} itself, we proceed similarly, using the filtration (3.1) on the complement Y of \tilde{X}^1 together with Lemma 3.2. In the commutative diagram

$$\begin{array}{ccccccc} \mathrm{CH}(Y^{i+1})_K & \longrightarrow & \mathrm{CH}(Y^i)_K & \longrightarrow & \mathrm{CH}(Y^i \setminus Y^{i+1})_K & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{CH}(Y^{i+1})_F & \longrightarrow & \mathrm{CH}(Y^i)_F & \longrightarrow & \mathrm{CH}(Y^i \setminus Y^{i+1})_F & \longrightarrow & 0 \end{array}$$

with exact rows, it suffices to show that the specialization map on the right is surjective. This follows from the commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}(Y^i \setminus Y^{i+1})_K & \longrightarrow & \mathrm{CH}(Y^i \setminus Y^{i+1})_F \\
\downarrow \text{onto} & & \downarrow \text{isomorphism} \\
\mathrm{CH}(U_K) & \longrightarrow & \mathrm{CH}(U_F) \\
\uparrow & & \uparrow \text{onto} \\
\mathrm{CH}(X'_K) & \xrightarrow{\text{onto}} & \mathrm{CH}(X'_F)
\end{array}$$

(Since $\mathrm{CH}(X'_K) = \mathbb{Z} = \mathrm{CH}(X'_F)$, the lower map is actually an isomorphism.) The right lower arrow in the diagram is onto by Corollary 3.3. \square

Proof of Theorem 0.7. Having done Proposition 5.1 already, we only need to check the statement on the kernel of the specialization homomorphism. We first show that the odd Chern classes are in the kernel. In other terms, we show that the odd Chern classes

$$c_3, c_5, \dots, c_{2n-1} \in \mathrm{CH}(\mathcal{B}\mathrm{SO}(2n))$$

vanish over a field of characteristic 2.

Indeed, over a field of characteristic 2, the standard representation of $\mathrm{SO}(2n)$ factors through the standard representation of the symplectic group:

$$\mathrm{SO}(2n) \hookrightarrow \mathrm{Sp}(2n) \hookrightarrow \mathrm{GL}(2n).$$

The odd Chern classes of the standard representation for $\mathrm{Sp}(2n)$ vanish (see [17, §15] or [13, §3]). Therefore they vanish for $\mathrm{SO}(2n)$.

It follows by Theorem 0.5 and Proposition 5.1 that – for $\mathrm{SO}(2n)$ considered over a field of characteristic 2 – the ring $\mathrm{CH}(\mathcal{B}\mathrm{SO}(2n))$ is generated by the even Chern classes c_2, c_4, \dots, c_{2n} and the specialization of the Edidin-Graham characteristic class $y \in \mathrm{CH}(\mathcal{B}\mathrm{SO}(2n))$. These generators satisfy the relation

$$(5.2) \quad y^2 = (-1)^n 2^{2n-2} c_{2n}.$$

To finish the proof of Theorem 0.7, it remains to show that there are no further relations.

Let $T \subset \mathrm{SO}(2n)$ be the standard split maximal torus. The ring $\mathrm{CH}(\mathcal{B}T)$ is canonically isomorphic to the symmetric \mathbb{Z} -algebra $\mathcal{S}(\hat{T})$ of the character group \hat{T} of T . Taking the standard basis x_1, \dots, x_n of \hat{T} , we identify $\mathrm{CH}(\mathcal{B}T)$ with the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$. For every $i = 1, \dots, n$, the image of c_{2i} under the restriction ring homomorphism $\mathrm{CH}(\mathcal{B}\mathrm{SO}(2n)) \rightarrow \mathrm{CH}(\mathcal{B}T)$ equals $(-1)^i p_i$ (see, e.g., [9, Examples 2.8 and 2.9]), where p_i is the i th Pontryagin class defined as the i th elementary symmetric polynomial in the squares x_1^2, \dots, x_n^2 of the variables. The image of y equals $2^{n-1}e$ (see, e.g., [15, Lemma 4.8]), where $e := x_1 \dots x_n$. The elements p_1, \dots, p_n, e satisfy the unique relation $e^2 = p_n$. Therefore the generators $c_2, c_4, \dots, c_{2n}, y$ satisfy no further relations besides (5.2). \square

Remark 5.3. The specialization homomorphism $\mathrm{CH}(\tilde{X}_K) \rightarrow \mathrm{CH}(\tilde{X}_F)$ turns out to be an isomorphism.

As a byproduct, we extended to the arbitrary base field the main result of [7]:

Corollary 5.4. *For any field L (of any characteristic), the ring $\mathrm{CH}(\tilde{X}_L)$ is generated by a single element $y \in \mathrm{CH}^n(\tilde{X}_L)$ subject to the unique relation $y^2 = 0$. \square*

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