ENVELOPES AND CLASSIFYING SPACES

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ABSTRACT. For a split semisimple algebraic group H with its split maximal torus S, let $f\colon \mathrm{CH}(\mathsf{B}H) \to \mathrm{CH}(\mathsf{B}S)^W$ be the restriction homomorphism of the Chow rings CH of the classifying spaces B of H and S, where W is the Weyl group. A constraint on the image of f, given by the Steenrod operations, has been applied to the spin groups in [9]. Here we describe and apply to the spin groups another constraint, which is given by the reductive envelopes of H. We also recover this way some older results on orthogonal groups.

1. Constraint

Let H be a split semisimple algebraic group (over an arbitrary field) with a split maximal torus S and the Weyl group W. The image of the restriction homomorphism $\mathrm{CH}(\mathsf{B}H) \to \mathrm{CH}(\mathsf{B}S)$, where $\mathrm{CH}(\mathsf{B}H)$ is the Chow ring of the classifying space $\mathsf{B}H$ of H, defined in [14], consists of W-invariant elements. The resulting homomorphism of graded rings

$$f\colon \mathrm{CH}(\mathsf{B}H) \to \mathrm{CH}(\mathsf{B}S)^W$$

is rationally an isomorphism: its kernel and cokernel are killed by the torsion index of H, see [15, Theorem 1.3(1)]. Note that $\mathrm{CH}(\mathsf{B}S)$ is canonically identified with the symmetric ring $\mathsf{S}(\hat{S})$ of the character group $\hat{S}$ of S, [3, §3.2] (see also [7, §3]). In particular, the group $\mathrm{CH}(\mathsf{B}S)$ is free of torsion so that the kernel of f actually coincides with the ideal $\mathrm{Tors}\,\mathrm{CH}(\mathsf{B}H)$ of torsion elements in $\mathrm{CH}(\mathsf{B}H)$. Therefore determination of the image $\mathrm{Im}\, f$ of f is equivalent to determination of the quotient ring $\mathrm{CH}(\mathsf{B}H)/\mathrm{Tors}\,\mathrm{CH}(\mathsf{B}H)$. This quotient ring is relevant for some applications (e.g., [2] and [10]), where the torsion in $\mathrm{CH}(\mathsf{B}H)$ is irrelevant.

In [9], a (given by the Steenrod operations) constraint on $\mathrm{Im}\, f$ has been described. There is another constraint on $\mathrm{Im}\, f$ which is given by any *envelope* of H – a split reductive group G such that H is its semisimple part. Namely, G contains H as a normal subgroup, has a split maximal torus T with $T \cap H = S$, the Weyl group of G coincides with the Weyl group W of H, and the square

$$\begin{array}{ccc} \mathrm{CH}(\mathsf{B}G) & \longrightarrow & \mathrm{CH}(\mathsf{B}T)^W \\ \big\downarrow & & \big\downarrow g \\ \mathrm{CH}(\mathsf{B}H) & \xrightarrow{\ f\ } & \mathrm{CH}(\mathsf{B}S)^W, \end{array} \tag{1.1}$$

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formed by restriction homomorphisms of the Chow rings of the classifying spaces, commutes. By [8, Proposition 4.1], since the quotient G/H is a split torus, the left arrow in (1.1) is surjective. Therefore

Proposition 1.2. *The image of f is contained in the image of g.* □

Note that the homomorphism of the character groups $\hat{T} \to \hat{S}$, induced by the embedding $S \hookrightarrow T$, is surjective, implying that the ring homomorphism

$$\mathrm{CH}(\mathcal{B}T) = \mathcal{S}(\hat{T}) \to \mathcal{S}(\hat{S}) = \mathrm{CH}(\mathcal{B}S),$$

is also surjective. However, as we will see, the homomorphism g of the subrings of W-invariants can fail to be so. In other terms, the constraint of Proposition 1.2 is nontrivial in general.

An envelope is *strict*, if its center is a torus. A strict envelope of H exists for any H (see [12, Example 9.7]) and provides the strongest constraint among all envelopes of H. Indeed, given an envelope G and a strict envelope $\tilde{G}$, there exists by [12, Lemma 9.8] a homomorphism $G \to \tilde{G}$ identical on H. A given split maximal torus T of G, containing H, is mapped into some split maximal torus $\tilde{T}$, and the restriction homomorphism $\tilde{g}\colon \mathrm{CH}(\mathcal{B}\tilde{T})^W \to \mathrm{CH}(\mathcal{B}S)^W$ factors through g. Therefore the image of $\tilde{g}$ is contained in $\operatorname{Im} g$ and so the constraint on $\operatorname{Im} f$ provided by $\tilde{G}$ is stronger than that provided by G.

