FILTRATIONS ON THE REPRESENTATION RING OF AN AFFINE ALGEBRAIC GROUP

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ABSTRACT. Let G be an affine algebraic group over a field. The representation ring $\mathrm{R}(G)$ has three standard filtrations, defining the same topology on $\mathrm{R}(G)$: *augmentation*, *Chern*, and *Chow*, each of which contained in the next one. For split reductive G, motivated by potential applications related to spin groups, we introduce and study one more filtration, containing the previous ones, which we call *induced* because it is induced by any of the filtrations on the representation ring of a maximal split torus of G. In the case of semisimple simply connected G, this fourth filtration turns out to be equivalent (in the above topological sense) to the previous three. However, for the spin group $G = \mathrm{Spin}(d)$ over the complex numbers with $d = 7, 8$, the new filtration is shown to be strictly larger than the others. It is also shown that for $G = \mathrm{Spin}(d)$ over an arbitrary field and with any $d \geq 7$, the Chern and Chow filtrations on $\mathrm{R}(G)$ are not the same, giving new counter-examples to an extension of Atiyah's conjecture.

CONTENTS

1. INTRODUCTION

Let G be an affine algebraic group of finite type over a field. The representation ring $\mathrm{R}(G)$ has three standard filtrations: *augmentation*, *Chern*, and *Chow*, each of which is smaller than the next one, see, e.g., [14]. These filtrations are known to be equivalent to each other; in particular, they define the same topology on $\mathrm{R}(G)$, see [14, Corollary 4.8].

For split reductive G, motivated by potential applications related to spin groups (see the explanation after Question 3.11), we introduce in §2 and then study one more filtration,

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containing the previous ones. We call it the *induced filtration* because it is induced by any of the filtrations on the representation ring of a maximal split torus of G. In the case of semisimple simply connected G, this fourth filtration turns out to be equivalent to the previous three (see Proposition 3.1). However, for $G = \mathrm{Spin}(d)$ over the complex numbers with $d = 7, 8$, it is shown to be strictly larger than the others (Theorem 4.9). It is also shown (in Theorem 4.5) that for $G = \mathrm{Spin}(d)$ over an arbitrary field and with any $d \geq 7$, the Chern and Chow filtrations on $\mathrm{R}(G)$ are not the same, giving refined counter-examples to an extension of Atiyah's conjecture. The case of $d = 7$ has already been observed (over an algebraically closed field) in [30, Theorem 1.1].

Indeed, for a finite group G viewed as a constant algebraic group over the complex numbers, Atiyah conjectured in [1] that the Chern filtration on $\mathrm{R}(G)$ coincides with the *topological* filtration defined with a help of the topological K-theory. The topological filtration contains the Chow filtration (also called topological in some papers)– its algebraic analogue defined for an arbitrary base field with a help of the algebraic K-theory. Various counter-examples to this conjecture have been constructed by Weiss [29], Thomas [25] (this is the paper from which the title of the present paper is stolen), and Leary–Yagita [17]. Totaro in [28, Chapter 15] produced new enhanced counter-examples with a difference observed between the topological and the Chow filtration. He also extended Atiyah's question to more general algebraic groups and produced some (enhanced in the above sense) counter-examples to that extension too.

Looking at the difference between the Chow and the Chern filtrations (over an arbitrary field) is another enhancement of the extended question. Note that with rational coefficients, the two filtrations coincide (see Lemma A.1).

The first counter-example of that type with a split semisimple group is probably given by the special orthogonal group $\mathrm{SO}(2n)$ with $n \geq 3$. It was implicitly worked out in [5] (see Example 4.4 here). In fact, we use the special orthogonal group to demonstrate a lot of other related phenomena. One of the reasons why it provides so convenient source of examples is a computation of the Chow ring for the classifying space of $\mathrm{SO}(2n)$ made over a field of characteristic $\neq 2$ in [19] and, by a different method, over the complex numbers in [5].

Returning to the extended Atiyah's conjecture, the groups $\mathrm{Spin}(d \geq 7)$, treated by Theorem 4.5, seem to provide the first split semisimple *simply connected* counter-examples to it. The starting ingredient of the proof is a computation of the Chern filtration for quadrics made by the author in [7] over 30 years ago.

In Appendix A, a definable for arbitrary affine algebraic group G replacement of the induced filtration is suggested and briefly discussed.

The spin groups $\mathrm{Spin}(d)$ and the special orthogonal group $\mathrm{SO}(d)$ we consider in the paper are defined using the standard *split* d-dimensional quadratic forms which, unlike to the tradition in topology, are *not* sums of squares. To distinguish with topology, some authors prefer the notation Spin_d and SO_d.

In this paper, for an affine algebraic group, the condition of being reductive includes the condition of being connected.

2. The induced filtration

Let G be a reductive group with a split maximal torus $T \subset G$. We would like to restrain the image of the homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(BT)$ of the Chow rings of the classifying spaces given by the inclusion. It follows from [27, Theorem 1.3(1)] that the kernel of this homomorphism is exactly the torsion ideal (i.e., the ideal consisting of all elements of finite order). The destination ring $\mathrm{CH}(BT)$ is canonically isomorphic to the symmetric $\mathbb{Z}$-algebra $S(\hat{T})$ of the character group $\hat{T}$ of T. In particular, $\mathrm{CH}(BT)$ is torsion free.

First of all, the image consists of elements invariant under the action of the Weyl group W of G with respect to T. Therefore, we actually have a homomorphism

$$f\colon\ \mathrm{CH}(BG) \to \mathrm{CH}(BT)^W.$$

Next, as discussed in [13], the image satisfies certain constraints given by Steenrod operations on Chow groups modulo a prime. Besides, for semisimple G, constraints given by reductive envelopes of G have been considered in [8].

