A-UPPER MOTIVES OF REDUCTIVE GROUPS

CHARLES DE CLERCQ, NIKITA KARPENKO, AND ANNE QUÉGUINER-MATHIEU

Abstract. Given a prime number p, we introduce the notion of a *p'-inner* reductive algebraic group G over a field F, opposite to an older notion of *p-inner* group. (Any absolutely simple group of type not D_4 is p- or p'-inner; a reductive group is simultaneously p- and p'-inner if and only if it is of inner type.) For such G, the degree of the minimal field extension E/F, over which G becomes of inner type, is prime to p. (In the p-inner case, the degree is a p-power.) In the category of Chow motives with coefficients $\mathbb{Z}/p\mathbb{Z}$ over the field F, we define the *A-upper motives* of G; they are naturally related to those *Artin motives* which are indecomposable direct summands in the motives of spectra of intermediate fields in E/F. We show that the motives of any projective G-homogeneous variety is isomorphic to a direct sum of Tate shifts of A-upper motives. Based on that, we get a motivic classification of such varieties by means of their *higher Artin-Tate traces*. We also show that the *higher Tits p-indexes* determine motivic equivalence classes of the p'-inner groups.

Contents

1. Introduction

Envisioned by Alexander Grothendieck in the sixties, Chow motives provide powerful invariants to study arithmetic and geometry of smooth projective varieties over fields. The case of projective homogeneous varieties has received a lot of attention over the years and numerous breakthroughs and solutions to classical conjectures were obtained through the study of their motives. Most of these results are proved in the framework of semisimple algebraic groups of inner type, i.e., such that the $*$-action of the absolute Galois group of

Date: 1 Nov 2024 (with minor revision of 20 Nov 2024).

Key words and phrases. Affine algebraic groups; projective homogeneous varieties; Chow rings and motives. *Mathematical Subject Classification (2020):* 20G15; 14C25.

the base field on the associated Dynkin diagram is trivial. In this work, we initiate the study of motives and motivic decompositions for projective homogeneous varieties under arbitrary reductive groups.

An extensive study of motives of projective quadrics, which were essential to Voevodsky's proof of the Milnor conjecture [21], was carried out by Alexander Vishik in [19]. This milestone led notably to advances on the Kaplansky problem [20] and a proof of Hoffmann's conjecture [11]. Vishik provides on the way a qualitative description of motivic structure of projective quadrics through the motives of Čech simplicial schemes associated to orthogonal Grassmannians. Working now with coefficients in $\mathbb{Z}/p\mathbb{Z}$ and motivated by the case of generalized Severi-Brauer varieties, the second author then obtains a description of indecomposable summands in the motives of projective G-homogeneous varieties for G a reductive group of inner type: the indecomposable summands are Tate shifts of *upper motives* of G [14]. (More generally, the description holds for *p-inner* reductive groups – the reductive groups which acquire inner type over a finite base field extension of a p-power degree, see [12].) This result led to many applications, notably on the isotropy of orthogonal involutions [13] and the classification of motivic decompositions for exceptional groups [8] as well as of motives of projective homogeneous varieties under p-inner groups [6].

We continue to work with coefficients in $\mathbb{Z}/p\mathbb{Z}$, where p is a fixed prime. Let G be an arbitrary reductive algebraic group over a field F and let E/F be a minimal field extension over which G acquires inner type. The extension E/F is finite Galois, and its isomorphism class is determined by G. We say that G is *p'-inner* if the degree of E/F is not divisible by p and the following additional condition holds: for any intermediate field L in E/F and any projective G_L-homogeneous variety Y (over L), there exists a projective G-homogeneous variety $\hat{Y}$ (over F) such that the L-varieties Y and $\hat{Y}_L$ are equivalent in the sense of [6, §2] meaning that there exist multiplicity $1 \in \mathbb{F}$ correspondences $\hat{Y}_L \rightsquigarrow Y$ and $Y \rightsquigarrow \hat{Y}_L$.

For Y as above, we define the *A-upper motives of* Y (see Definition 5.3). Each of them is an indecomposable F-motive naturally related with an Artin F-motive – a summand in the F-motive of the spectrum of L. This leads to the notion of the *A-upper motives of* G (see Definition 5.4). If G is of inner type, the A-upper motives of G are the upper motives of G considered previously.

Assume that G is p'-inner, and pick a projective G-homogeneous F-variety X. Theorem 7.1 provides a qualitative analysis of the motivic structure of X, stating that the motive of X decomposes (in a unique way) as a direct sum of Tate shifts of A-upper motives of G. As a consequence of this structural result we obtain a complete classification of motives of such projective homogeneous varieties through their *higher Artin-Tate traces* (Theorem 8.6). In fact, a complete classification is obtained for arbitrary sums of Tate shifts of A-upper motives (of possibly distinct p'-inner groups). Note that by Remark 3.6, Tate traces of [6] are not sufficient for this purpose.

Finally, in §9, we provide criteria of motivic equivalence for p'-inner groups by means of *higher Tits p-indexes* of [5]. These results expound how higher isotropy of such reductive groups determines motives of projective homogeneous varieties and extend results of [6].

2. Notation

Let F be a field and $\bar{F}$ a separable closure of F. We denote by Γ_F the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$ of F. Throughout the paper, p is a prime number, $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$, and $\mathrm{Ch}(\cdot)$ denotes the Chow group with coefficients in $\mathbb{F}$. We also let $\mathrm{CM}(F, \mathbb{F})$ be the category of Chow motives over F with coefficients in $\mathbb{F}$ (see [7, §64]), while $\mathrm{CM}_{\mathrm{eff}}(F, \mathbb{F})$ stands for its full subcategory of effective motives. A *variety* is a separated scheme of finite type over a field. For any smooth projective F-variety X, we denote by $M(X)$ its motive in both categories. We use the notation $\mathrm{AM}(F, \mathbb{F})$ for the full subcategory of $\mathrm{CM}_{\mathrm{eff}}(F, \mathbb{F})$ which consists of direct summands of motives of 0-dimensional varieties, that is, the category of Artin (Chow) motives, see §3.

A *complete decomposition* of a motive M is a finite direct sum decomposition with indecomposable summands. We say that the Krull-Schmidt property holds for a motive M if any direct sum decomposition of M can be refined into a complete one, and M admits a unique complete decomposition up to permutation and isomorphism of the summands. Since we work with finite coefficients, this property holds for direct summand of motives of geometrically split varieties satisfying the nilpotence principle, see [14, §2.I]. This covers, in particular, the case of projective homogeneous varieties under the action of a reductive algebraic group. More generally, by [12, Theorem 2.1], this includes the variety X^F, defined in the next paragraph, when X is a projective homogeneous variety over a finite separable extension field L/F. In particular, taking $X = \operatorname{Spec} L$, one sees that the Artin motives have the Krull-Schmidt property.

Given an F-variety X and a field extension L/F, X_L is the L-variety given by the product of the F-schemes X and $\operatorname{Spec} L$; we also let $\bar{X} = X_{\bar{F}}$. The functor $X \mapsto X_L$ for smooth projective X extends to motives; given an F-motive M, we write M_L for the corresponding L-motive.

If L/F is finite and Y is an L-variety, we let Y^F be the F-variety given by the scheme Y endowed with the composition $Y \to \operatorname{Spec} L \to \operatorname{Spec} F$. In practice, we will only consider smooth projective varieties Y and finite separable field extensions L/F, in which case the F-variety Y^F is also smooth and projective. By [12, §3], the functor $Y \mapsto Y^F$ extends to motives, the resulting functor $\mathrm{CM}(L, \mathbb{F}) \to \mathrm{CM}(F, \mathbb{F})$ is called the *corestriction* functor; given an L-motive M, we write M^F for the corresponding F-motive.

By default, the spectrum of a field is the variety over this very field; for a finite field extension L/F, we use the notation $(\operatorname{Spec} L)^F$ for the F-variety given by the spectrum of L. We write $M(L)^F$ for the motive of $(\operatorname{Spec} L)^F$, and $\mathbb{F} = M(F) = M(\operatorname{Spec} F)$ for the Tate motive.

Let E/F be a Galois field extension with a Galois group Γ. Given an E-variety Y and an automorphism $\gamma \in \Gamma$, we write Y_γ for the E-variety obtained from Y by the base change via γ. Thus Y_γ is the scheme Y viewed as an E-variety via the composition

$$Y \to \operatorname{Spec} E \xrightarrow{\gamma^{-1}} \operatorname{Spec} E,$$

for which we also write $Y^{\gamma^{-1}} = Y_\gamma$. This base change is invertible: $(Y_\gamma)_{\gamma^{-1}} = Y$. Hence the variety Y_γ has a rational point if and only if Y has one. We use the similar notation for motives, and for the same reason, a motive M_γ is indecomposable if and only if M is.

3. Artin motives

A-upper motives, defined in §5, are an essential tool in this paper. The letter "A" in their name indicates their relationship with the *Artin motives*. In this section, we recall known facts about Artin motives.

By definition (cf. [22, Definitions 1.2, 1.3]), an Artin motive over F is a direct summand in the Chow motive of the spectrum of an *étale F-algebra* (that is, up to an isomorphism, a finite direct product of finite separable field extensions of F). An Artin-Tate motive is a Tate shift of an Artin motive.[1] In particular, the Tate motive $\mathbb{F}$, as well as $M(L)^F$ for all finite separable field extensions L/F, are Artin motives. Artin motives form an additive subcategory of $\mathrm{CM}_{\mathrm{eff}}(F,\mathbb{F})$, denoted by $\mathrm{AM}(F,\mathbb{F})$. Here is a simple example of an indecomposable Artin motive that is not isomorphic to $\mathbb{F}$:

Example 3.1 (see Example 3.4 for more details)**.** Consider an odd prime number p, a field F, and a separable quadratic field extension L/F. In $\mathrm{CM}_{\mathrm{eff}}(F,\mathbb{F})$ (as well as in $\mathrm{CM}(F,\mathbb{F})$), the complete decomposition of the motive $M(L)^F$ consists of two summands: $\mathbb{F}$ and A, where the motive A (whose isomorphism class is uniquely determined by the Krull-Schmidt property) satisfies

$$\mathrm{Hom}(\mathbb{F},A) = 0 = \mathrm{Hom}(A,\mathbb{F}).$$

In particular, A is not isomorphic to $\mathbb{F}$.