Using [12, §9], one can formalize Proposition 1.2. Namely, let B be an abstract finitely generated abelian group with a surjective homomorphism $\mu\colon B \to \hat{S}/\Lambda_{\mathrm{r}}$, where Λ_{r} is the root lattice of H. Assume that $\operatorname{Ker}\mu$ is free of torsion. Let A be the kernel of the homomorphism $B \oplus \hat{S} \to \hat{S}/\Lambda_{\mathrm{r}}$, given by the difference of μ and the quotient map $\hat{S} \to \hat{S}/\Lambda_{\mathrm{r}}$. We consider $A \subset B \oplus \hat{S}$ with the action of W, induced by its usual action on $\hat{S}$ and its trivial action on B. Then, by [12, Proposition 9.4], the W-module A is identified with $\hat{T}$ for an appropriate envelope $G \supset T \supset S$ of H the way that the projection $A \to \hat{S}$ is identified with the homomorphism $\hat{T} \to \hat{S}$ given by the embedding $S \hookrightarrow T$. Moreover, any envelope G of H with a split maximal torus $T \supset S$ is obtained this way. Finally, G is strict if and only if B is free of torsion.

In §2 and §3, we apply Proposition 1.2 to spin groups and answer in Theorems 2.2 and 3.2 the question raised in [9, Remark 7]. In §4, we apply Proposition 1.2 to orthogonal groups and obtain this way (see Theorem 4.1) a simpler and valid in arbitrary characteristic proof of earlier results obtained in in [13] and [4] over fields of characteristic different from 2. We hope that Proposition 1.2 will help to resolve [9, Question 9], see Remark 3.4.

2. Odd spin groups

Here is an example of successful application for Proposition 1.2. In particular, this is an example of non-surjective g and nontrivial constraint.

Let us take for H the standard split spin group $\mathrm{Spin}(n)$ with odd $n = 2l+1$. For $n < 7$, the torsion index of H is 1 so that the homomorphism f is surjective. By this reason, below we assume that $n \geq 7$.

Let $S \subset H$ be the standard split maximal torus. The graded ring $\mathrm{CH}(\mathcal{B}S)$ is identified with the polynomial ring $\mathbb{Z}[a, y_1, \dots, y_l]$ in $l+1$ variables (graded in the usual way with each variable having degree 1) modulo the unique relation

$$2a = y_1 + \ldots + y_l.$$

As a subgroup in the automorphism group of $\mathrm{CH}(BS)$, the Weyl group W is generated by the permutations of $y_1, \ldots, y_l$ and their individual sign changes. Note that the permutations act trivially on a; for any $i = 1, \ldots, l$, the sign change of y_i moves a to $a - y_i$.

The ring $\mathrm{CH}(BS)^W$ has certain standard generators found in [1], where the topological analogue of f has been studied for spin groups. The first group of generators of $\mathrm{CH}(BS)^W$ consists of the elementary symmetric polynomials $p_1, \ldots, p_l$ in the squares of $y_1, \ldots, y_l$. They are called Pontryagin classes and lie in the image of f, because, up to signs, they are Chern classes of the representation of H given by the standard representation of the special orthogonal group $\mathrm{SO}(n)$, see, e.g., [9].

For every $i \geq 1$, an element $q_i \in \mathrm{CH}^{2^i}(BS)^W$ is constructed in [1, Proposition 3.3]. The second group of generators is $q_1, \ldots, q_{l-2}$. It has been shown in [9] that, unlike the situation in topology, several first elements of this group are outside the image of f.

Finally, there is one last generator $\alpha \in \mathrm{CH}^{2^{l-1}}(BS)^W$ defined as

$$\alpha = \prod_I (a - \sum_{i \in I} y_i), \tag{2.1}$$

where I runs over the subsets of $\{1, \ldots, l-1\}$. The square α^2 of α is the product of the elements in the W-orbit of a, see [1, Proposition 4.1(i)]. This orbit product is the highest Chern class of the spin representation of H and therefore lies in $\operatorname{Im} f$. By the topological results of [1], α itself is not in the image of f if $n \equiv \pm 3 \pmod 8$. More precisely, for such n the image of f is contained in the subring of $\mathrm{CH}(BS)^W$ generated by all the generators with α replaced by 2α and α^2. Now we can show that, unlike the situation in topology, the latter statement holds for *any* odd n (including $n \equiv \pm 1 \pmod 8$):

Theorem 2.2. *For $H = \mathrm{Spin}(n)$ with any odd $n \geq 7$, the generator α is not in the image of f. The image of f is contained in the subring of $\mathrm{CH}(BS)^W$ generated by all the standard generators with α replaced by 2α and α^2.*