If G is defined over the complex numbers, the cycle class map to the integral singular cohomology delivers the commutative square

$$\begin{array}{ccc} \mathrm{CH}(BG) & \longrightarrow & \mathrm{CH}(BT) \\ \downarrow & & \downarrow\wr \\ \mathrm{H}(BG) & \longrightarrow & \mathrm{H}(BT) \end{array}$$

with an isomorphism on the right. Therefore f factors as

$$\mathrm{CH}(BG) \longrightarrow \mathrm{H}(BG) \xrightarrow{f_{\mathrm{H}}} \mathrm{H}(BT)^W = \mathrm{CH}(BT)^W.$$

So, one more restraint on $\operatorname{Im} f$ is the inclusion $\operatorname{Im} f \subset \operatorname{Im} f_{\mathrm{H}}$. This inclusion is of practical use because for many groups G, the image of f_{H} is known.

Passing to an arbitrary base field, let us consider the similar restraint given by the representation ring. Namely, the representation ring $\mathrm{R}(G)$ of G can be viewed as the Grothendieck ring of BG and as such is endowed with the *Chow filtration*

$$\mathrm{R}(G) = \mathrm{R}^{(0)}(G) \supset \mathrm{R}^{(1)}(G) \supset \dots$$

(also known under several different names, e.g., the filtration by codimension of support), see [28] and [14]. For the associated graded ring $\mathrm{ChowR}(G)$, we have a canonical surjective homomorphism of graded rings $\mathrm{CH}(BG) \twoheadrightarrow \mathrm{ChowR}(G)$ whose kernel is contained in the torsion ideal. In particular, $\mathrm{CH}(BT) \twoheadrightarrow \mathrm{ChowR}(T)$ is an isomorphism. So, it follows from the commutative square

$$\begin{array}{ccc} \mathrm{CH}(BG) & \longrightarrow & \mathrm{CH}(BT) \\ \downarrow & & \downarrow\wr \\ \mathrm{ChowR}(G) & \longrightarrow & \mathrm{ChowR}(T) \end{array}$$

that f factors through $\mathrm{ChowR}(G)$:

$$f\colon\ \mathrm{CH}(BG) \xrightarrow{\text{onto}} \mathrm{ChowR}(G) \xrightarrow{f_{\mathrm{R}}} \mathrm{ChowR}(T)^W = \mathrm{CH}(BT)^W. \tag{2.1}$$

We proceed by collecting some information on ChowR(G) starting with R(G). The homomorphism $\mathrm{R}(G) \to \mathrm{R}(T)$, induced by the embedding, is injective with the image $\mathrm{R}(T)^W$ (see [22, Théorème 4] or [18, Theorem 22.38]). So, the ring R(G) is identified with $\mathrm{R}(T)^W$. The representation ring $R(T)$ is the group ring $\mathbb{Z}[\hat{T}]$. We follow the exponential tradition of writing e^x for the element in $\mathbb{Z}[\hat{T}]$ corresponding to $x \in \hat{T}$ to handle the issue that the addition in $\hat{T}$ becomes multiplication in the group ring. The elements e^x all together form an additive basis of $\mathbb{Z}[\hat{T}]$ and the action of W permutes this basis. Therefore $\mathbb{Z}[\hat{T}]^W$ is additively generated by the orbit sums of the basis elements. (This situation drastically differs from $S(\hat{T})$ for which determination of W-invariants often represents a considerable difficulty.)

For semisimple and simply connected G there is even better classical description of $\mathbb{Z}[\hat{T}]^W$: it turns out to be the polynomial ring on the orbit sums of the fundamental weights of G ([3, Theorem 1 and Example 1 of §3 of Chapter VI]). These orbit sums are the classes in R(G) of the fundamental representations of G.

Example 2.2. For the special orthogonal group $G = \mathrm{SO}(2n+1)$ with $n \geq 2$, let $T \subset G$ be its standard split maximal torus and let $x_1, \ldots, x_n$ be the standard basis of $\hat{T}$. The action of the Weyl group W on $\hat{T}$ makes it a subgroup of the automorphism group $\operatorname{Aut}\hat{T}$ of $\hat{T}$ generated by the permutations of $x_1, \ldots, x_n$ and their sign changes.

The ring R(T) is the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $t_i := e^{x_i}$. The action of W permutes $t_1, \ldots, t_n$ and exchanges t_i with t_i^{-1} for individual i so that $\mathrm{R}(T)^W$ is generated by the elementary symmetric polynomials in $t_1 + t_1^{-1}, \ldots, t_n + t_n^{-1}$.

Switching from $G = \mathrm{SO}(2n+1)$ to the spin group $G = \mathrm{Spin}(2n+1)$ with its standard split maximal torus T, we "enlarge" $\hat{T}$ by adjoining the element $(x_1 + \cdots + x_n)/2$. (Therefore $\mathbb{Z}[\hat{T}]$ is "enlarged" by adjoining a square root of $t_1 \ldots t_n$.) The fundamental weights are the partial sums $x_1 + \cdots + x_i$ for $i = 1, \ldots, n-1$ together with the adjoint element $(x_1 + \cdots + x_n)/2$, see [3, Table II].

Example 2.3. For $G = \mathrm{SO}(2n)$ with $n \geq 3$, let $T \subset G$ be its standard split maximal torus and let $x_1, \ldots, x_n$ be the standard basis of $\hat{T}$. The action of the Weyl group W on $\hat{T}$ makes it a subgroup of $\operatorname{Aut}\hat{T}$ generated by the permutations of $x_1, \ldots, x_n$ and the sign changes of any even number of them.