Artin motives may be described in terms of Galois permutation modules, as we now proceed to recall. Note that for any F-variety X, the Chow group $\mathrm{Ch}^0(\bar{X})$ is a Γ_F-module. To better understand the structure of this module, one can use the anti-equivalence of categories between étale F-algebras and finite sets with a (continuous) left Γ_F-action, see [15, (18.4)]. The Γ_F-set corresponding to an étale F-algebra L is the set of F-algebra homomorphisms from L to $\bar{F}$. Its cardinality is equal to the dimension of L over F. The direct product (respectively, tensor product) of étale F-algebras corresponds to the disjoint union (respectively, direct product) of Γ_F-sets. Note that L is a field if and only if the corresponding Γ_F-set is transitive.

Let $L \subset \bar{F}$ be a finite separable field extension of F embedded into $\bar{F}$, and let $\Gamma_L = \mathrm{Gal}(\bar{F}/L)$. The set of F-algebra homomorphisms from L to $\bar{F}$ is identified with the set of left cosets Γ_F/Γ_L, on which Γ_F acts by left multiplication. For the F-variety $X := (\operatorname{Spec} L)^F$, consider the $\bar{F}$-variety $\bar{X} = \operatorname{Spec}(L \otimes_F \bar{F})$. Using the identification of $\bar{F}$-algebras

$$L \otimes_F \bar{F} = \prod_{\Gamma_F/\Gamma_L} \bar{F}, \quad x \otimes \lambda \mapsto (\gamma(x)\lambda)_{\gamma\Gamma_L \in \Gamma_F/\Gamma_L}, \tag{3.2}$$

we see that $\bar{X}$ is a disjoint union of base points identified with Γ_F/Γ_L. To the Artin motive $M(L)^F$, we associate the Chow group $\mathrm{Ch}^0(\bar{X})$, which is a transitive permutation $\mathbb{F}[\Gamma_F]$-module isomorphic to the $\mathbb{F}[\Gamma_F]$-module $\mathbb{F}[\Gamma_F/\Gamma_L]$ given by the Γ_F-set Γ_F/Γ_L. (The choice of the embedding $L \hookrightarrow \bar{F}$ influences the isomorphism.) By *permutation module* over a group ring $\mathbb{F}[\Gamma]$ we mean a module possessing a finite base over $\mathbb{F}$ permuted by Γ; in particular, all our permutation modules are finite dimensional vector spaces over $\mathbb{F}$.

[1] Thanks to Stefan Gille and Alexander Vishik for suggestion to consider the Artin and Artin-Tate motives in this context.

By the same arguments as in [3, §7], we obtain an anti-equivalence of additive categories between the category of Artin motives $\mathrm{AM}(F,\mathbb{F})$ and the category of direct summands in permutation $\mathbb{F}[\Gamma_F]$-modules. This anti-equivalence is compatible with the tensor products in these two categories. Indeed, the tensor product of two étale F-algebras corresponds to the direct product of the associated Γ_F-sets, which in turn gives rise to the tensor product of the corresponding permutation modules.

Remark 3.3. Restricting to the subcategory $\mathrm{AM}(F,\mathbb{F}) \subset \mathrm{CM}(F,\mathbb{F})$ the duality functor $\mathrm{CM}(F,\mathbb{F}) \to \mathrm{CM}(F,\mathbb{F})^{\mathrm{op}}$ of [7, §65], we get a functor $\mathrm{AM}(F,\mathbb{F}) \to \mathrm{AM}(F,\mathbb{F})^{\mathrm{op}}$ (which is identity on the motives of varieties). Composing it with the above anti-equivalence, one gets the equivalence of additive categories between the category of Artin motives $\mathrm{AM}(F,\mathbb{F})$ and the category of direct summands of permutation $\mathbb{F}[\Gamma_F]$-modules, obtained in [3, §7] directly using the Chow functor Ch_0 in place of Ch^0.

By construction, the motive $M(L)^F$ corresponds to the permutation module $\mathbb{F}[\Gamma_F/\Gamma_L]$; in particular, the Tate motive $\mathbb{F} = M(F)$ corresponds to $\mathbb{F}[\Gamma_F/\Gamma_F]$, that is, the 1-dimensional module $\mathbb{F}$ with the trivial Γ_F-action.

Let $E \subset \bar{F}$ be a finite Galois extension field of F containing L, and let Γ be its Galois group $\mathrm{Gal}(E/F)$. The action of Γ_F on $\mathbb{F}[\Gamma_F/\Gamma_L]$ factors through Γ. We may consider $\mathbb{F}[\Gamma_F/\Gamma_L]$ as an $\mathbb{F}[\Gamma]$-module rather than an $\mathbb{F}[\Gamma_F]$-module (without affecting, say, its endomorphism ring). This applies notably when L/F itself is Galois and $E = L$.

Example 3.4. In the settings of Example 3.1, we have $\Gamma_F/\Gamma_L = \Gamma = \{1,\sigma\} \simeq \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{F}[\Gamma]$-module $\mathbb{F}[\Gamma]$, associated to $M(L)^F$, decomposes as

$$\mathbb{F}[\Gamma] = \mathbb{F}\cdot(1+\sigma) \oplus \mathbb{F}\cdot(1-\sigma).$$

The action of Γ is trivial on the first summand, and non trivial on the second one. So $\mathbb{F}\cdot(1+\sigma)$ corresponds to the Tate summand $\mathbb{F}$ in $M(L)^F$ whereas $\mathbb{F}\cdot(1-\sigma)$ corresponds to A.

Example 3.5. The previous example can be extended as follows. Let p be an arbitrary prime number. Consider a finite Galois field extension L/F of some degree n prime to p. The $\mathbb{F}[\Gamma]$-module $\mathbb{F}[\Gamma]$ contains a submodule of dimension 1 over $\mathbb{F}$ with trivial Γ-action, namely, $\mathbb{F}\cdot(\sum_{\gamma\in\Gamma}\gamma)$. Since n is invertible in $\mathbb{F}$, this submodule splits off as a direct summand, where the complementary summand is given by the submodule B of linear combinations $\sum_{\gamma\in\Gamma}\lambda_\gamma\cdot\gamma$ satisfying $\sum_{\gamma\in\Gamma}\lambda_\gamma = 0$. As a result, $M(L)^F$ contains an indecomposable direct summand isomorphic to the Tate motive $M(F) = \mathbb{F}$. We get a direct sum decomposition $M(L)^F = \mathbb{F}\oplus A$, where the Artin motive A corresponds to the $\mathbb{F}[\Gamma]$-module B.

(i) Assume $p = 2$ and $n = 3$, so that L/F is a cubic field extension. The $\mathbb{F}[\Gamma]$-module B has no proper stable submodule in this case, so that $M(L)^F = \mathbb{F}\oplus A$ with A indecomposable. Over L, the motive A is isomorphic to $\mathbb{F}\oplus\mathbb{F}$.

(ii) Assume now $p = 7$ and $n = 3$. Pick a generator σ of Γ. The module B admits a basis given by the elements $v_1 := 1+2\sigma-3\sigma^2$ and $v_2 := 1-3\sigma+2\sigma^2$, which satisfy $\sigma v_1 = -3v_1$ and $\sigma v_2 = 2v_2$. Therefore, $B = B_1\oplus B_2$ with $B_i := F\cdot v_i$, and $A = A_1\oplus A_2$ with A_i corresponding to B_i. The motives A_1 and A_2 are indecomposable Artin motives, non-isomorphic to $\mathbb{F}$ over F and becoming isomorphic to $\mathbb{F}$ over L. Moreover, since σ

acts on $\mathbb{F}v_1$ and $\mathbb{F}v_2$ by multiplication by two different scalars, the modules B_1 and B_2 are not isomorphic, so that the motives A_1 and A_2 are not isomorphic. The action of Γ on the tensor products $B_1^{\otimes 3}$, $B_2^{\otimes 3}$, and $B_1 \otimes B_2$ is trivial. Therefore each of the motives $A_1^{\otimes 3}$, $A_2^{\otimes 3}$, and $A_1 \otimes A_2$ is isomorphic to $\mathbb{F}$, i.e., the motives A_1 and A_2 are invertible, and their classes in the Picard group of isomorphism classes of invertible motives in $\mathrm{CM}(F, \mathbb{F})$ (with multiplication induced by tensor product, [9, Definition A.2.7]) are inverse to each other elements of order 3.

Remark 3.6. The motives A_1 and A_2 defined in Example 3.5(ii) have the same *Tate trace* (defined in [6]) over any extension field of F, even though they are not isomorphic over F. Indeed, none of the indecomposable motives A_1 and A_2 is isomorphic to $\mathbb{F}$, hence each of them has the trivial Tate trace. This remains true over any extension field K of F such that the tensor product $L \otimes_F K$ is a field. On the contrary, if $L \otimes_F K$ is not a field, it is the split étale K-algebra $K \times K \times K$. Hence the motive $M(L)^F$ becomes isomorphic to a direct sum of three copies of $\mathbb{F}$ over such K, so that both A_1 and A_2 become isomorphic to $\mathbb{F}$ meaning that $\mathbb{F}$ is the Tate trace of A_1 as well as of A_2 over K. This example demonstrates limitations for possible generalizations of [6, Theorem 4.3], and is a strong motivation to introduce the *Artin-Tate traces* below.

4. A retraction

We now construct a functor $\mathbf{m}$ from the category $\mathrm{CM}_{\mathrm{eff}}(F, \mathbb{F})$ of effective Chow motives to its subcategory $\mathrm{AM}(F, \mathbb{F})$ of Artin motives, which is a crucial ingredient in the definition of A-upper motives.

The category $\mathrm{CM}_{\mathrm{eff}}(F, \mathbb{F})$ is the idempotent completion of the category $\mathrm{CC}(F, \mathbb{F})$ of degree 0 Chow correspondences. We first define a functor on $\mathrm{CC}(F, \mathbb{F})$.

By definition, the objects of $\mathrm{CC}(F, \mathbb{F})$ are given by smooth projective varieties over F; we write $M(X)$ for the object given by such a variety X. The morphisms from $M(X)$ to $M(X')$ are the degree 0 correspondences $X \rightsquigarrow X'$ from X to X' with coefficients in $\mathbb{F}$, where the degree of a correspondence is defined as in [7, §63]. (Note a difference with the definition of degree used in [16].)