Proof. The abelian group Λ_{r} is free with a basis $y_1, \ldots, y_l$, where $2l + 1 = n$. The Weyl group W acts on Λ_{r} by permutations and sign changes of $y_1, \ldots, y_l$. The abelian group $\hat{S} \supset \Lambda_{\mathrm{r}}$ is generated by Λ_{r} and a, satisfying $2a = y_1 + \ldots + y_l$, so that $\hat{S}/\Lambda_{\mathrm{r}} = \mathbb{Z}/2\mathbb{Z}$. We take $B = \mathbb{Z}$ with μ the quotient homomorphism to $\mathbb{Z}/2\mathbb{Z}$. (This way we get a strict envelope G of H which is actually the even Clifford group $\Gamma^+(n)$, see [11, §23].) The subgroup

$$\hat{T} = A \subset B \oplus \hat{S}$$

is free, a basis is given by $y_1, \ldots, y_l$ and $z := x + a$, where x is a generator of B. The epimorphism $A \twoheadrightarrow \hat{S}$ maps z to a and y_i to y_i for every $i = 1, \ldots, l$.

The ring

$$\mathrm{CH}(BT) = S(\hat{T}) = S(A)$$

is the polynomial ring $\mathbb{Z}[z, y_1, \ldots, y_l]$ in $l+1$ (independent) variables. The Weyl group W acts by permuting $y_1, \ldots, y_l$ and by changing their signs. The permutations act trivially on z, the ith change of sign transforms z to $z - y_i$.

Determination of the ring $\mathrm{CH}(BT)^W$ has been started in [2] and finished in [6, Proposition 2.4]. The generators are:

- the elementary symmetric polynomials $p_1, \dots, p_l$ in $y_1^2, \dots, y_l^2$ (Pontryagin classes);
- $f_0 := 2z - (y_1 + \dots + y_l)$;
- certain $f_1, \dots, f_{l-1}$, where f_i is homogeneous of degree 2^i;
- and the orbit product of z (homogeneous of degree 2^l) denoted $\tilde{z}$.

Under the homomorphism $\mathrm{CH}(BT) \to \mathrm{CH}(BS)$, the images of these generators respectively are (where for every $i \geq 1$ we write φ_i for the image of f_i):

- the Pontryagin classes $p_1, \dots, p_l$;
- 0;
- $\varphi_1, \dots, \varphi_{l-1}$;
- and α^2.

By Proposition 1.2, $\mathrm{Im}\, f$ is contained in the subring of $\mathrm{CH}(BS)$ generated by these images. By Lemma 2.3 below, the generators $\varphi_1, \dots, \varphi_{l-1}$ can be replaced by $q_1, \dots, q_{l-1}$. The generator q_{l-1} can be replaced by 2α because, by [1, Corollary 7.2(i)], $2\alpha - q_{l-1}$ is in the subring generated by $p_2, \dots, p_l$ and $q_1, \dots, q_{l-2}$.

In order to see that α is not in the image of f, notice that every element of $\mathbb{Z}[a, y_1, \dots, y_l]$ can be written as an integral polynomial in $a, y_1, \dots, y_{l-1}$; moreover, such integral polynomial is unique. Let us view it as a polynomial in a with coefficients in $\mathbb{Z}[y_1, \dots, y_{l-1}]$. By definition (2.1), α has coefficient 1 at $a^{2^{l-1}}$. If α would be in the image of f, it could be written as a polynomial in the elements $p_1, \dots, p_l$, $q_1, \dots, q_{l-2}$, 2α. But for each of these elements the coefficient at any positive power of a is even. □

Here is the assertion on the images $\varphi_1, \varphi_2, \dots \in \mathrm{CH}(BS)^W$ under the homomorphism

$$\mathrm{CH}(BT)^W \to \mathrm{CH}(BS)^W$$

of the generators $f_1, f_2, \dots \in \mathrm{CH}(BT)^W$, used in the above proof:

Lemma 2.3. *For every $i \geq 1$, the subring in $\mathrm{CH}(BS)^W$ generated by the elements $\varphi_1, \dots, \varphi_i$, is contained in the subring generated by $q_1, \dots, q_i$. For $i \leq l-2$ these two subrings coincide.*

Proof. Comparing the construction of q_i, given in [1, Proof of Proposition 3.3], with the construction of f_i, given in [2, §3] (as well as in [6, §2]), one sees that $\varphi_1 + q_1 = 0$ so that the statement of Lemma 2.3 holds for $i = 1$.