Switching from $G = \mathrm{SO}(2n)$ to $G = \mathrm{Spin}(2n)$ with its standard split maximal torus T, we "enlarge" $\hat{T}$ as in Example 2.2 – by adjoining the element $(x_1 + \cdots + x_n)/2$. The fundamental weights are the partial sums $x_1 + \cdots + x_i$ for $i = 1, \ldots, n-2$ together with $(x_1 + \cdots + x_{n-1} - x_n)/2$ and $(x_1 + \cdots + x_{n-1} + x_n)/2$, see [3, Table II]. Note that since the Weyl group here is smaller than in Example 2.2, the orbit sum

$$\sqrt{t_1 \ldots t_n} + \sqrt{t_1 \ldots t_n}/(t_1 t_2) + \cdots \in \mathbb{Z}[\hat{T}]$$

of the last fundamental weight we have here is different from the orbit sum

$$\sqrt{t_1 \ldots t_n} + \sqrt{t_1 \ldots t_n}/t_1 + \cdots \in \mathbb{Z}[\hat{T}]$$

of Example 2.2.

So, we can calculate the ring $\mathrm{R}(G) = \mathrm{R}(T)^W$. If we could also calculate its Chow filtration, then we would be able to determine $\operatorname{Im} f$ precisely because $\operatorname{Im} f = \operatorname{Im} f_{\mathrm{R}}$ as we see from decomposition (2.1). Unfortunately, we cannot calculate the filtration in general. But we can calculate the Chow filtration on $\mathrm{R}(T)$. And this will provide us with an estimate on the filtration on $\mathrm{R}(G) \subset \mathrm{R}(T)$.

Let I be the kernel of the (*augmentation* or *rank*) ring homomorphism $\mathrm{R}(T) \to \mathbb{Z}$, mapping every e^x to 1. Clearly, I is additively generated by the differences $1 - e^x$. Since $\mathrm{CH}(BT)$ is generated by $\mathrm{CH}^1(BT)$, it follows by [14, Corollary 3.4] that the Chow filtration on $\mathrm{R}(T)$ is the filtration

$$\mathrm{R}(T) \supset I \supset I^2 \supset \dots$$

by the powers of the augmentation ideal. Since the homomorphism

$$\mathrm{R}(G) = \mathrm{R}(T)^W \hookrightarrow \mathrm{R}(T)$$

respects the filtrations, we have for any i the inclusion $\mathrm{R}^{(i)}(G) \subset \mathrm{R}(G) \cap I^i = (I^i)^W$. The terms on the right define a filtration on $\mathrm{R}(G)$ which we call the *induced filtration* because it is induced by any of the augmentation = Chern = Chow filtration of $\mathrm{R}(T)$. (There also is a much smaller and less interesting *augmentation filtration* on $\mathrm{R}(G)$ – the filtration by the powers $(I^W)^i$ of the augmentation ideal $\mathrm{R}(G) \cap I = I^W$ of $\mathrm{R}(G)$.) We write $\mathrm{IndR}(G)$ for the graded ring associated to the induced filtration. The homomorphism f decomposes as

$$f\colon \mathrm{CH}(BG) \xrightarrow{\text{onto}} \mathrm{ChowR}(G) \longrightarrow \mathrm{IndR}(G) \hookrightarrow \mathrm{CH}(BT)^W. \tag{2.4}$$

This gives the following estimation for the image of f:

Proposition 2.5. *The image of f is contained in* $\mathrm{IndR}(G)$. □

Remark 2.6. Proposition 2.5 can also be explained via the equivariant connective K-theory CK of [15], which yields the commutative square

$$\begin{array}{ccc} \mathrm{CK}(BG) & \longrightarrow & \mathrm{CK}(BT)^W \\ \big\downarrow{\scriptstyle \text{onto}} & & \big\downarrow \\ \mathrm{CH}(BG) & \xrightarrow{f} & \mathrm{CH}(BT)^W \end{array}$$

with surjection on the left, showing that the image of f is contained in the image of the map on the right. Note that $\mathrm{CK}^i(BT) = I^i$ for any $i \in \mathbb{Z}$, where the non-positive powers of I are defined to be equal to the entire $\mathrm{R}(T)$. Therefore, for any $i \geq 0$, $\mathrm{CK}^i(BT)^W = (I^i)^W$ is the ith term of the induced filtration on $\mathrm{R}(G)$.

Remark 2.7. The representation ring $\mathrm{R}(G)$, the induced filtration on it, and therefore the ring $\mathrm{IndR}(G)$ can be constructed in terms of the character lattice $\hat{T}$, Weyl group W, and the action of W on $\hat{T}$. In particular, these objects do not depend on the base field. One can use [18, Theorem 23.74] to state it accurately.

Remark 2.8. Example 3.9 below shows that the inclusion $\mathrm{IndR}(G) \hookrightarrow \mathrm{CH}(BT)^W$ can be strict. In other terms, the estimate of Proposition 2.5 is in general nontrivial.

We describe in Proposition 3.5 below a situation where the estimate of Proposition 2.5 is the exact value of $\mathrm{Im}\, f$.

3. Relations between filtrations

There are 4 filtrations on the representation ring $\mathrm{R}(G)$ of a split reductive group G, related by a chain of inclusions. The smallest one is the augmentation filtration which is followed by the Chern, the Chow, and the induced ones. Recall that by [14, Corollary 4.8], the first 3 filtrations are equivalent to each other. In particular, they define the same topology on $\mathrm{R}(G)$. Besides, the Chern and Chow filtrations coincide rationally.