Any smooth connected F-variety X determines a finite separable field extension L/F and a smooth geometrically connected L-variety Y with $Y^F = X$. The underlying scheme of the variety Y is just the scheme of X. The field L coincides with the algebraic closure of F inside the function field $F(X)$ of X and is called the *field of constants* of X. Let us choose an embedding $L \hookrightarrow \bar{F}$. Since

$$\bar{X} = Y \times_{\operatorname{Spec} L} \operatorname{Spec}(L \otimes_F \bar{F}),$$

the isomorphism (3.2) provides an identification

$$\bar{X} = \coprod_{\gamma\Gamma_L \in \Gamma_F/\Gamma_L} \bar{Y}_\gamma, \tag{4.1}$$

where $\bar{Y}_\gamma$ is $\bar{Y}$ modified by $\gamma \in \Gamma_F$. (See notation introduced in §2.) Note that since Y is defined over L, we have $\bar{Y}_\sigma = \bar{Y}$ for all $\sigma \in \Gamma_L$ so that $\bar{Y}_\gamma$ only depends on the coset $\gamma\Gamma_L$. Therefore, the Γ_F-set of connected components of $\bar{X}$ is identified with the Γ_F-set

Γ_F/Γ_L (the identification depends on the choice of the embedding $L \hookrightarrow \bar{F}$). It follows that $\mathrm{Ch}^0(\bar{X})$ is a transitive permutation $\mathbb{F}[\Gamma_F]$-module isomorphic to $\mathbb{F}[\Gamma_F/\Gamma_L]$.

Dropping the assumption that X is connected, we see that $\mathrm{Ch}^0(\bar{X})$ is the permutation $\mathbb{F}[\Gamma_F]$-module $\mathrm{Ch}^0(\bar{X}_1) \oplus \cdots \oplus \mathrm{Ch}^0(\bar{X}_n)$, where $X_1, \dots, X_n$ are the connected components of X.

Lemma 4.2. *The additive contravariant functor from the category of correspondences* $\mathrm{CC}(F, \mathbb{F})$ *to the category of abelian groups, which maps to* $\mathrm{Ch}^0(\bar{X})$ *the motive* $M(X)$ *of a smooth projective* F*-variety* X*, yields an additive contravariant functor from* $\mathrm{CC}(F, \mathbb{F})$ *to the category of permutation* $\mathbb{F}[\Gamma_F]$*-modules.*

Proof. We already noticed that $\mathrm{Ch}^0(\bar{X})$ is a permutation $\mathbb{F}[\Gamma_F]$-module. We need to check that the respective homomorphisms of abelian groups respect this structure. Given a degree 0 correspondence $\alpha : X \rightsquigarrow Y$ for smooth projective F-varieties X and Y, the induced homomorphism of abelian groups $\mathrm{Ch}^0(\bar{Y}) \mapsto \mathrm{Ch}^0(\bar{X})$ coincides with the composition

$$\mathrm{Ch}^0(\bar{Y}) \xrightarrow{\bar{p}_2^\star} \mathrm{Ch}^0(\bar{X} \times \bar{Y}) \xrightarrow{\cdot\bar{\alpha}} \mathrm{Ch}^d(\bar{X} \times \bar{Y}) \xrightarrow{\bar{p}_{1\star}} \mathrm{Ch}^0(\bar{X}),$$

where d is the dimension of Y and p_1 and p_2 are the projections from $X \times Y$ to X and Y. Since p_1, p_2, and α are defined over F, this composition commutes with the action of Γ_F, concluding the proof. □

Taking the idempotent completion of both categories, and combining with the anti-equivalence of categories between direct summands of permutation modules and Artin motives, described in §3, we get an additive functor

$$\mathbf{m}\colon \mathrm{CM}_{\mathrm{eff}}(F, \mathbb{F}) \to \mathrm{AM}(F, \mathbb{F}). \tag{4.3}$$

We now prove some useful properties of this functor.

Lemma 4.4. *The functor* $\mathbf{m}$ *maps the motive* $M(X)$ *of a smooth projective connected* F*-variety* X *to the Artin motive* $M(L)^F$*, where* L *is the field of constants of* X*.*

If the field extension L/F *is Galois with Galois group* Γ*, and* Y *is the* L*-variety with* $Y^F = X$*, the additive group of the ring* $\mathrm{End}_{\mathrm{CM}(F,\mathbb{F})}(M(X))$ *is identified with the direct sum* $\bigoplus_{\sigma\in\Gamma} \mathrm{Ch}_{\dim Y}(Y_\sigma \times Y)$*, and* $\mathbf{m}$ *sends an element* $\alpha \in \mathrm{Ch}_{\dim Y}(Y_\sigma \times Y)$ *to*

$$\mathrm{mult}(\alpha) \cdot \sigma \in \mathbb{F}[\Gamma] = \mathrm{End}_{\mathrm{AM}(F,\mathbb{F})}\big(M(L)^F\big),$$

where $\mathrm{mult}(\alpha)$ *is the* multiplicity *(see* [7, §75]*) of the degree* 0 *correspondence* $\alpha\colon Y_\sigma \rightsquigarrow Y$*.*

Proof. Since L/F is finite separable, we may assume $L \subset \bar{F}$; let $\Gamma_L = \mathrm{Gal}(\bar{F}/L)$. The first assertion of Lemma 4.4 is a direct consequence of the definition of $\mathbf{m}$, since, as noticed at the beginning of this section, $\mathrm{Ch}^0(\bar{X})$ is isomorphic to the $\mathbb{F}[\Gamma_F]$-module $\mathbb{F}[\Gamma_F/\Gamma_L]$, which corresponds to the Artin motive $M(L)^F$.

Assume now L/F is Galois. Using the identification

$$L \otimes_F L = \prod_\Gamma L, \quad x \otimes y \mapsto (\sigma(x)y)_{\sigma\in\Gamma}, \tag{4.5}$$

we get that $X \times X = Y \times \mathrm{Spec}(L \otimes_F L) \times Y = \coprod_{\sigma\in\Gamma} Y_\sigma \times Y$. Therefore, we have

$$\mathrm{End}\, M(X) = \mathrm{Ch}_d(X \times X) = \bigoplus_{\sigma\in\Gamma} \mathrm{Ch}_d(Y_\sigma \times Y),$$

where d is the dimension of X.

By (4.1), we have $\bar{X} = \coprod_{\sigma\in\Gamma} \bar{Y}_\sigma$, hence $\mathrm{Ch}_d(\bar{X}\times\bar{X}) = \bigoplus_{(\sigma,\sigma')\in\Gamma^2} \mathrm{Ch}_d\big(\bar{Y}_\sigma \times \bar{Y}_{\sigma'}\big)$. To determine the image of α in $\mathrm{Ch}_d(\bar{X}\times\bar{X})$, we use the following identifications:

$$\bar{X}\times\bar{X} = (X\times X)\times_{\mathrm{Spec}\, F} \mathrm{Spec}\,\bar{F} = \coprod_{\sigma\in\Gamma} (Y_\sigma \times Y)\times \mathrm{Spec}(L\otimes \bar{F}).$$

Using again the identification (3.2), we get $\bar{X}\times\bar{X} = \coprod_{\sigma\in\Gamma}\big(\coprod_{\tau\in\gamma} \bar{Y}_{\sigma\tau}\times\bar{Y}_\tau\big)$. Hence, an element $\alpha\in\mathrm{Ch}_d(Y_\sigma\times Y)\subset\mathrm{Ch}_d(X\times X)$ satisfies $\bar{\alpha}\in\bigoplus_{\tau\in\Gamma}\mathrm{Ch}_d(\bar{Y}_{\sigma\tau}\times\bar{Y}_\tau)\subset\mathrm{Ch}_d(\bar{X}\times\bar{X})$. The endomorphism of $M(L)^F$, induced by α, corresponds to the endomorphism of $\mathbb{F}[\Gamma]$-modules defined by

$$\mathrm{Ch}^0(\bar{X}) \xrightarrow{\bar{p}_2^\star} \mathrm{Ch}^0(\bar{X}\times\bar{X}) \xrightarrow{\cdot\bar{\alpha}} \mathrm{Ch}_d(\bar{X}\times\bar{X}) \xrightarrow{\bar{p}_{1\star}} \mathrm{Ch}^0(\bar{X}).$$

The intersection product of the image under p_2^* of $[\bar{Y}]\in\mathrm{Ch}_0(\bar{X})$ with $\bar{\alpha}$ is the projection of $\bar{\alpha}$ to the summand $\mathrm{Ch}_d(\bar{Y}_\sigma\times\bar{Y})$. This element maps under p_{1*} to $\mathrm{mult}(\alpha)[\bar{Y}_\sigma]$. Identifying $\mathrm{Ch}^0(\bar{X})$ with $\mathbb{F}[\Gamma]$, we get that the endomorphism of $\mathrm{Ch}^0(\bar{X})$, induced by α, maps 1 to $\mathrm{mult}(\alpha)\cdot\sigma$; hence, it is the left multiplication by $\mathrm{mult}(\alpha)\cdot\sigma$, as claimed. □

Remark 4.6. If X is geometrically connected, its field of constants is F, and we get that $\mathbf{m}(M(X)) = \mathbb{F}$. The homomorphism $\mathrm{End}\, M(X)\to\mathrm{End}\,\mathbb{F}$ of the endomorphism rings is the multiplicity homomorphism $\mathrm{Ch}_{\dim X}(X\times_F X)\to\mathbb{F}$.

Recall that for a finite separable field extension K/F and a K-motive M, we denote by M^F its corestriction to F, defined as in [12, §3].

Lemma 4.7. *The functor* $\mathbf{m}$ *commutes with the corestriction functor. In particular, for every finite separable field extension* K/F*, and every motive* $M\in\mathrm{CM}(K,\mathbb{F})$*, we have* $\mathbf{m}(M^F) = \mathbf{m}(M)^F$*.*

Proof. Let Y be a connected smooth projective K-variety, and let L be its field of constants. There exists an L-variety Z such that $Y = Z^K$. It follows that $Y^F = Z^F$. Therefore,

$$\mathbf{m}\big(M(Y)^F\big) = M(L)^F = \big(M(L)^K\big)^F = \mathbf{m}\big(M(Y)\big)^F. \qquad \square$$

Remark 4.8. The functor $\mathbf{m}$ also commutes with the restriction functors (given by arbitrary base field extensions) and respects tensor products.

Remark 4.9. The restriction of $\mathbf{m}$ to the subcategory of Artin motives is the identity, so $\mathbf{m}$ is a "retraction" of the entire category of the effective Chow motives to its subcategory of Artin motives.

5. A-upper motives

Let G be a reductive algebraic group over F. Given a finite separable field extension L/F, we consider a projective G_L-homogeneous L-variety Y. The upper motive of the F-variety Y^F, denoted by $U(Y^F)$, is the indecomposable summand in the F-motive of Y^F satisfying $\mathrm{Ch}^0(U(Y^F))\neq 0$ or, equivalently, $\mathrm{Ch}^0(U(Y^F)) = \mathrm{Ch}^0(Y)$. It is uniquely defined up to isomorphism. In particular, the L-motive $U(Y)$ is defined. As explained in [12, §3], the corestriction $U(Y)^F$ of the L-motive $U(Y)$ contains the motive $U(Y^F)$ as a direct summand. But in general, $U(Y)^F$ and $U(Y^F)$ are not isomorphic, that is, $U(Y)^F$ is not always indecomposable.