For any $i \geq 2$, one sees that $\varphi_i + q_i = 2\psi$, where ψ is a homogeneous integral polynomial in $a, y_1, \dots, y_l$ of degree 2^i. Since φ_i and q_i are W-invariant, ψ is also W-invariant. Therefore, if $i \leq l-2$, by [1, Theorem 7.1(i)], ψ is a polynomial in the Pontryagin classes and $q_1, \dots, q_i$. By [1, Proposition 3.3(iv)], $2q_i - q_{i-1}^2$ is an integral polynomial in the Pontryagin classes. This proves the statement of Lemma 2.3 for $i \leq l-2$.

If $i \geq l-1$, again by [1, Theorem 7.1(i)], ψ is a polynomial in the Pontryagin classes, $q_1, \dots, q_i$, and α. By [1, Corollary 7.2(i)], 2α is a polynomial in the Pontryagin classes and $q_1, \dots, q_i$. □

3. Even spin groups

A similar example of successful application for Proposition 1.2 occurs with the even spin groups: $H = \mathrm{Spin}(n)$ with $n = 2l$. For $n < 7$ again, the torsion index of H is 1 so that the homomorphism f is surjective. By this reason, below we assume that $n \geq 8$.

Let $S \subset H$ be the standard split maximal torus. Exactly as in the case of odd $n = 2l+1$, the graded ring $\mathrm{CH}(BS)$ is identified with the polynomial ring $\mathbb{Z}[a, y_1, \ldots, y_l]$ in $l+1$ variables (graded in the usual way with each variable having degree 1) modulo the unique relation

$$2a = y_1 + \ldots + y_l.$$

As a subgroup in the automorphism group of $\mathrm{CH}(BS)$, the Weyl group W is generated by the permutations of $y_1, \ldots, y_l$ and the sign changes of any even number of them. So, the Weyl group we have now is smaller than the Weyl group of §2.

The ring $\mathrm{CH}(BS)^W$ has certain standard generators found in [1]. The first group of generators of $\mathrm{CH}(BS)^W$ consists of the Pontryagin classes – the elementary symmetric polynomials $p_1, \ldots, p_l$ in the squares of $y_1, \ldots, y_l$. By the same reason as in the case $n = 2l + 1$, they lie in the image of f.

There is an additional (with respect to the $n = 2l + 1$ case) generator $e := y_1 \ldots y_l$ called the Euler class. For $n \neq 10$, it has been shown in [9, Theorem 3] and [5, Appendix A] that (unlike the situation in topology) e is outside the image of f.

For every $i \geq 1$, the element $q_i \in \mathrm{CH}^{2^i}(BS)$ from §2 is W-invariant. The next group of generators is $q_1, \ldots, q_{l-2}$ for odd l and $q_1, \ldots, q_{l-3}$ for even l. By [9, Theorem 3] several first elements of this group are outside the image of f. (All elements of this group are in the image of f in topology.)

Finally, there is one last generator $\alpha \in \mathrm{CH}(BS)^W$ defined as

$$\alpha = \prod_I (a - \sum_{i \in I} y_i). \tag{3.1}$$

Here, if l is odd, then I runs over the *even* (i.e., consisting of an even number of elements) subsets of $\{1, \ldots, l\}$. In particular, $\alpha \in \mathrm{CH}^{2^{l-1}}(BS)$, α is the orbit product of a and lies in the image of f (as it is the image of the highest Chern class of a half-spin representation of S).

However, if l is even, then I runs over the even subsets of $\{1, \ldots, l-1\}$ so that $\alpha \in \mathrm{CH}^{2^{l-2}}(BS)$. In this case, the orbit product of a (lying in the image of f) is equal to α^2, see [1, Proposition 4.1(ii)]. We show that α itself is not in the image of f (although in topology $\alpha \in \mathrm{Im}\, f$ for l divisible by 4):

Theorem 3.2. *For $H = \mathrm{Spin}(n)$ with any $n \geq 8$ divisible by 4, the generator α is not in the image of f. The image of f is contained in the subring of $\mathrm{CH}(BS)^W$ generated by all the standard generators with α replaced by 2α and α^2.*

Proof. As in §2 (where $n = 2l + 1$), the abelian group $\hat{S}$ is still generated by $y_1, \ldots, y_l$ (where now $n = 2l$) and a subject to the relation $2a = y_1 + \ldots + y_l$; but the root lattice $\Lambda_{\mathrm{r}} \subset \hat{S}$ is smaller: it consists of the linear combinations $a_1 y_1 + \ldots + a_l y_l$ with integer coefficients $a_1, \ldots, a_l$ satisfying the condition $a_1 + \ldots + a_l \in 2\mathbb{Z}$. Since n is divisible by 4, the integer l is even and we have $\hat{S}/\Lambda_{\mathrm{r}} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, where the first summand is generated by the class of a and the second summand is generated by the class of y_1 coinciding with the class of y_i for any i. (For odd l, $\hat{S}/\Lambda_{\rm r} = \mathbb{Z}/4\mathbb{Z}$ is generated by the class of a and for any i the class of y_i coincides with the class of $2a$.) The Weyl group W acts on $\hat{S}$ by permutations and even sign changes of $y_1, \dots, y_l$.