Proposition 3.1. *For a simply connected split semisimple group G, all four filtrations on $\mathrm{R}(G)$ are equivalent.*

Proof. By [20, Theorem 2.2(2)], the kernel of the canonical (surjective since G is simply connected) homomorphism of $R(T)$ to the Grothendieck ring of the quotient variety G/T is the ideal generated by I^W. Since the homomorphism respects the Chow filtration, I^i is in the kernel, where $i := 1 + \dim(G/T)$. It follows that $I^i \subset I^W \cdot \mathrm{R}(T)$. Therefore

$$I^{ij} \subset (I^W)^j \cdot \mathrm{R}(T) \tag{3.2}$$

for any $j \geq 0$. Since G is simply connected, the embedding of $\mathrm{R}(T)^W$-modules $\mathrm{R}(T)^W \hookrightarrow \mathrm{R}(T)$ splits, see [23, Theorem 2.2], so that (3.2) implies $(I^{ij})^W \subset (I^W)^j$. This inclusion proves equivalence of the induced filtration (the largest of the four) with the augmentation filtration (which is the smallest of the four). □

Corollary 3.3. *Let G be as in Proposition 3.1. For any $i \geq 0$, the ith term $(I^i)^W$ of the induced filtration on $\mathrm{R}(G)$ consists of all elements in $\mathrm{R}(G)$ having a positive multiple in $\mathrm{R}^{(i)}(G)$.*

Proof. If $a \in \mathrm{R}(G)$ has a positive multiple in $\mathrm{R}^{(i)}(G) \subset (I^i)^W$, then $a \in (I^i)^W$ because the quotient $\mathrm{R}(G)/(I^i)^W \subset \mathrm{R}(T)/I^i$ is torsion free. On the other hand, if $a \in (I^i)^W$, then a determines an element of $(I^i)^W/(I^{i+1})^W \subset (I^i/I^{i+1})^W = \mathrm{CH}^i(BT)^W$, which after multiplication by the torsion index t of G comes to the image of f. Here we use [27, Theorem 1.3(1)] telling that the cokernel of f is killed by t.

It follows that

$$ta \in \mathrm{R}^{(i)}(G) + (I^{i+1})^W.$$

Applying the same procedure to the $(I^{i+1})^W$-component of ta and continuing this way, we show that for any j, a positive multiple of a is in $\mathrm{R}^{(i)}(G) + (I^j)^W$. Taking sufficiently large j we get by Proposition 3.1 that a positive multiple of a is in $\mathrm{R}^{(i)}(G)$. □

Since the homomorphism $\mathrm{ChowR}(G) \to \mathrm{ChowR}(T)$ is the composition

$$\mathrm{ChowR}(G) \to \mathrm{IndR}(G) \hookrightarrow \mathrm{ChowR}(T),$$

the kernel of $\mathrm{ChowR}(G) \to \mathrm{IndR}(G)$ coincides with the kernel of the homomorphism $\mathrm{ChowR}(G) \to \mathrm{ChowR}(T)$.

Lemma 3.4. *The kernel of the homomorphism $\mathrm{ChowR}(G) \to \mathrm{IndR}(G)$ coincides with the torsion ideal of $\mathrm{ChowR}(G)$.*

Proof. Since $\mathrm{IndR}(G)$ is free of torsion, the kernel contains the torsion ideal.

Any element of $\mathrm{ChowR}(G)$ is the image of an element of $\mathrm{CH}(BG)$. Since the kernel of $\mathrm{CH}(BG) \to \mathrm{CH}(BT) = \mathrm{ChowR}(T)$ is killed by the torsion index of G ([27, Theorem 1.3(1)]), the statement follows. □

Proposition 3.5. *For G and T given, conditions* (2) *and* (3) *below are equivalent and imply condition* (1)*. If G is as in Proposition 3.1, then all the three conditions are equivalent.*

(1) *the homomorphism* $\mathrm{ChowR}(G) \to \mathrm{IndR}(G)$ *is surjective (i.e.,* $\mathrm{Im}\, f = \mathrm{IndR}(G)$*);*
(2) *the homomorphism* $\mathrm{ChowR}(G) \to \mathrm{IndR}(G)$ *is injective (i.e.,* $\mathrm{ChowR}(G)$ *is torsion free, see Lemma 3.4);*
(3) *the induced filtration on* $\mathrm{R}(G)$ *coincides with the Chow filtration.*

Proof. Condition (3) clearly implies (1) and (2).

Assuming condition (2), one shows by induction on $i \geq 0$ that the ith terms of the two filtrations of (3) coincide. Therefore conditions (2) and (3) are equivalent.

The surjectivity (1) implies that for any $i \geq 0$, the ith term $(I^i)^W$ of the induced filtration on $\mathrm{R}(G)$ equals the sum $\mathrm{R}^{(i)}(G) + (I^j)^W$ with any $j \geq i$. By Proposition 3.1, $(I^j)^W \subset \mathrm{R}^{(i)}(G)$ for large enough j provided that G is as in Proposition 3.1. Consequently, (1)⇒(3) for such G. □

Despite of the examples constructed in [28, Chapter 5], the following question seems to be open:

Question 3.6. *Does there exist a split reductive G with torsion in* $\mathrm{ChowR}(G)$*?*

This question is answered by positive in Theorem 4.9 below.

Example 3.7. Assume that the torsion index of a split reductive group G is equal to 1. Then $\mathrm{CH}(BG)$ is torsion free. It follows by Proposition 3.5 that the induced filtration on $\mathrm{R}(G)$ coincides with the Chow filtration and $\mathrm{IndR}(G) = \mathrm{Im}\, f$. However, the homomorphism f is surjective in this case. Therefore we do not get an interesting application ground for Proposition 2.5 with such G.