Proposition 5.1. *Let G be a reductive group over F, and let Y be a projective G_L-homogeneous L-variety for some finite separable field extension L/F. Then*

$$\mathbf{m}\big(U(Y)^F\big) = M(L)^F.$$

Proof. Let $p \in \operatorname{End}\big(M(Y)\big)$ be an idempotent defining $U(Y)$. We have $U(Y)^F = \big(M(Y)^F, p^F\big)$, hence $\mathbf{m}(U(Y)^F)$ is the summand of $M(L)^F$ determined by the image of p^F in $\operatorname{End}(M(L)^F)$. By [14, Lemma 2.8], p has multiplicity 1; therefore p maps to $1 \in \operatorname{End}(M(L))$, see Remark 4.6. Since $\mathbf{m}$ commutes with the corestriction, we get that p^F also maps to $1 \in \operatorname{End}(M(L)^F)$, so that $\mathbf{m}(U(Y)^F) = M(L)^F$. □

We get the following commutative diagram, where i is the natural inclusion, j is the surjective map defined by $j(f) = p^F f p^F$ for all $f \in \operatorname{End}\big(M(Y)^F\big)$, and the commutativity follows from the fact that p^F maps to 1.

$$\tag{5.2} \begin{array}{ccc} \operatorname{End}\big(U(Y)^F\big) & \searrow & \\ {\scriptstyle j}\uparrow\;\downarrow{\scriptstyle i} & & \operatorname{End}\big(M(L)^F\big) \\ \operatorname{End}\big(M(Y)^F\big) & \nearrow & \end{array}$$

The arrow i in the diagram is an additive homomorphism; the remaining arrows are ring homomorphisms.

A motive M satisfying $\mathbf{m}(M) \neq 0$ is called *sustainable*.

Definition 5.3. Let G be a reductive group over F, and Y be a projective G_L-homogeneous variety for some finite separable field extension L/F. The *A-upper* F-motives of Y are the sustainable F-motives isomorphic to indecomposable summands of $U(Y)^F$. (The letter “A” in their name honors Emil Artin and Artin motives.)

Definition 5.4. Let G be a reductive group over F and E/F be a minimal field extension such that G_E is of inner type. An *A-upper motive of* G is an F-motive isomorphic to an A-upper motive of a projective G_L-homogeneous variety, defined over an intermediate field L of E/F.

Remark 5.5. Note that for a given G, the field extension E/F in Definition 5.4 is uniquely determined up to an isomorphism so that its choice does not influence the notion of the A-upper motives of G.

6. Groups of p'-inner type

Theorem 6.1. *Let G be a reductive group over a field F and Y a projective G_L-homogeneous variety for some finite separable field extension L/F. Assume that the degree of the normal closure of L/F is prime to p and there exists a projective G-homogeneous variety $\hat{Y}$ such that $\hat{Y}_L$ is* equivalent *to Y in the sense of* [6, §2] *meaning that there exist multiplicity $1 \in \mathbb{F}$ correspondences $\hat{Y}_L \rightsquigarrow Y$ and $Y \rightsquigarrow \hat{Y}_L$. Then the following holds:*

(1) *every summand in* $M(L)^F$ *is isomorphic to the image under* $\mathbf{m}$ *of a summand in* $U(Y)^F$*;*
(2) *two summands in* $U(Y)^F$ *with isomorphic images under* $\mathbf{m}$ *are isomorphic;*
(3) *a summand in* $U(Y)^F$ *is indecomposable if and only if its image under* $\mathbf{m}$ *is so.*

Theorem 6.1 implies that under its hypotheses, the set of isomorphism classes of A-upper motives of Y is in bijection with the set of isomorphism classes of indecomposable Artin motives which are direct summands in $M(L)^F$. In equivalent terms, given such an Artin motive A there is a unique up to isomorphism A-upper motive $U_A(Y)$ of Y whose image under the functor $\mathbf{m}$ is isomorphic to A, and any A-upper motive of Y is obtained this way.

Remark 6.2. The base field F of the motive $U_A(Y)$ does not show up in its notation because it is concealed in the motive A. With these notation in hand, Theorem 6.1 implies that if $A_1 \oplus \cdots \oplus A_r$ is a complete decomposition for $M(L)^F$, then $U_{A_1}(Y) \oplus \cdots \oplus U_{A_r}(Y)$ is a complete decomposition for $U(Y)^F$.

The rest of the section is devoted to the proof of Theorem 6.1.

Proof of Theorem 6.1 in the Galois case. Let us assume that L/F is Galois and write Γ for its Galois group Γ_F/Γ_L.

Lemma 6.3. *The ring homomorphism*

$$\mathbf{m} : \operatorname{End}\big(U(Y)^F\big) \to \operatorname{End}\big(M(L)^F\big)$$

given by the functor $\mathbf{m}$ *is surjective; its kernel consists of nilpotents.*

Proof. Since Y is equivalent to $\hat{Y}_L$, for every $\sigma \in \Gamma$, there exists a multiplicity 1 correspondence $Y_\sigma \rightsquigarrow Y$. Viewed as an element of $\operatorname{End}\big(M(Y)^F\big)$, it maps under $\mathbf{m}$ to $\sigma \in \mathbb{F}[\Gamma]$ – by the description of $\mathbf{m}$ given in Lemma 4.4. This proves that $\operatorname{End}\big(M(Y)^F\big)$ maps surjectively onto $\operatorname{End}\big(M(L)^F\big) = \mathbb{F}[\Gamma]$, and the surjectivity statement of Lemma 6.3 follows by diagram (5.2).

To prove the statement on the kernel, let us take some $f \in \operatorname{End}\big(U(Y)^F\big)$ with $\mathbf{m}(f) = 0$. Since we work with finite coefficients, by [14, Corollary 2.2], some power of f is a projector q, which also satisfies $\mathbf{m}(q) = 0$. To show that f is nilpotent, it is enough to show that $q = 0$. The projector q determines a summand M of $U(Y)^F$, and since $\mathbf{m}(q) = 0$, it satisfies $\mathbf{m}(M) = 0$. On the other hand, L/F is Galois, so by (4.5) we have

$$(Y^F)_L = \coprod_{\sigma\in\Gamma} Y_\sigma, \text{ and } (U(Y)^F)_L = \bigoplus_{\sigma\in\Gamma} U(Y)_\sigma,$$

with $U(Y)_\sigma = U(Y_\sigma)$ indecomposable for all $\sigma \in \Gamma$. By the Krull-Schmidt property, we get that $M_L = \bigoplus_{\sigma\in S} U(Y)_\sigma$ for some subset S of Γ. By Proposition 5.1, $\mathbf{m}(U(Y_\sigma))$ is the copy of $M(L)$ indexed by σ in $\mathbf{m}\big(M(Y^F)_L\big) = \bigoplus_{\sigma\in\Gamma} M(L)$. The condition $\mathbf{m}(M_L) = 0$ implies that S is empty, i.e., $M_L = 0$. Consequently, $M = 0$ and $q = 0$ by the nilpotence principle [12, Theorem 2.1]. □

The three statements of Theorem 6.1 in the Galois case are now proved as follows.

(1) By Lemma 6.3, the projector p defining a given summand in $M(L)^F$ lifts to an element of $\operatorname{End}\big(U(Y)^F\big)$. By [14, Corollary 2.2], an appropriate power of this element is a projector which maps to p under $\mathbf{m}$.

(2) Let M_1 and M_2 be summands of $U(Y)^F$. Any morphism between $\mathbf{m}(M_1)$ and $\mathbf{m}(M_2)$ is given by an endomorphism of $M(L)^F$ and therefore, by Lemma 6.3, can be lifted to a morphism between M_1 and M_2. In particular, if $\mathbf{m}(M_1)$ and $\mathbf{m}(M_2)$ are isomorphic, mutually inverse isomorphisms lift to some morphisms $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_1$. By Lemma 6.3 once again, each of the compositions $g \circ f$ and $f \circ g$ has the form $\mathrm{id} + \varepsilon$ with some nilpotent ε and so is an isomorphism (with the inverse given by the finite sum $\mathrm{id} - \varepsilon + \varepsilon^2 - \dots$).

(3) This is a consequence of (1) and (2). □

Proof of Theorem 6.1 in the general case.

Lemma 6.4. *$U(Y) \simeq U(Y^F)_L$.*

Proof. Let us recall that $Y \approx \hat{Y}_L$ for certain projective homogeneous F-variety $\hat{Y}$. Moreover, $Y^F \approx \hat{Y}$ because $[L:F]$ is prime to p. The statement of Lemma 6.4 therefore reads as $U(\hat{Y}_L) \simeq U(\hat{Y})_L$.

The indecomposable motive $U(\hat{Y}_L)$ is a direct summand in $U(\hat{Y})_L$. So, the two motives are isomorphic provided that the second one is also indecomposable.

To get the indecomposability of $U(\hat{Y})_L$ for the field extension L/F, it is enough to check the indecomposability of $U(\hat{Y})_E$ for its normal closure E/F. Therefore we may assume that the extension L/F itself is Galois.

Let us write $N \hookrightarrow M$ for motives N and M to indicate that N is a direct summand in M. Since $\mathbb{F} \hookrightarrow M(L)^F$ (see Example 3.5), using the already proven Galois case of Theorem 6.1, we obtain a motive $U_{\mathbb{F}}(Y) \hookrightarrow U(Y)^F$, which is indecomposable and whose image under $\mathbf{m}$ is isomorphic to $\mathbb{F}$. Since $U(Y)^F \hookrightarrow M(Y)^F = M(Y^F)$, we have $U_{\mathbb{F}}(Y) \hookrightarrow M(Y^F)$. Since $\mathrm{Ch}^0(U_{\mathbb{F}}(Y)) = \mathrm{Ch}^0(\mathbf{m}(U_{\mathbb{F}}(Y))) \neq 0$, there is an isomorphism $U_{\mathbb{F}}(Y) \simeq U(Y^F)$, implying that $U(Y^F) \hookrightarrow U(Y)^F$. Since $U(\hat{Y}) \simeq U(Y^F)$, it follows that

$$U(\hat{Y})_L \hookrightarrow (U(Y)^F)_L \simeq \bigoplus_{\mathrm{Gal}(L/F)} U(Y)$$

and so, $U(\hat{Y})_L$ is isomorphic to a direct sum of several copies of $U(Y) \simeq U(\hat{Y}_L)$. Since $\mathrm{Ch}^0(U(\hat{Y})_L) = \mathbb{F} = \mathrm{Ch}^0(U(\hat{Y}_L))$, or – equivalently – $\mathbf{m}(U(\hat{Y})_L) = \mathbb{F} = \mathbf{m}(U(\hat{Y}_L))$, we finally conclude that $U(\hat{Y})_L \simeq U(\hat{Y}_L)$. □

Lemma 6.5. *$U(Y)^F \simeq U(Y^F) \otimes M(L)^F$.*

Proof. The second isomorphism in the chain

$$U(Y)^F \simeq \big(U(Y^F)_L \otimes M(L)\big)^F \simeq U(Y^F) \otimes M(L)^F$$

is a particular case of the following general formula that holds for any finite separable field extension L/F, an F-motive M, and an L-motive N:

$$(M_L \otimes N)^F = M \otimes N^F. \tag{6.6}$$ □

Let A be an indecomposable direct summand in $M(L)^F$. Let us consider the tensor product of F-motives $M := U(Y^F) \otimes A$. Note that $\mathbf{m}(M) = A$.