Taking $B = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with μ mapping $1 \in \mathbb{Z}$ to the class of a in $\hat{S}/\Lambda_{\rm r}$ and mapping $1 \in \mathbb{Z}/2\mathbb{Z}$ to the class of y_1, we get the envelope given by the even Clifford group. The subgroup

$$\hat{T} = A \subset B \oplus \hat{S}$$

is free, a basis is given by $y'_1 := y + y_1, \dots, y'_l := y + y_l$, where y is a generator of the second summand ($\mathbb{Z}/2\mathbb{Z}$) of $B = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and $z := x + a$, where x is a generator of the first summand ($\mathbb{Z}$) of B. The epimorphism $A \twoheadrightarrow \hat{S}$ maps z to a and y'_i to y_i for every $i = 1, \dots, l$.

The ring

$$\mathrm{CH}(\mathsf{B}T) = \mathsf{S}(\hat{T}) = \mathsf{S}(A)$$

is the polynomial ring $\mathbb{Z}[z, y'_1, \dots, y'_l]$ in $l+1$ (independent) variables. To simplify notation, we write y_i for y'_i below.

The Weyl group W acts by permuting $y_1, \dots, y_l$ and by changing the signs of even numbers of them. The permutations act trivially on z, the sign change of y_i and y_j for $i \neq j$ transforms z to $z - y_i - y_j$.

The ring $\mathrm{CH}(\mathsf{B}T)^W$ of W-invariants has been computed in [10, Proposition 5.1] (based on [2]). The generators are:

- the elementary symmetric polynomials $p_1, \dots, p_l$ in $y_1^2, \dots, y_l^2$ (Pontryagin classes);
- the Euler class $e = y_1 \dots y_l$;
- $f_0 := 2z - (y_1 + \dots + y_l)$;
- the elements $f_1, \dots, f_{l-2}$ of §2;
- and the orbit product of z (homogeneous of degree 2^{l-1}) denoted $\check{z}$.

Under the homomorphism $\mathrm{CH}(\mathsf{B}T) \to \mathrm{CH}(\mathsf{B}S)$, the images of these generators respectively are:

- the Pontryagin classes $p_1, \dots, p_l$;
- the Euler class e;
- 0;
- $\varphi_1, \dots, \varphi_{l-2}$;
- and α^2.

By Proposition 1.2, $\operatorname{Im} f$ is contained in the subring of $\mathrm{CH}(\mathsf{B}S)$ generated by these images. By Lemma 2.3, the generators $\varphi_1, \dots, \varphi_{l-2}$ can be replaced by $q_1, \dots, q_{l-2}$. The generator q_{l-2} can be then replaced by 2α because, by [1, Corollary 7.2(ii)], $2\alpha - q_{l-2}$ is in the subring generated by $p_2, \dots, p_{l-1}$, e, and $q_1, \dots, q_{l-3}$.

In order to see that α is not in the image of f, notice that every element of $\mathbb{Z}[a, y_1, \dots, y_l]$ can be written as an integral polynomial in $a, y_1, \dots, y_{l-1}$; moreover, such integral polynomial is unique. Let us view it as a polynomial in a with coefficients in $\mathbb{Z}[y_1, \dots, y_{l-1}]$. By definition (3.1), α has coefficient 1 at $a^{2^{l-2}}$. If α would be in the image of f, it could be written as a polynomial in $p_1, \dots, p_l$, e, $q_1, \dots, q_{l-3}$, and 2α. But for each of these elements the coefficient at any positive power of a is even. □

Remark 3.3. Theorems 2.2 and 3.2 answer the question raised in [9, Remark 7].

Remark 3.4. It is plausible that Proposition 1.2 may help to resolve [9, Question 9] on the group $H = \mathrm{Spin}(n)$ with even $n \geq 12$. This question constitutes the obstacle for determination of the indexes of generic orthogonal grassmannians given by the spin groups.