Example 3.8. Let G be the special orthogonal group $\mathrm{SO}(2n+1)$ with $n \geq 1$. It has been shown in [14, §5] that $\mathrm{ChowR}(G)$ is torsion free. Note that over a field of characteristic not 2, $\mathrm{CH}(BG)$ is not torsion free (see the computation of $\mathrm{CH}(BG)$ made in [26, §16] and [19, Theorem 5.1]): its torsion ideal is generated by the odd Chern classes (each of order 2 besides the vanishing first one) of the standard G-representation. In contrast, over a field of characteristic 2, $\mathrm{CH}(BG)$ is torsion free by [14, Appendix B] (the odd Chern classes vanish all together).

Since $\mathrm{Im}\, f$ is known to coincide with $\mathrm{CH}(BT)^W$, we do not get an interesting application ground for Proposition 2.5 with this G. To recall why f is surjective, let us write $x_1, \dots, x_n$ for the standard basis of $\hat{T}$ and identify $\mathrm{CH}(BT) = S(\hat{T})$ with the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$. The Weyl group W of G permutes $x_1, \dots, x_n$ and changes the sign of any of them. Therefore the ring of W-invariants $\mathbb{Z}[x_1, \dots, x_n]^W$ is generated by the Pontryagin classes – the elementary symmetric polynomials in $x_1^2, \dots, x_n^2$. The Pontryagin classes are in the image of f: up to signs, they are images of the even Chern classes $c_2, c_4, \dots, c_{2n} \in \mathrm{CH}(BG)$ of the standard G-representation. Concretely, for $i = 1, 2, \dots, n$, the image of c_{2i} is equal to $(-1)^i p_i$ because $\pm x_1, \pm x_2, \dots, \pm x_n$ are the roots of the standard representation meaning that for $i = 1, 2, \dots, c_{2n}$ the image of c_i is the ith elementary symmetric polynomial in $\pm x_1, \pm x_2, \dots, \pm x_n$. Note that the images of the odd Chern classes vanish.

Example 3.9. Let G be the special orthogonal group $\mathrm{SO}(2n)$ with $n \geq 2$ and let T be its standard split maximal torus. Writing $x_1, \ldots, x_n$ for the standard basis of $\hat{T}$, we identify $\mathrm{CH}(BT) = S(\hat{T})$ with the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. The Weyl group W of G permutes $x_1, \ldots, x_n$ and change the signs of any even number of them. The ring of W-invariants $\mathbb{Z}[x_1, \ldots, x_n]^W$ is generated by the Euler class $e := x_1 \ldots x_n$ and the Pontryagin classes – the elementary symmetric polynomials in $x_1^2, \ldots, x_n^2$ (see, e.g., [8, Lemma 4.5]). The Pontryagin classes are in the image of f: up to a sign as in Example 3.8, they are images of the even Chern classes of the standard G-representation. (The images of the odd Chern classes vanish.) Since the cokernel of f is killed by the torsion index of G (by [27, Theorem 1.3(1)]), which is equal to 2^{n-1} (see [27, Theorem 3.2]), the multiple $2^{n-1}e$ is also in the image of f.

In fact, the image of f is generated by the Pontryagin classes and $2^{n-1}e$. In particular, it is strictly smaller than $\mathrm{CH}(BT)^W$.

The above description of $\mathrm{Im}\, f$ in characteristic $\neq 2$ can be obtained from the computation of $\mathrm{CH}(BG)$ made over a field of characteristic $\neq 2$ in [19] and [5]. Over a field of characteristic 2, a computation of $\mathrm{CH}(BG)$ is not available. But one can refer to [8, Proposition 4.2], where the image of f has been determined in a characteristic free setting.

In turns out that one can determine $\mathrm{Im}\, f$ with the help of Proposition 2.5 as well. Indeed, by Lemma 3.10 below, in characteristic $\neq 2$, the group $\mathrm{ChowR}(G)$ is torsion free. Since by Remark 2.7 the ring $\mathrm{IndR}(G)$ does not depend on the base field (and, in particular, on its characteristic), $\mathrm{IndR}(G) = \mathrm{Im}\, f$ in any characteristic.

Lemma 3.10. *For $G = \mathrm{SO}(2n)$ over a field of characteristic $\neq 2$, the group* $\mathrm{ChowR}(G)$ *is torsion free.*

Proof. By [19] and [5], In characteristic $\neq 2$, the torsion ideal of $\mathrm{CH}(BG)$ is generated by the odd Chern classes of the standard G-representation. Since they are restrictions of odd Chern classes of the standard representation of $\mathrm{SO}(2n+1) \supset \mathrm{SO}(2n) = G$, they vanish in $\mathrm{ChowR}(G)$. □

Question 3.11. *It would be interesting to compute the ring* $\mathrm{IndR}(G)$ *for the spin groups* $G = \mathrm{Spin}(d)$.

The case of even d in Question 3.11 is especially interesting. Depending on the answer, it may help to resolve the problem of determination of the indexes of generic orthogonal grassmannians for even spin groups. For odd spin groups the similar problem has been resolved in [9] – a continuation of [4].

4. Spin groups

Let us recall that on the Grothendieck ring $\mathrm{K}(X)$ of a smooth variety X, the *Chern filtration*

$$\mathrm{K}(X) = \mathrm{K}^{[0]}(X) \supset \mathrm{K}^{[1]}(X) \supset \ldots$$

is defined (also called *gamma filtration* in the literature). It is defined using the Chern classes (with values in $\mathrm{K}(X)$) of the elements of $\mathrm{K}(X)$ and satisfies for any i the relation $\mathrm{K}^{[i]}(X) \subset \mathrm{K}^{(i)}(X)$ with the Chow filtration. For $i \leq 2$, the inclusion is actually the equality.

Similarly, for any affine algebraic group G we have the Chern filtration on $\mathrm{R}(G)$ related the same way to the Chow filtration on $\mathrm{R}(G)$, see [15]. We write $\mathrm{ChernK}(X)$ and $\mathrm{ChernR}(G)$ for the associated graded ring of the corresponding Chern filtration.