Proposition 6.7. *The motive $M := U(Y^F) \otimes A$ is indecomposable.*

Proof. Let E/F be the normal closure of L/F. The motive M_E is the direct sum of $\mathrm{rk}(A)$ copies of the indecomposable by Lemma 6.4 motive $U(Y^F)_E$, where $\mathrm{rk}(A)$ stands for the rank of A defined as the number of (Tate) summands in the complete decomposition of A over its splitting field (e.g., the field E). To prove indecomposability of M, we take any its nonzero direct summand U and check that U_E is still the sum of $\mathrm{rk}(A)$ copies of $U(Y^F)_E$. Since $\dim \mathrm{Ch}^0(U(Y^F)_E) = 1$, it suffices to show that $\dim \mathrm{Ch}^0(U_E) \geq \mathrm{rk}(A)$.

Let X be the F-variety of Borel subgroup in G. The motive $\big(U(Y^F) \otimes A\big)_{F(X)}$ is a direct sum of the Artin motive $A_{F(X)}$ and some positive shifts of some effective motives. It follows that the (indecomposable by Corollary 7.5) motive $A_{F(X)}$ is a summand of $U_{F(X)}$. Therefore

$$\dim \mathrm{Ch}^0(U_E) \geq \dim \mathrm{Ch}^0(A_{E(X)}) = \mathrm{rk}(A).$$ □

The proof of Theorem 6.1 follows: Lemma 6.5 and Proposition 6.7 give that under our hypothesis, any complete motivic decomposition $M(L)^F \simeq A_1 \oplus ... \oplus A_k$ of the spectrum of L yields a complete decomposition

$$U(Y)^F \simeq U(Y^F) \otimes M(L)^F \simeq \bigoplus_{i=1}^{k} (U(Y^F) \otimes A_i). \tag{6.8}$$

Since $\mathbf{m}$ is an additive functor and $\mathbf{m}(U(Y^F) \otimes A_i) = A_i$, any summand of $M(L)^F$ is isomorphic to the image of some summand of $U(Y)^F$, proving (1). Assertions (2) and (3) follow directly from decomposition (6.8) as well. □

Remark 6.9. It follows by Proposition 6.7 that the motive $U_A(Y)$ defined right after Theorem 6.1 is isomorphic to the tensor product $U(Y^F) \otimes A$.

7. Motivic decompositions

The following result generalizes [14, Theorem 3.5] (dealing with G of inner type). Another generalization of [14, Theorem 3.5] (going in a different direction) is given in [12, Theorem 1.1] (dealing with p-inner G).

Theorem 7.1. *Let G be a p'-inner reductive group. Every summand in the complete decomposition of the Chow motive with coefficients in $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ of any projective G-homogeneous variety X is a Tate shift of an A-upper motive of G.*

Proof. Since the center of G acts on X trivially, we may assume that G is semisimple and adjoint.

We write D_G (or simply D) for the set of vertices of the Dynkin diagram of G. We write F for the base field of G and let E/F be a minimal field extension with inner G_E. The field extension E/F is Galois and its Galois group $\Gamma = \mathrm{Gal}(E/F)$ acts on D. For a field L with $F \subset L \subset E$, the set D_{G_L} is identified with $D = D_G$. Any $\mathrm{Gal}(E/L)$-stable subset τ in D determines a projective G_L-homogeneous variety $Y_{G_L,\tau}$ the way described in [14, §3] (which is opposite to the original convention of [18, §1.6]). For instance, $Y_{G_L,D}$ is the variety of Borel subgroups of G_L, and $Y_{G_L,\emptyset} = \operatorname{Spec} L$. Any projective G_L-homogeneous variety is isomorphic to $Y_{G_L,\tau}$ for some $\mathrm{Gal}(E/L)$-stable $\tau \subset D$.

We prove Theorem 7.1 simultaneously for all F, G, X using induction on $n := \dim X$. The base of the induction is $n = 0$ where $X = \operatorname{Spec} F$ and the statement is trivial.

From now on we are assuming that $n \geq 1$ and that Theorem 7.1 is already proven for varieties of dimension $< n$.

For any field extension L/F, we write $\tilde{L}$ for the function field $L(X)$ (note that any projective homogeneous variety and, in particular X, is geometrically integral). Let G' be the semisimple group over the field $\tilde{F} = F(X)$ given by the semisimple anisotropic kernel of the group $G_{\tilde{F}}$. We note that the group $G'_{\tilde{E}}$ is of inner type. The field extension $\tilde{E}/\tilde{F}$ is Galois with the Galois group

$$\Gamma = \mathrm{Gal}(\tilde{E}/\tilde{F}) = \mathrm{Gal}(E/F)$$

(see Lemma 7.4). In particular, any of its intermediate fields is of the form $\tilde{L}$ for some intermediate field L of the extension E/F. The set $D_{G'}$ is identified with a Γ-invariant subset in D_G; the complement $D_G \setminus D_{G'}$ contains the subset in D_G corresponding to X.

Let M be an indecomposable summand of the motive of X. We are going to show that M is isomorphic to a shift of a direct summand in $U(Y_{G_L,\tau})^F$ for some intermediate field L of E/F and some $\mathrm{Gal}(E/L)$-stable subset $\tau \subset D_G$ containing the complement of $D_{G'}$. This will prove Theorem 7.1.

According to [1, Theorem 4.2] (an enhancement of [2, Theorem 7.5]), the motive of $X_{\tilde{F}}$ decomposes into a sum of shifts of motives of projective $G'_{\tilde{L}}$-homogeneous (where L runs over intermediate fields of the extension E/F) varieties Y, satisfying $\dim Y < \dim X = n$. It follows by the induction hypothesis that each summand of the complete motivic decomposition of $X_{\tilde{F}}$ is a shift $N'\{i\}$ of a summand N' in $U(Y')^{\tilde{F}}$ for some $L/F \subset E/F$, some $\mathrm{Gal}(E/L)$-stable $\tau' \subset D_{G'}$, and $Y' := Y_{G'_{\tilde{L}},\tau'}$. By the Krull-Schmidt property [12, Corollary 2.2], the summands of the complete decomposition of $M_{\tilde{F}}$ are also of this shape.

In the complete decomposition of $M_{\tilde{F}}$, let us choose a summand $N'\{i\}$ with minimal i. We set $\tau := \tau' \cup (D_G \setminus D_{G'}) \subset D_G$ with the corresponding $\tau' \subset D_{G'}$. The subset τ is $\mathrm{Gal}(E/L)$-stable. To prove Theorem 7.1, it is enough to show that M is isomorphic to a direct summand in $U(Y)^F\{i\}$ for these L, τ, i, and $Y := Y_{G_L,\tau}$.

Since M is indecomposable, it suffices to construct morphisms

$$\alpha : U(Y)^F\{i\} \to M \ \text{ and } \ \beta : M \to U(Y)^F\{i\}$$

such that no power of the composition $\alpha \circ \beta$ vanishes. (We recall that by [14, Corollary 2.2], an appropriate power of any endomorphism of M is a projector.)

We first construct certain, defined over the field $\tilde{F}$, predecessors $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β. Recall that $N'\{i\}$ is a summand in $M_{\tilde{F}}$ and note that $U(Y')^{\tilde{F}}$ is a summand in $(U(Y)^F)_{\tilde{F}}$. Using projections to and inclusions of direct summands, we define $\tilde{\alpha}$ and $\tilde{\beta}$ as the compositions

$$\tilde{\alpha}\colon U(Y)^F\{i\}_{\tilde{F}} \twoheadrightarrow U(Y')^{\tilde{F}}\{i\} \twoheadrightarrow N'\{i\} \hookrightarrow M_{\tilde{F}} \quad \text{and}$$
$$\tilde{\beta}\colon M_{\tilde{F}} \twoheadrightarrow N'\{i\} \hookrightarrow U(Y')^{\tilde{F}}\{i\} \hookrightarrow U(Y)^F\{i\}_{\tilde{F}},$$

where $\twoheadrightarrow$ is a sign for a projection onto a direct summand and $\hookrightarrow$ means an inclusion of a direct summand. The composition $\tilde{\alpha}\circ\tilde{\beta}$ is the projector which yields the summand $N'\{i\}$ of $M_{\tilde{F}}$.

Now we construct α and β starting with α. Note that $\tilde{\alpha}$ is an element of the Chow group $\mathrm{Ch}(Y^F\times X)_{\tilde{F}}$ over $\tilde{F}$. We take for α an element of the Chow group $\mathrm{Ch}(Y^F\times X)$ over F such that its image under the surjective ring homomorphism

$$\mathrm{Ch}(Y^F\times X)\to \mathrm{Ch}(X_{F(Y^F)})$$

(from [7, Corollary 57.11]) followed by the change of field homomorphism for the field extension $\tilde{F}(Y^F)/F(Y^F)$, coincides with the image of $\tilde{\alpha}$ under the surjective ring homomorphism

$$\mathrm{Ch}(Y^F\times X)_{\tilde{F}}\to \mathrm{Ch}(X_{\tilde{F}(Y^F)}).$$

Such α exists because the field extension $\tilde{F}(Y^F)/F(Y^F)$ is purely transcendental and therefore the change of field homomorphism $\mathrm{Ch}(X_{F(Y^F)})\to \mathrm{Ch}(X_{\tilde{F}(Y^F)})$ is surjective as follows from the homotopy invariance of Chow groups (see [7, Theorem 57.13] or [7, Corollary 52.11]) and [7, Corollary 57.11].