Recall that

$$\hat{S}/\Lambda_{\mathrm{r}} = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is divisible by } 4; \\ \mathbb{Z}/4\mathbb{Z}, & \text{otherwise.} \end{cases}$$

To get a strict envelope in the second case, one can take $B = \mathbb{Z}$; in the first case, one can take $B = \mathbb{Z} \oplus \mathbb{Z}$. The envelope used in the proof of Theorem 3.2 being not strict, it probably does not provide the full constraint of Proposition 1.2.

We could see if the constraint given by a strict envelope G is strong enough, if we had a computation of the W-invariants $\mathrm{CH}(BT)^W$ for the split maximal torus T of G. However such a computation is not yet available.

4. Orthogonal groups

Here is one more example of application for Proposition 1.2.

We consider the standard split special orthogonal group $H := \mathrm{SO}(2l)$ for some $l \geq 2$ with its standard split maximal torus S. The graded ring $\mathrm{CH}(BS) = S(\hat{S})$ is identified with the polynomial ring $\mathbb{Z}[y_1, \dots, y_l]$ in l variables $y_1, \dots, y_l$. The Weyl group W of H acts on $\mathbb{Z}[y_1, \dots, y_l]$ by permuting the variables and changing the signs of any even number of them. The ring $\mathbb{Z}[y_1, \dots, y_l]^W$ of the W-invariant elements is generated by the Euler class $e := y_1 \dots y_l$ and the Pontryagin classes $p_1, \dots, p_l$, defined as the elementary symmetric polynomials in the squares of the variables (see Lemma 4.5).

Since the image of the restriction homomorphism $\mathrm{CH}(BH) \to \mathrm{CH}(BS)$ consists of W-invariant elements only, it is contained in the subring of $\mathrm{CH}(BS)$ generated by e and $p_1, \dots, p_l$. In fact, due to a computation of $\mathrm{CH}(BH)$, made in [13] over a field of characteristic different from 2 (see also [4]), this image is known to be generated by $2^{l-1}e$ and $p_1, \dots, p_l$. We give a characteristic-free proof of this statement not relying on a computation of $\mathrm{CH}(BH)$:

Theorem 4.1. *The image of the restriction homomorphism* $f\colon \mathrm{CH}(BH) \to \mathrm{CH}(BS)^W$ *is generated by* $2^{l-1}e$ *and* $p_1, \dots, p_l$.

Actually, the fact that $2^{l-1}e$ is in $\mathrm{Im}\, f$ is a consequence of the general result [15, Theorem 1.3(1)] telling that the torsion index of H (which is equal to 2^{l-1}, see [15, Theorem 3.2]) annihilates the cokernel of f. As to the Pontryagin classes, they are, up to signs, the images under f of the Chern classes of the standard representation $\mathrm{SO}(2l) \hookrightarrow \mathrm{GL}(2l)$ (see [9] for more details). Therefore we only need to show that $\mathrm{Im}\, f$ is contained in the subring of $\mathrm{CH}(BT)$ generated by $2^{n-1}e$ and $p_1, \dots, p_n$.

The proof of this inclusion is based on the technique of strict envelopes considered in §1. Instead of the formal approach, suggested there, we proceed here with a direct construction of a strict envelope.

Set $G := (\mathbb{G}_\mathrm{m} \times H)/\mu_2$, where μ_2 is embedded into the center $\mathbb{G}_\mathrm{m} \times \mu_2$ of $\mathbb{G}_\mathrm{m} \times H$ via the composition

$$\mu_2 \hookrightarrow \mu_2 \times \mu_2 \hookrightarrow \mathbb{G}_\mathrm{m} \times \mu_2, \ \ \xi \mapsto (\xi, \xi)$$

with the diagonal. Then G is a reductive group containing H as its semisimple part. The product of S with the center $\mathbb{G}_\mathrm{m}$ of G yields a split maximal torus $T \subset G$, containing S. The Weyl group of G with respect to T coincides with W. Since the quotient G/H is isomorphic to $\mathbb{G}_\mathrm{m}$, the left arrow in the commutative square of restriction maps

$$\begin{array}{ccc} \mathrm{CH}(\mathsf{B}G) & \longrightarrow & \mathrm{CH}(\mathsf{B}T)^W \\ \big\downarrow & & \big\downarrow g \\ \mathrm{CH}(\mathsf{B}H) & \xrightarrow{f} & \mathrm{CH}(\mathsf{B}S)^W \end{array}$$

is surjective (see [8, Proposition 4.1]). It follows that the image of f is contained in the image of g (cf. Proposition 1.2).

Proposition 4.2. *The image of* $g\colon \mathrm{CH}(\mathsf{B}T)^W \to \mathrm{CH}(\mathsf{B}S)^W$ *is generated by* $2^{l-1}e$ *and* $p_1, \dots, p_l$.