The two filtrations on $\mathrm{K}(X)$ coincide (in every term) if and only if the ring $\mathrm{ChowK}(X)$ is generated by Chern classes; this condition is implied by but is weaker than the condition for $\mathrm{CH}(X)$ to be generated by Chern classes. The same holds for the two filtrations on $\mathrm{R}(G)$. Indeed, by its very construction, the ring $\mathrm{ChernR}(G)$ is generated by Chern classes. The condition that $\mathrm{ChowR}(G)$ is generated by Chern classes means surjectivity of the homomorphism $\mathrm{ChernR}(G) \to \mathrm{ChowR}(G)$ and implies that for any $i \geq 0$, the ith term $\mathrm{R}^{(i)}(G)$ of the Chow filtration equals the sum $\mathrm{R}^{[i]}(G) + \mathrm{R}^{(j)}(G)$ with any $j \geq i$. By equivalence of the filtrations shown in [14, Corollary 4.8], we know that $\mathrm{R}^{(j)}(G) \subset \mathrm{R}^{[i]}(G)$ for sufficiently large j.

Remark 4.1. For a split reductive group G, similarly to the induced filtration as in Remark 2.7, the Chern filtration on the ring $\mathrm{R}(G)$ and therefore the associated graded ring $\mathrm{ChernR}(G)$ can be constructed in terms of $\hat{T}$, W, and its action on $\hat{T}$. In contrast to this, the Chow filtration on $\mathrm{R}(G)$, situated between the Chern and the induced filtration, and its associated graded ring $\mathrm{ChowR}(G)$ are much more subtle invariants of G.

Example 4.2. Assume that a split reductive group G is *special*, i.e., every G-torsor over every field extension of the base field is trivial. (The torsion index of G is 1 in this case.) Then by [12, Proposition 5.5], the ring $\mathrm{CH}(BG)$ is generated by Chern classes. It follows that the Chow and the Chern filtrations on $\mathrm{R}(G)$ coincide.

Example 4.3. The torsion index of $G = \mathrm{SO}(2n+1)$ is 2^n (see [27, Theorem 3.2]). However the ring $\mathrm{CH}(BG)$ is still generated by Chern classes (of the standard G-representation – see the references provided in Example 3.8) and so, the two filtrations on $\mathrm{R}(G)$ coincide.

Example 4.4. For $G = \mathrm{SO}(2n)$ with $n \geq 3$ and over a field of characteristic not 2, according to [5, Corollary 2], the ring $\mathrm{CH}(BG)$ is not generated by Chern classes. Since the kernel of the homomorphism $\mathrm{CH}(BG) \to \mathrm{ChowR}(G)$ is generated by Chern classes, the ring $\mathrm{ChowR}(G)$ is not generated by Chern classes and the two filtrations on $\mathrm{R}(G)$ differ each from the other.

In characteristic 2, each of the homomorphisms $\mathrm{CH}(BG) \to \mathrm{ChowR}(G) \to \mathrm{IndR}(G)$ is surjective. The ring $\mathrm{IndR}(G)$ is "the same" in any characteristic (see Remark 2.7) and is not generated by Chern classes. Therefore in characteristic 2 neither the ring $\mathrm{ChowR}(G)$, nor the ring $\mathrm{CH}(BG)$ is generated by Chern classes. The Chern and Chow filtrations on $\mathrm{R}(G)$ differ each from the other as well.

Theorem 4.5. *For $G = \mathrm{Spin}(d)$ with $d \geq 7$, the Chern and Chow filtrations on $\mathrm{R}(G)$ differ each from the other. More exactly, their codimension 3 terms are different. This statement also holds with the coefficients $\mathbb{Z}_{(2)}$ (the ring of integers $\mathbb{Z}$ localized at the prime ideal generated by 2) in place of $\mathbb{Z}$.*

We prove Theorem 4.5 in three steps, the first two being Lemma 4.6 and Proposition 4.8.

Lemma 4.6. *Let X be a smooth projective quadric of dimension $d \geq 7$ defined by a quadratic form of trivial discriminant and Clifford invariant. Then* $\mathrm{K}^{[3]}(X) \neq \mathrm{K}^{(3)}(X)$. *This statement also holds with the coefficients* $\mathbb{Z}_{(2)}$.

Proof. We write $\bar{X}$ for X with the scalars extended to an algebraic closure of its base field. By the conditions on the discriminant and Clifford invariant, we have $\mathrm{K}(X) = \mathrm{K}(\bar{X})$ (see [24] or [20]). More correctly, the change of field homomorphism $\mathrm{K}(X) \to \mathrm{K}(\bar{X})$ is an isomorphism. Note that the Chern filtrations on $\mathrm{K}(X)$ and on $\mathrm{K}(\bar{X})$ coincide, but the Chow filtrations differ each from the other in general.

Let $l \in \mathrm{K}(\bar{X})$ be the class of a maximal totally isotropic subspace. As an element of $\mathrm{K}(X)$, l belongs to the codimension 2 term $\mathrm{K}^{[2]}(X) = \mathrm{K}^{(2)}(X)$ of both filtrations. Note that some positive multiple of l is in $\mathrm{K}^{[3]}(X) \subset \mathrm{K}^{(3)}(X)$. In particular, l yields a torsion element (possibly zero) in $\mathrm{Chow}^2\mathrm{K}(X)$.

For odd $d = 2n+1$, it has been shown in [11] (see also [7, Chapter IV]) that $l \notin \mathrm{K}^{[3]}(X)$. The assumption that the characteristic of the base field if not 2, made in [11] and in [7], can be omitted because the ring $\mathrm{K}(X)$ along with the Chern filtration does not depend on the characteristic.