In order to define β, we note that $\tilde{\beta}$ is an element of $\mathrm{Ch}(X\times Y^F)_{\tilde{F}}$ and let β' be an element of $\mathrm{Ch}(X\times X\times Y^F)$ mapped to $\tilde{\beta}$ under the surjection (from [7, Corollary 57.11])

$$\mathrm{Ch}(X\times X\times Y^F)\to \mathrm{Ch}(X\times Y^F)_{\tilde{F}}$$

given by the generic point of the *second* factor in the product $X\times X\times Y^F$. We consider β' as a correspondence $X\rightsquigarrow X\times Y^F$ and let β'' be the composition of correspondences $\beta'\circ\mu$, where $\mu\in\mathrm{Ch}(X\times X)$ is the projector which yields the motivic summand M of X. Finally, we define β as the pullback of β'' with respect to the closed embedding

$$X\times Y^F\hookrightarrow X\times X\times Y^F,\ \ (x,y)\mapsto (x,x,y)$$

given by the diagonal of X.

An appropriate power of the endomorphism $\alpha\circ\beta$ of the motive M is a projector which defines a summand in M isomorphic to $N\{i\}$ for certain summand N in $U(Y)^F$. By construction, $\mathbf{m}(N')$ is a summand in $\mathbf{m}(N)_{\tilde{F}}$. Since $N'\neq 0$, we have $\mathbf{m}(N')\neq 0$ by Theorem 6.1. Consequently, $\mathbf{m}(N)_{\tilde{F}}\neq 0$. It follows that N is nonzero and therefore isomorphic to M. □

Remark 7.2. Instead of [1, Theorem 4.2], the weaker result [2, Theorem 7.5] can be used in the proof of Theorem 7.1. To do so, it suffices to take for G' the semisimple part of the parabolic subgroup defining $X_{\tilde{F}}$.

Remark 7.3. As follows from the proof of Theorem 7.1, the A-upper motives of G, whose Tate shifts actually appear as direct summands of $M(X)$ in Theorem 7.1, are associated with varieties Y with Y^F *dominating* X in the sense of [4] (see also [6, Lemma 2.2]).

The following lemma and corollary have been applied in the proof of Theorem 7.1:

Lemma 7.4. *Let X be a geometrically integral variety over a field F and let E/F be a finite Galois field extension. Then $E(X)/F(X)$ is also a finite Galois field extension and its Galois group $\tilde{\Gamma}$ is isomorphic to $\Gamma := \mathrm{Gal}(E/F)$.*

Proof. The extension $E(X)/F(X)$ is algebraic, normal, and separable; therefore it is Galois. Since E is algebraically closed in $E(X)$, any element of $\tilde{\Gamma}$ maps E to E. Since the subfields E and $F(X)$ both together generate the field $E(X)$, the group homomorphism $\tilde{\Gamma} \to \Gamma$, $\sigma \mapsto \sigma|_E$ is injective. Since any element of E, which is stable under the image of $\tilde{\Gamma}$, belongs to $E \cap F(X) = F$, the image of $\tilde{\Gamma}$ is the entirety of Γ. □

Corollary 7.5. *Let X be a geometrically integral F-variety. Let L/F be a subextension of a finite Galois field extension E/F. For any direct summand $\tilde{A}$ of the motive $M(L(X))^{F(X)}$, there is one and only one direct summand A of $M(L)^F$ satisfying $A_{F(X)} = \tilde{A}$. The motive $\tilde{A}$ is indecomposable if and only if A is. Direct summands A and A' of $M(L)^F$ with isomorphic $A_{F(X)}$ and $A'_{F(X)}$ are isomorphic.* □

8. Criteria of isomorphism

The notion of a reductive group of *p'-inner type* (or simply a *p'-inner* group), introduced in §1, is opposite to the notion of *p-inner* group introduced in [12]. Any absolutely simple group of type different from ${}^3D_4$ and ${}^6D_4$ is p- or p'-inner (the case of E_6 follows from [5]). Direct product of p'-inner groups is p'-inner. Here is an additional source of p'-inner groups:

Example 8.1. Let L/F be a *p'-extension*, i.e., a finite separable field extension such that the degree of its normal closure is prime to p. Given an inner reductive group H over F, the group $G := R_{L/F}(H_L)$, where $R_{L/F}$ is the Weil transfer, is p'-inner.

Note that for any p'-inner group G over F and any projective G_L-homogeneous variety Y over an arbitrary extension field L/F, there exists a projective G-homogeneous variety $\hat{Y}$ (over F) with $\hat{Y}_L$ equivalent to Y. Therefore, A-upper motives of p'-inner reductive groups fit in with the conditions of Theorem 6.1. We keep using the notation $U_A(Y)$ for them introduced right after the theorem.

In this section, we produce criteria of isomorphism for A-upper motives of p'-inner reductive groups (see Theorem 8.3 and Corollary 8.5) and their direct sums (Theorem 8.6), the latter formulated in terms of *higher Artin-Tate traces*.

Let G be a p'-inner reductive group over a field F, L/F a p'-extension, Y a projective G_L-homogeneous variety, and A an indecomposable direct summand in $M(L)^F$. Let L'/F, G', Y', and A' be another set of such data. We are going to formulate (below in Theorem 8.3) our first criterion of isomorphism for the A-upper F-motives $U_A(Y)$ and $U_{A'}(Y')$. We start with

Proposition 8.2. *If $U_A(Y) \simeq U_{A'}(Y')$, then $A \simeq A'$.*

Proof. Applying the functor $\mathbf{m}$ to an isomorphism $U_A(Y) \to U_{A'}(Y')$, we get an isomorphism $A \to A'$. □

Recall that the variety Y^F *dominates* Y'^F if there is a multiplicity 1 correspondence $Y^F \rightsquigarrow Y'^F$. The varieties Y^F and Y'^F are *equivalent*, $Y^F \approx Y'^F$, if each of them dominates the other.

Theorem 8.3. *The motives $U_A(Y)$ and $U_{A'}(Y')$ are isomorphic if and only if $A \simeq A'$ and $Y^F \approx Y'^F$.*

Proof. By Proposition 8.2, we may assume that $A \simeq A'$.

The Artin motive $A_{F(Y^F)}$ is a direct summand in $U_A(Y)_{F(Y^F)}$. Assuming $U_A(Y) \simeq U_{A'}(Y')$, we conclude that the Artin motive $A_{F(Y^F)} \simeq A'_{F(Y^F)}$ is also a direct summand in $U_{A'}(Y')_{F(Y^F)}$. With Remark 6.9 taken into account, this implies that the Tate motive $\mathbb{F}$ is a direct summand in $U(Y'^F)_{F(Y^F)}$ and so the variety $(Y'^F)_{F(Y^F)}$ is isotropic (i.e., has a 0-cycle of degree $1 \in \mathbb{F}$), which means that Y^F dominates Y'^F. Similarly, Y'^F dominates Y^F and we conclude that $Y^F \approx Y'^F$.

Conversely, assume $Y^F \approx Y'^F$. Then $U(Y^F) \simeq U(Y'^F)$. It follows by Remark 6.9 that

$$U_A(Y) \simeq U(Y^F) \otimes A \simeq U(Y'^F) \otimes A' \simeq U_{A'}(Y'). \qquad \square$$

We write $R_{L/F}(Y)$ for the F-variety given by the Weil transfer of the L-variety Y.

Proposition 8.4. *One has $Y^F \approx R_{L/F}(Y)$.*

Proof. Since the degree of L/F is prime to p, the statement of Proposition 8.4 can be reformulated as equivalence $Y \approx R_{L/F}(Y)_L$.

Since the functor $R_{L/F}$ is right adjoint to the base change functor (see [17, (4.2.2)]), we have

$$\mathrm{Mor}(R_{L/F}(Y)_L, Y) = \mathrm{Mor}(R_{L/F}(Y), R_{L/F}(Y)) \ni \mathrm{id}$$

showing that there is a morphism $R_{L/F}(Y)_L \to Y$ and, in particular, $R_{L/F}(Y)_L$ dominates Y.

To obtain the other domination, recall that by our assumption on Y, there exists a projective homogeneous F-variety $\hat{Y}$ with $\hat{Y}_L \approx Y$. Applying the Weil transfer of [10, §3] to multiplicity one correspondences between $\hat{Y}_L$ and Y, we get multiplicity one correspondences between the varieties $R_{L/F}(\hat{Y}_L)$ and $R_{L/F}(Y)$ witnessing their equivalence. Since

$$\mathrm{Mor}(\hat{Y}, R_{L/K}(\hat{Y}_L)) = \mathrm{Mor}(\hat{Y}_L, \hat{Y}_L) \ni \mathrm{id},$$

there is a morphism $\hat{Y} \to R_{L/K}(\hat{Y}_L)$. It follows that $\hat{Y}$ dominates $R_{L/K}(Y)$ and therefore $Y \approx \hat{Y}_L$ dominates $R_{L/K}(Y)_L$. $\square$

Corollary 8.5. *The motives $U_A(Y)$ and $U_{A'}(Y')$ are isomorphic if and only if $A \simeq A'$ and $R_{L/F}(Y) \approx R_{L'/F}(Y')$.* $\square$

With Corollary 8.5 at hand, we can now prove Theorem 8.6 below giving a criterion of isomorphism for direct sums of A-upper motives in terms of higher Artin-Tate traces defined just next.

Let M and M' be F-motives which are finite direct sums such that each summand N is a shift of the motive $U_A(Y)$ for some p'-inner algebraic group G over F, some projective G_L-homogeneous variety Y over a p'-extension L/F, and for an indecomposable summand A in $M(L)^F$ (where G, L, Y, A may vary with N). The *Artin-Tate trace* of M is defined as the part of the above (complete) decomposition of M consisting of Artin-Tate motives. We say that M and M' have *isomorphic higher Artin-Tate traces*, if over any extension field of F, the Artin-Tate trace of M is isomorphic to the Artin-Tate trace of M'.

The following result shows that the isomorphism class of the motive M is determined by its higher Artin-Tate trace. We recall that by Remark 3.6 the higher Tate trace of [6] is insufficient for this purpose.

Theorem 8.6. *The motives M and M' are isomorphic if and only if they have isomorphic higher Artin-Tate traces.*

Proof. If M and M' are isomorphic, then by the Krull-Schmidt property, they have isomorphic higher Artin-Tate traces.