For the proof of Proposition 4.2 we describe $\mathrm{CH}(\mathsf{B}T) = \mathsf{S}(\hat{T})$, starting with a description of $\hat{T}$. Since

$$T = (\mathbb{G}_\mathrm{m} \times S)/\mu_2,$$

the character group $\hat{T}$ is the kernel of the homomorphism $\mathbb{Z}^{\oplus(l+1)} \to \mathbb{Z}/2\mathbb{Z}$, mapping to 1 each of the standard basic elements $y, y_1, \dots, y_l$. The action of the Weyl group W on $\hat{T}$ is the restriction of its action on $\mathbb{Z}^{\oplus(l+1)} = \hat{\mathbb{G}}_\mathrm{m} \oplus \hat{S}$ given by the trivial action on the summand $\hat{\mathbb{G}}_\mathrm{m}$ (containing the element y) and its standard action on the summand $\hat{S}$ (containing the elements $y_1, \dots, y_l$).

We choose the basis $t := 2y$, $t_1 := y + y_1$, $\dots$, $t_l := y + y_l$ of the lattice $\hat{T}$. Here t is W-invariant and $t_1, \dots, t_l$ are permuted by W; moreover, for any $i = 1, \dots, l$, the sign change of y_i transforms t_i into $t - t_i$ and does not affect the rest of the basis.

The ring homomorphism

$$\mathrm{CH}(\mathsf{B}T) = \mathsf{S}(\hat{T}) = \mathbb{Z}[t, t_1, \dots, t_l] \to \mathbb{Z}[y_1, \dots, y_l] = \mathsf{S}(\hat{S}) = \mathrm{CH}(\mathsf{B}S)$$

kills t and maps t_i to y_i for all i.

Let us construct some W-invariant elements in $\mathbb{Z}[t, t_1, \dots, t_l]$. For $i = 1, \dots, l$, let P_i be the ith elementary symmetric polynomial in $t_1(t_1 - t), \dots, t_l(t_l - t)$. Clearly, P_i is W-invariant and has p_i as its image in $\mathrm{CH}(\mathsf{B}S)$.

Note that for any i, the difference $2t_i - t$ changes the sign under $t_i \mapsto t - t_i$. It follows that

$$E := \frac{1}{2}\Big((-1)^{l-1}t^l + \prod_{i=1}^{l}(2t_i - t)\Big) \in \mathbb{Z}[t, t_1, \dots, t_l] = \mathrm{CH}(\mathsf{B}T)$$

is W-invariant. The image of E in $\mathrm{CH}(\mathsf{B}S)$ is equal to $2^{l-1}e$.

Proposition 4.2 is a consequence of the following computation:

Lemma 4.3. *The ring* $\mathrm{CH}(\mathsf{B}T)^W = \mathbb{Z}[t, t_1, \dots, t_l]^W$ *is generated by* E *and* $P_1, \dots, P_l$.

Proof. It suffices to prove that any *homogeneous* W-invariant polynomial in $\mathbb{Z}[t, t_1, \dots, t_l]$ is in the subring generated by E and $P_1, \dots, P_l$. Setting $t = 1$, we come to the following equivalent problem: show that any W-invariant polynomial in $\mathbb{Z}[t_1, \dots, t_l]$ is in the subring generated by E and $P_1, \dots, P_l$. Note that the ith sign change element of W now transforms t_i to $1 - t_i$. The new polynomials $E, P_1, \dots, P_l \in \mathbb{Z}[t_1, \dots, t_l]$ are obtained from their previous versions by the substitution $t = 1$ so that

$$E := \frac{1}{2}\Big((-1)^{l-1} + \prod_{i=1}^{l}(2t_i - 1)\Big) \in \mathbb{Z}[t_1, \dots, t_l]$$

and $P_1, \dots, P_l$ are the elementary symmetric polynomials in $t_1(t_1 - 1), \dots, t_l(t_l - 1)$.