The case of even $d = 2n+2$ was discussed in [11] and [7, Chapter IV] with less details. However, taking a smooth subquadric $X' \subset X$ of the odd dimension $d - 1 = 2n + 1$ and writing $l' \in \mathrm{K}(X')$ for the class in $\mathrm{K}(X')$ of a maximal totally isotropic subspace, we get the pull-back homomorphism $\mathrm{K}(X) \to \mathrm{K}(X')$ respecting the Chern filtrations and mapping l to l'. Since $l' \notin \mathrm{K}^{[3]}(X')$, it follows that $l \notin \mathrm{K}^{[3]}(X)$.

Now, turning to the Chow filtration and arbitrary (odd or even) $d \geq 7$, we use the fact that the group $\mathrm{CH}^2(X)$ is torsion free. The condition $d \geq 7$ is needed in this result; the discriminant and the Clifford invariant conditions are superfluous. The result was proven in [10, Theorem 6.1] for characteristic $\neq 2$ and in [2, Theorem A.1] for characteristic 2. It implies that $\mathrm{Chow}^2\mathrm{K}(X)$ is torsion free and therefore $l \in \mathrm{K}^{(3)}(X)$. In particular, $\mathrm{K}^{[3]}(X) \neq \mathrm{K}^{(3)}(X)$.

At this point, we proved Lemma 4.6 for coefficients $\mathbb{Z}$. To prove if for coefficients $\mathbb{Z}_{(2)}$, we show that no odd multiple of l is in $\mathrm{K}^{[3]}(X)$. For odd $d = 2n + 1$ this follows from the inclusion $2l \in \mathrm{K}^{[3]}(X)$, which is a consequence of the formula

$$2l = h \cdot (h^n + l) \tag{4.7}$$

(see [10, §3.2] or – for more details – [7, Lemma 2.2.9]), where $h \in \mathrm{K}^{[1]}(X)$ is the class of a hyperplane section, or, equivalently, the first Chern class of the dual to the tautological line bundle on X. For even d the formula (4.7) does not hold, but we can reduce the initial statement “no odd multiple of l is in $\mathrm{K}^{[3]}(X)$” to the odd-dimensional codimension 1 subquadric the same way as above. □

Now we are going to consider the even Clifford group $\Gamma^+(d)$ defined as in [16, §23] out of the standard split nondegenerate quadratic form of dimension d. This is a split reductive group and $\mathrm{Spin}(d)$ is its semisimple part.

Proposition 4.8. *For $d \geq 7$, the Chern and Chow filtrations on the representation ring* $\mathrm{R}(\Gamma^+(d))$ *differ each from the other. More exactly, their codimension* 3 *terms are different. This statement also holds with the coefficients* $\mathbb{Z}_{(2)}$.

Proof. Let P be the standard maximal parabolic subgroup in $\operatorname{Spin}(d+2)$ corresponding to the first vertex of the Dynkin diagram. The quotient variety $\operatorname{Spin}(d+2)/P$ is a (split d-dimensional) quadric. Since $\Gamma^+(d)$ is the reductive part of P, the representation rings with their Chern and Chow filtrations for the two groups are identical, see [12, Proof of Proposition 6.1].

Let E be a generic $\operatorname{Spin}(d+2)$-torsor (i.e., the generic fiber of the quotient morphism $\mathrm{GL}(N) \to \mathrm{GL}(N)/\operatorname{Spin}(d+2)$ given by an embedding of $\operatorname{Spin}(d+2)$ into the general linear group $\mathrm{GL}(N)$ for some N) and let X be the quotient variety E/P. Then X is a smooth d-dimensional projective quadric defined by the associated with E quadratic form of trivial discriminant and Clifford invariant. We have a natural surjective ring homomorphism $\mathrm{R}(P) \to \mathrm{K}(X)$ mapping the terms of the Chern and Chow filtrations on $\mathrm{R}(P)$ surjectively onto corresponding terms of the filtrations on $\mathrm{K}(X)$. Since $\mathrm{K}^{[3]}(X) \neq \mathrm{K}^{(3)}(X)$ by Lemma 4.6, it follows that $\mathrm{R}^{[3]}(P) \neq \mathrm{R}^{(3)}(P)$. □

Proof of Theorem 4.5 – the third and final step. In order to show that $\mathrm{R}^{[3]}(G) \neq \mathrm{R}^{(3)}(G)$, we assume that $\mathrm{R}^{[3]}(G) = \mathrm{R}^{(3)}(G)$ and get in contradiction with Proposition 4.8.

The group $G = \operatorname{Spin}(d)$ is the semisimple part of the group $G' := \Gamma^+(d)$. In particular, G is a normal subgroup in G'. The quotient G'/G is a rank 1 split torus. The homomorphism $\mathrm{R}(G') \to \mathrm{R}(G)$, induced by the embedding of the groups, is surjective (see, e.g., [12, Lemma 5.4]) and its kernel is generated by certain element $y \in \mathrm{R}^{[1]}(G') \setminus \mathrm{R}^{[2]}(G')$ – the first Chern class of the image under the homomorphism $\mathrm{R}(G'/G) \to \mathrm{R}(G')$ of the tautological character of G'/G. It follows that $\mathrm{R}(G')^{[i]} \to \mathrm{R}(G)^{[i]}$ is surjective for any $i \geq 0$. Given any $x \in \mathrm{R}^{(3)}(G')$, its image in $\mathrm{R}^{(3)}(G)$ belongs to $\mathrm{R}^{[3]}(G)$ so that we can find $x' \in \mathrm{R}^{[3]}(G')$ with the difference $x - x'$ vanishing in $\mathrm{R}(G)$. Therefore we have $x - x' = yz$ for some $z \in \mathrm{R}(G')$. If we show that $z \in \mathrm{R}^{[2]}(G')$, then we get that $\mathrm{R}^{[3]}(G') = \mathrm{R}^{(3)}(G')$ which is a contradiction with Proposition 4.8, finishing the proof of Theorem 4.5.