Conversely, assume that M and M' have isomorphic higher Artin-Tate traces. We prove that M and M' are isomorphic by induction on the maximum of the numbers of summands in their complete motivic decompositions. If this maximum is zero, both M and M' are trivial. If it is nonzero, write

$$M = U_{A_1}(X_1)\{n_1\} \oplus .. \oplus U_{A_k}(X_k)\{n_k\} \text{ and } M' = U_{B_1}(Y_1)\{m_1\} \oplus ... \oplus U_{B_s}(Y_s)\{m_s\}.$$

We may assume that $n = \min_{1\leq i\leq k} n_i$ is not higher than $m = \min_{1\leq j\leq s} m_j$. Pick an integer $1 \leq \alpha \leq k$ such that the Weil transfer $R(X_\alpha)$ to F of X_α is minimal for the domination relation among the $R(X_i)$'s such that $U_{A_i}(X_i)\{n\}$ is a direct summand in the above decomposition of M (to lighten notation, we write here $R(\cdot)$ for Weil transfers, dismissing the associated finite separable extensions).

By assumption on the higher Artin-Tate traces of M and M', since over the function field of $R(X_\alpha)$ the motive M contains as a summand the Artin-Tate motive $A\{n\}$ with $A := (A_\alpha)_{F(R(X_\alpha))}$, the motive M' over the same function field also contains $A\{n\}$. It follows that $n = m$ and that for some $1 \leq \beta \leq s$, the summand $U_{B_\beta}(Y_\beta)\{m_\beta\}$ of M' is such that $R(X_\alpha)$ dominates $R(Y_\beta)$, $(B_\beta)_{F(R(X_\alpha))}$ is isomorphic to A, and $m_\beta = n$. The Artin motives A_α and B_β are isomorphic by Corollary 7.5. The same reasoning over the function field of $R(Y_\beta)$ yields some $1 \leq \gamma \leq k$ such that $R(X_\gamma)$ is dominated by $R(Y_\beta)$, $A_\gamma \simeq A$, and $n_\gamma = n$.

By minimality of $R(X_\alpha)$, the varieties $R(X_\alpha)$ and $R(Y_\beta)$ are equivalent. The A-upper motives $U_A(X_\alpha)$ and $U_A(Y_\beta)$ are then isomorphic by Corollary 8.5. Induction, applied to the complementary summands in M and M' of $U_A(X_\alpha)\{n\}$ and $U_A(Y_\beta)\{n\}$, proves that M and M' are isomorphic. □

9. Motivic equivalence

In this section we produce a criterion of motivic equivalence for p'-inner reductive algebraic groups which are inner forms of each other. We remind that absolutely simple algebraic groups of any type other than ${}^3D_4$ and ${}^6D_4$ are p'-inner or p-inner, the latter case treated in [6].

Recall that a projective homogeneous variety is called *isotropic* (with coefficients in $\mathbb{F}$) if it possesses a closed point of degree prime to p.

Proposition 9.1. *Let X be a projective homogeneous F-variety.*

i) *If L/F is a finite separable field extension and A is an indecomposable direct summand in the F-motive $M(L)^F$, then the $F(X)$-motive $A_{F(X)}$ is indecomposable.*

ii) *Let G and G' be reductive algebraic groups over F, let L/F, L'/F be two p'-extensions, and let Y, Y' be projective homogeneous varieties over L and L' under G_L and $G'_{L'}$, respectively, which are equivalent to the restrictions of some projective homogeneous F-varieties and such that Y^F and Y'^F both dominate X. If the A-upper $F(X)$-motives $U_{A_{F(X)}}(Y_{L(X)})$ and $U_{A'_{F(X)}}(Y'_{L(X)})$ are isomorphic, then the F-motives $U_A(Y)$ and $U_{A'}(Y')$ are isomorphic as well.*

Proof. Given an indecomposable summand A of $M(L)^F$, the $F(X)$-motive $A_{F(X)}$ is indecomposable by Corollary 7.5, proving *i)*.

We prove *ii)* using Theorem 8.3. First, by Corollary 7.5 once again, if the Artin $F(X)$-motives $A_{F(X)}$ and $A'_{F(X)}$ are isomorphic, then the F-motives A and A' are isomorphic.

Assume that $U_{A_{F(X)}}(Y_{L(X)})$ and $U_{A'_{F(X)}}(Y'_{L(X)})$ are isomorphic. By Theorem 8.3, the $F(X)$-varieties $(Y^F)_{F(X)}$ and $(Y'^F)_{F(X)}$ are equivalent and $A_{F(X)} \simeq A'_{F(X)}$, hence $A \simeq A'$. By [4, Proof of Proposition 9], Y^F and Y'^F are equivalent and so $U_A(Y) \simeq U_{A'}(Y')$ by Theorem 8.3. □

Let G be a reductive group over F. Recall that we write D_G for its Dynkin diagram, which can be canonically attached to G using the generic point of the variety of pairs $T \subset B$ with T a maximal torus and B a Borel subgroup. Sometimes, depending on the context, D_G stands for the set of vertices of the Dynkin diagram.

Any Γ_F-invariant subset of D_G, yields a projective G-homogeneous variety (we keep the same convention as in the proof of Theorem 7.1). This induces a bijection between the Γ_F-invariant subsets of D_G and the isomorphism classes of projective G-homogeneous varieties. An invariant subset $\tau \subset D_G$ is *p-distinguished*, if the associated projective G-homogeneous variety $X_{G,\tau}$ is isotropic. The union of all p-distinguished orbits yields the largest p-distinguished subset, denoted D^p_G (see [5]).

We are going to consider two reductive groups G and G' each of which is an inner form of the other, that is, both of them are inner forms of the same quasi-split group. In such a situation, the Dynkin diagrams D_G and $D_{G'}$ are Γ_F-equivariant isomorphic and we will be fixing one of the possible isomorphisms.

Proposition 9.2. *Let G and G' be p'-inner reductive groups over F, inner forms of each other. Fix an equivariant isomorphism of their Dynkin diagrams $\varphi\colon D_G \longrightarrow D_{G'}$ and an invariant subset τ_0 of D_G. The following conditions on G, G', τ_0, and φ are equivalent:*

i) *for any field extension K/F, one has $\tau_0 \subset D^p_{G_K}$ (i.e., τ_0 is p-distinguished over K) if and only if $\varphi(\tau_0) \subset D^p_{G'_K}$; moreover, $\varphi(D^p_{G_K}) = D^p_{G'_K}$ in this case;*

ii) *for any minimal field extension E/F such that G_E (and G'_E) are of inner type, any field extensions L/F contained in E, any indecomposable summand A of the motive $M(L)^F$, and any* Gal(E/L)*-invariant subset $\tau \subset D_G$ containing τ_0, the A-upper motives $U_A(X_{G_L,\tau})$ and $U_A(X_{G'_L,\varphi(\tau)})$ are isomorphic.*

Proof. *i)* $\Rightarrow$ *ii)* Assuming *i)*, fix a field extension L/F contained in E, an Artin motive A, and a subset $\tau \supset \tau_0$ as in *ii)*. The subset τ_0 is p-distinguished for G over the function field

$\tilde{L}$ of the variety $X_{G_L,\tau}$. It follows from $i)$ that the subset $\varphi(\tau) \subset D_{G'}$ is p-distinguished over $\tilde{L}$. The L-variety $X_{G_L,\tau}$ thus dominates $X_{G'_L,\varphi(\tau)}$. The same reasoning with $\varphi(\tau)$ and the inverse of φ implies that the L-varieties $X_{G_L,\tau}$ and $X_{G'_L,\varphi(\tau)}$ are equivalent. It follows that the F-varieties $X^F_{G_L,\tau}$ and $X^F_{G'_L,\varphi(\tau)}$ are equivalent and hence the A-upper motives $U_A(X_{G_L,\tau})$ and $U_A(X_{G'_L,\varphi(\tau)})$ are isomorphic by Theorem 8.3.

ii) $\Rightarrow$ *i)* First, given a field extension K/F, the variety X_{G_K,τ_0} is isotropic if and only if $X_{G_K,\varphi(\tau_0)}$ is isotropic as well, since by assumption *ii)* (with $L = F$) the upper motives $U(X_{G,\tau_0})$ and $U(X_{G',\varphi(\tau_0)})$ are isomorphic. This means that τ_0 is p-distinguished over K if and only if $\varphi(\tau_0)$ is.

Now fix a field extension K/F such that τ_0 is p-distinguished over K. Fix a minimal subextension L/F of K such that $D^p_{G_K} \subset D_G$ is $\mathrm{Gal}(E/L)$-invariant, for some minimal field extension E/F over which G (and G') become of inner type. By assumption *ii)*, the A-upper motives $U_A(X_{G_L,D^p_{G_K}})$ and $U_A(X_{G'_L,\varphi(D^p_{G_K})})$ are isomorphic for any A, thus, by Theorem 8.3, the L-varieties $X_{G_L,D^p_{G_K}}$ and $X_{G'_L,\varphi(D^p_{G_K})}$ are equivalent. As L is contained in K, it follows that the K-varieties $X_{G_K,D^p_{G_K}}$ and $X_{G'_K,\varphi(D^p_{G_K})}$ are also equivalent.

Since the first of the two equivalent K-varieties is isotropic, the second one is also isotropic (see, e.g., [6, Lemma 2.2] and [14, Corollary 2.15]) which means that the subset $\varphi(D^p_{G_K})$ is p-distinguished for G' over K. The same reasoning with G replaced by G', τ_0 by $\varphi(\tau_0)$, and φ by its inverse, gives that $\varphi^{-1}(D^p_{G'_K}) \subset D^p_{G_K}$. Hence $\varphi(D^p_{G_K}) = D^p_{G'_K}$. $\square$

Let G be a reductive algebraic group over a field F. Recall that the classical Tits index of G is its Dynkin diagram D_G, endowed with the action of the absolute Galois group of F, together with the subset D^0_G of *distinguished vertices*. A vertex of D_G is distinguished if it is contained in a Galois orbit τ such that the projective homogeneous variety $X_{G,\tau}$ has a rational point.

For any subset τ of D_G, let us consider the minimal subextension F_τ/F inside of a fixed separable closure $\bar{F}/F$ such that τ is Γ_{F_τ}-invariant. The F-motive $M_{G,\tau} := M(X_{G_{F_\tau},\tau})^F$ is called the *standard motive of G of type τ*. Up to isomorphism, the motive $M_{G,\tau}$ does not depend on the choice of the separable closure $\bar{F}/F$. If τ is Γ_F-invariant, it is simply the motive of the projective G-homogeneous variety $X_{G,\tau}$.

We now introduce a set of integers describing motivic decompositions. Let G be a p'-inner reductive group, E/F a Galois p'-extension such that G_E is of inner type, X a projective G-homogeneous variety, and M a direct summand in $M(X)$. For any A-upper F-motive $U_A(Y)$ and any integer n, we write $l_{A,Y,n}(M)$ for the number of indecomposable summands isomorphic to $U_A(Y)\{n\}$ in a complete decomposition of M.