Let us invert $2 \in \mathbb{Z}$ by passing to the ring $\mathbb{Z}' := \mathbb{Z}[1/2]$. The $\mathbb{Z}'$-algebra $\mathbb{Z}'[t_1, \dots, t_l]$ is generated by the algebraically independent elements

$$2t_1 - 1, \dots, 2t_l - 1 \tag{4.4}$$

on which W acts by permutations and sign changes so that (by Lemma 4.5)

the ring $\mathbb{Z}[2t_1 - 1, \dots, 2t_l - 1]^W$ as well as the $\mathbb{Z}'$-algebra $\mathbb{Z}'[t_1, \dots, t_l]^W$

are generated by the product of the elements in (4.4) and the elementary symmetric polynomials in the squares of the elements in (4.4). Therefore, the $\mathbb{Z}'$-algebra $\mathbb{Z}'[t_1, \dots, t_l]^W$ is generated by $E, P_1, \dots, P_l$. So, any element $a \in \mathbb{Z}[t_1, \dots, t_l]^W$ is a polynomial b in $E, P_1, \dots, P_l$ with $\mathbb{Z}'$-coefficients. Since

$$E^2 = (-1)^{l-1}E + P_1 + 4P_2 + 4^2P_3 + \ldots + 4^{l-1}P_l,$$

we may assume that in b there are no monomials containing E in a power higher than 1. We prove Lemma 4.3 by showing that the coefficients of any such $\mathbb{Z}'$-polynomial b in $E, P_1, \dots, P_l$ are integers.

We may assume that a is not divisible by 2. Under this assumption, let $r \geq 0$ be the smallest integer such that the coefficients of $2^r b$ are integers. It suffices to show that $r = 0$.

Note that $a \in \mathbb{Z}[t_1, \dots, t_l]$ is symmetric in $t_1, \dots, t_l$ and therefore $a \in \mathbb{Z}[c_1, \dots, c_l]$, where $c_1, \dots, c_l$ are the elementary symmetric polynomials in $t_1, \dots, t_l$. If $r > 0$, then the element $2^r a \in \mathbb{Z}[c_1, \dots, c_l]$ vanishes in $\mathbb{F}_2[c_1, \dots, c_l]$, where $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. Therefore, in order to show that $r = 0$, it suffices to show that $2^r b$ does not vanish in $\mathbb{F}_2[c_1, \dots, c_l]$.

Recall that $2^r b$ is a linear combination with integer coefficients of the monomials $E^\alpha P_1^{\alpha_1} \dots P_l^{\alpha_l}$ with $\alpha \leq 1$. By minimality of r, at least one of the coefficients is odd. Reducing the coefficients modulo 2, we get a linear combination of the images in $\mathbb{F}_2[c_1, \dots, c_l]$ of the monomials $E^\alpha P_1^{\alpha_1} \dots P_l^{\alpha_l}$ (with coefficient in $\mathbb{F}_2$) and with at least one nonzero coefficient. Note that the image of E in $\mathbb{F}_2[c_1, \dots, c_l]$ is c_1 and the image of P_i is c_i^2 plus terms of smaller degree, where $\deg c_i := i$. Since the elements $c_1, c_2^2, \dots, c_l^2$ are algebraically independent, the images in $\mathbb{F}_2[c_1, \dots, c_l]$ of the monomials $E^\alpha P_1^{\alpha_1} \dots P_l^{\alpha_l}$ with $\alpha \leq 1$ are linearly independent. This proves that $2^r b$ does not vanish in $\mathbb{F}_2[c_1, \dots, c_l]$. □

Proposition 4.2, and therefore Theorem 4.1, are proved.

For completeness, we provide the following classical computation:

Lemma 4.5. *The ring* $\mathbb{Z}[y_1, \dots, y_l]^W$ *is generated by* $e, p_1, \dots, p_l$.

Proof. Let $A \subset W$ be the subgroup of even sign changes. It suffices to show that the ring $\mathbb{Z}[y_1, \dots, y_l]^A$ is generated by $e, y_1^2, \dots, y_l^2$. We show this by induction on l starting with the case $l = 1$, where A is trivial. For $l \geq 2$, let $A' \subset A$ be the subgroup of even sign changes of $y_1, \dots, y_{l-1}$. Any A-invariant element is a polynomial in y_l with coefficients in $\mathbb{Z}[y_1, \dots, y_{l-1}]^{A'}$ and by the induction hypothesis is equal to

$$(g_0 + h_0 e') + (g_1 + h_1 e')y_l + \dots + (g_r + h_r e')y_l^r \tag{4.6}$$

for some $r \geq 0$, where $e' := y_1 \dots y_{l-1}$ and where $g_0, h_0, g_1, h_1, \dots, g_r, h_r$ are some uniquely determined polynomials in $y_1^2, \dots, y_{l-1}^2$. The sign change of y_1 and y_l transforms (4.6) to

$$(g_0 - h_0 e') + (g_1 - h_1 e')(-y_l) + \dots + (g_r - h_r e')(-y_l)^r.$$

It follows that $h_0 = h_2 = \dots = 0 = g_1 = g_3 = \dots$. Therefore (4.6) is an integral polynomial in $e, y_1^2, \dots, y_l^2$. □

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