Clearly, $z \in \mathrm{R}^{[1]}(G')$. Let T be a split maximal torus of G'. If $z \notin \mathrm{R}^{[2]}(G')$, then y and z yield two nonzero elements of $\mathrm{Chern}^1\mathrm{R}(G) \subset \mathrm{Chern}^1\mathrm{R}(T) = S^1(\hat{T})$ with zero product, a contradiction. □

Theorem 4.9. *For the group $G = \operatorname{Spin}(d)$ over the complex numbers with $d = 7, 8$, the ring* $\mathrm{ChowR}(G)$ *is* not *torsion free. More exactly, the component* $\mathrm{Chow}^3\mathrm{R}(G)$ *contains an element of order* 2.

Proof. Although the statement of Theorem 4.9 is for the integer coefficients, it suffices to prove it for the coefficients $\mathbb{Z}_{(2)}$. It has been shown in [6] that the Chow ring $\mathrm{CH}(B\operatorname{Spin}(7)) \otimes \mathbb{Z}_{(2)}$ is generated by Chern classes together with a single additional element $\xi \in \mathrm{CH}^3(B\operatorname{Spin}(7))$ of order 2. (The description of the integral Chow groups, worked out later in [21], has more generators which are not Chern classes.) Since by Theorem 4.5, the ring $\mathrm{ChowR}(\operatorname{Spin}(7)) \otimes \mathbb{Z}_{(2)}$ is not generated by Chern classes, the element ξ does not vanish there. This proves the $d = 7$ part of Theorem 4.9.

In [21], a 2-torsion element $\xi_{\operatorname{Spin}(8)} \in \mathrm{CH}^3(B\operatorname{Spin}(8))$ is constructed. It maps to ξ under the homomorphism induced by an embedding $\operatorname{Spin}(7) \hookrightarrow \operatorname{Spin}(8)$. It follows from the commutative square

$$\begin{array}{ccc} \mathrm{CH}^3(B\,\mathrm{Spin}(8)) & \longrightarrow & \mathrm{Chow}^3\mathrm{R}(\mathrm{Spin}(8)) \\ \downarrow & & \downarrow \\ \mathrm{CH}^3(B\,\mathrm{Spin}(7)) & \longrightarrow & \mathrm{Chow}^3\mathrm{R}(\mathrm{Spin}(7)) \end{array}$$

that $\xi_{\mathrm{Spin}(8)}$ does not vanish in $\mathrm{Chow}^3\mathrm{R}(\mathrm{Spin}(8))$. □

Appendix A. The reduced filtration

In this appendix we briefly discuss another filtration on the representation ring $\mathrm{R}(G)$ which is defined for arbitrary affine G and seems to be a suitable replacement for the induced filtration, the latter being defined for split reductive G only.

Let G be any affine algebraic group (of finite type over a field). The Chern filtration and containing it Chow filtration on $\mathrm{R}(G)$ also have the following relation:

Lemma A.1. *For any $i \geq$ and any $a \in \mathrm{R}^{(i)}(G)$, some positive multiple of a is in $\mathrm{R}^{[i]}(G)$.*

Proof. By [14, Corollary 4.9], after tensoring with $\mathbb{Q}$, the homomorphism

$$\mathrm{ChernR}(G) \to \mathrm{ChowR}(G)$$

becomes isomorphism. It follows that for any $j \geq i$ some positive multiple of a is in $\mathrm{R}^{[i]}(G) + \mathrm{R}^{(j)}(G)$. By [14, Corollary 4.8], $\mathrm{R}^{(j)}(G) \subset \mathrm{R}^{[i]}(G)$ for sufficiently large j. □

For an arbitrary affine algebraic group G, let us call *reduced* the filtration on $\mathrm{R}(G)$ the ith term of which – for any $i \geq 0$ – consists of all elements in $\mathrm{R}(G)$ possessing a positive multiple in $\mathrm{R}^{[i]}(G)$. We write $\mathrm{RedR}(G)$ for its associated graded ring. By Lemma A.1, the reduced filtration contains the Chow filtration and can be defined using the Chow filtration in place of the Chern one. The inclusion yields a ring homomorphism

$$\mathrm{ChowR}(G) \to \mathrm{RedR}(G).$$

By Corollary 3.3, for a split semisimple simply connected G, the reduced filtration coincides with the induced one. It seems to be a good replacement for the induced filtration also because, unlike the latter, it is defined for arbitrary G and has the following property in the case when G is split reductive:

Lemma A.2. *For any split reductive G, the reduced filtration is contained in the induced one. Moreover, the resulting homomorphism $\mathrm{RedR}(G) \to \mathrm{IndR}(G)$ is surjective.*

Proof. If for some $i \geq 0$, some a is in the ith term of the reduced filtration, then a positive multiple of a is in $\mathrm{R}^{(i)}(G)$ which is a subset in the ith term $(I^i)^W$ of the induced filtration. Since the ring $\mathrm{IndR}(G)$ is torsion free, it follows that $a \in (I^i)^W$.

By the argument of the proof of Corollary 3.3, for any $a \in (I^i)^W$, the multiple ta of a, where t is the torsion index of G, is in $\mathrm{R}^{(i)}(G) + (I^{i+1})^W$. This means surjectivity of $\mathrm{RedR}(G) \to \mathrm{IndR}(G)$. □

Remark A.3. Like the induced filtration (see Remark 2.7) and like the Chern filtration (see Remark 4.1), the reduced filtration on the representation ring of a split reductive G only depends on the type of G and, in particular, does not depend on the base field.

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