Theorem 9.3. *Let G and G' be p'-inner reductive groups over a field F which are inner forms of each other. Let τ_0 be a invariant subset in D_G. The equivalent conditions of Proposition 9.2 are satisfied if and only if for any subset $\tau \subset D_G$ containing τ_0, the motives $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic.*

Proof. The "if" part is clear: if the motives $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic, then for any intermediate field L of a minimal field extension E/F such that G_E and G'_E are of inner type, the varieties $X^F_{G_L,\tau}$ and $X^F_{G'_L,\varphi(\tau)}$ are equivalent. Hence, by Theorem 8.3, G and G' satisfy condition $ii)$ of Proposition 9.2.

We prove the opposite implication by induction on the (common) semisimple rank of G and G'. More concretely, assuming the conditions of Proposition 9.2, we will prove that for every $\tau \supset \tau_0$ the motives $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic. For $\tau = \emptyset$ the isomorphism trivially holds. This covers the rank zero case, which is the base of the induction. Below we assume that $\tau \neq \emptyset$.

We first show that $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic if τ and $\varphi(\tau)$ are $\mathrm{Gal}(E/F)$-invariant and the associated varieties both have a rational point (hence the reductive algebraic groups G and G' are isotropic).

Let $\tilde{G}$ be the semisimple part of a parabolic subgroup in G of type τ. The Dynkin diagram $D_{\tilde{G}}$ of $\tilde{G}$ is obtained by removing the subset τ from D_G, and $\tilde{G}_E$ is of inner type. By [1, Theorem 4.2], there is a motivic decomposition

$$M_{G,\tau} \simeq \bigoplus_{i\in\mathcal{I}} M^F_{\tilde{G}_{L_i},\tau_i}\{n_i\}$$

with some field extensions L_i/F contained in E and some $\mathrm{Gal}(E/L_i)$-invariant $\tau_i \subset D_{\tilde{G}}$. Note that the fields L_i, the projective $\tilde{G}_{L_i}$-homogeneous varieties $X_{\tilde{G}_{L_i},\tau_i}$, and the shifting numbers n_i in this decomposition are completely determined by the underlying combinatorics of G. The isomorphism $\varphi : D_G \longrightarrow D_{G'}$ from Proposition 9.2 yields an analogous decomposition of $M_{G',\varphi(\tau)}$ with respect to its semisimple part $\tilde{G}'$ of a parabolic subgroup in G' of type $\varphi(\tau)$ with the same $\mathcal{I}$, L_i, τ_i, and n_i:

$$M_{G',\varphi(\tau)} \simeq \bigoplus_{i\in\mathcal{I}} M^F_{\tilde{G}'_{L_i},\varphi(\tau_i)}\{n_i\}$$

Since G and G' are inner forms of each other and satisfy condition $i)$ of Proposition 9.2, so do $\tilde{G}$ and $\tilde{G}'$. Indeed, for any field extension K/F, we have disjoint union decompositions

$$D^p_{G_K} = D^p_{\tilde{G}_K} \sqcup \tau \quad \text{and} \quad D^p_{G'_K} = D^p_{\tilde{G}'_K} \sqcup \varphi(\tau).$$

Condition $i)$ of Proposition 9.2 for G and G' gives that $D^p_{G'_K} = \varphi(D^p_{G_K})$ and hence $D^p_{\tilde{G}'_K} = \varphi(D^p_{\tilde{G}_K})$. It follows that for any any $i \in \mathcal{I}$ and any field extension L_i/F, the reductive groups $\tilde{G}_{L_i}$ and $\tilde{G}'_{L_i}$ satisfy condition $i)$ of Proposition 9.2 with respect to the restriction of φ and the subset $\tilde{\tau}_0 = \emptyset$. By induction, the motives $M_{\tilde{G}_{L_i},\tau_i}$ and $M_{\tilde{G}'_{L_i},\varphi(\tau_i)}$ are thus isomorphic. Therefore, the motives $M^F_{\tilde{G}_{L_i},\tau_i}$ and $M^F_{\tilde{G}'_{L_i},\varphi(\tau_i)}$ are isomorphic as well and so $M_{G,\tau} \simeq M_{G',\varphi(\tau)}$.

We now treat the case of arbitrary $\mathrm{Gal}(E/F)$-invariant subsets τ and $\varphi(\tau)$. Assume that the motives of $X_{G,\tau}$ and $X_{G',\varphi(\tau)}$ are not isomorphic. By Theorem 7.1, since G and G' satisfy conditions of Proposition 9.2, this means that $l_{A,Y,n}(M_{G,\tau}) \neq l_{A,Y,n}(M_{G',\varphi(\tau)})$ for some indecomposable Artin motive A and some projective homogeneous variety Y defined over a field extension contained in E. Consider the minimal integer n for which such a non-equality occurs.

Over the function field K/F of the product $\Pi := X_{G,\tau} \times X_{G',\varphi(\tau)}$ both $X_{G,\tau}$ and $X_{G',\varphi(\tau)}$ have a rational point. The motive A_K is indecomposable (see Corollary 7.5) and so we can investigate the integer $l_{A_K,Y_K,n}(M_{G_K,\tau})$. To lighten notation (by abusing it), below we will write Y_K for the variety $Y_{L(\Pi)}$. If $U_{A_K}(Y_K)\{n\}$ is a direct summand of $M_{G_K,\tau}$, then by the Krull-Schmidt property and Theorem 7.1, it is a direct summand in the K/F-restriction $(U_B(Z))_K$ of an A-upper motive $U_B(Z)$ of G, shifted by some $k \leq n$.

Note that $(U_B(Z))_K \simeq U_{B_K}(Z_K) \oplus N$, where N is a direct sum of A-upper motives with Tate shifts at least 1. Since $X_{G,\tau}$ and $X_{G',\varphi(\tau)}$ are equivalent, any projective homogeneous variety which dominates $X_{G,\tau}$ (or $X_{G',\varphi(\tau)}$) dominates their product. In particular, Proposition 9.1 implies that a direct summand $U_{A_K}(Y_K)\{k\}$ of $M(X_{G_K,\tau})$ may only arise from a K/F-restriction $(U_B(Z)\{k\})_K$ (with the same shift) if $B \simeq A$ and $Z \approx Y$, that is from the A-upper motive $U_A(Y)\{k\}$ (see Theorem 8.3).

Write M for the direct summand of $M_{G,\tau}$ given by the sum of all its indecomposable summands with shifts strictly lower than n (in a fixed complete decomposition). Separating the summands $U_{A_K}(Y_K)\{n\}$ of $M_{G_K,\tau}$ which arise from M_K, we get thanks to the previous discussion the equality

$$l_{A_K,Y_K,n}(M_{G_K,\tau}) = l_{A,Y,n}(M_{G,\tau}) + l_{A_K,Y_K,n}(M_K).$$

Since by assumption the A-upper motives of G and G' are pairwise isomorphic, the same reasoning with $X_{G',\varphi(\tau)}$ ensures that

$$l_{A_K,Y_K,n}(M_{G'_K,\varphi(\tau)}) = l_{A,Y,n}(M_{G',\varphi(\tau)}) + l_{A_K,Y_K,n}(M'_K),$$

where M' is the direct sum of the summands in a complete motivic decomposition of $X_{G',\varphi(\tau)}$ with shifts strictly lower than n. By minimality of n, the motives M and M' are isomorphic, hence M_K and M'_K are isomorphic as well and $l_{A_K,Y_K,n}(M_K) = l_{A_K,Y_K,n}(M'_K)$. As by assumption $l_{A,Y,n}(M_{G,\tau}) \neq l_{A,Y,n}(M_{G',\varphi(\tau)})$, it follows that $l_{A_K,Y_K,n}(M_{G_K,\tau})$ and $l_{A_K,Y_K,n}(M_{G'_K,\varphi(\tau)})$ are not equal, a contradiction to the fact that the motives of $X_{G_K,\tau}$ and of $X_{G'_K,\varphi(\tau)}$ are isomorphic (recall that both of these varieties have a rational point).

We can now conclude: let τ be an arbitrary subset of D_G. The reductive groups G_{F_τ} and G'_{F_τ} satisfy condition $i)$ of Proposition 9.2. It follows from the Galois-invariant case that the motives $M_{G_{F_\tau},\tau}$ and $M_{G'_{F_\tau},\varphi(\tau)}$ are isomorphic, hence so are the motives $M_{G,\tau} = M^F_{G_{F_\tau},\tau}$ and $M_{G',\varphi(\tau)} = M^F_{G'_{F_\tau},\varphi(\tau)}$. □

A field is called *p-special* if every finite extension of this field has a p-power degree. Let G and G' be two reductive groups, inner forms of each other. Similarly to [4, Definition 1], we say that G and G' are *motivic equivalent* (with coefficients in $\mathbb{F}$) with respect to a Galois-equivariant isomorphism $\varphi : D_G \longrightarrow D_{G'}$, if for any subset τ of D_G, the motives $M_{\tau,G}$ and $M_{\varphi(\tau),G'}$ are isomorphic.

Corollary 9.4. *Let G and G' be p'-inner reductive algebraic groups over F, inner forms of each other. Let φ be a Γ_F-equivariant isomorphism of their Dynkin diagrams. The groups G and G' are motivic equivalent with respect to φ if and only if for any p-special field extension K/F, φ identifies the Tits indexes of G_K and G'_K.*

Proof. Theorem 9.3 with $\tau_0 = \emptyset$ states that G and G' are motivic equivalent with respect to φ if and only if for any field extension K/F, φ identifies the subsets of p-distinguished vertices of G_K and G'_K. Over p-special field K, this expresses as $\varphi(D^0_{G_K}) = D^0_{G'_K}$ (through classical Tits indexes), since a variety is isotropic if and only it has a rational point over a p-special closure of its base field [6, Proof of Lemma 4.11]. □

References

[1] Brosnan, P. On motivic decompositions arising from the method of Białynicki-Birula. *Invent. Math. 161*, 1 (2005), 91–111.
[2] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.
[3] Chernousov, V., and Merkurjev, A. Motivic decomposition of projective homogeneous varieties and the Krull-Schmidt theorem. *Transform. Groups 11*, 3 (2006), 371–386.
[4] De Clercq, C. Motivic equivalence of semisimple algebraic groups. *Compos. Math. 153*, 10 (2017), 2195–2213.
[5] De Clercq, C., and Garibaldi, S. Tits p-indexes of semisimple algebraic groups. *J. Lond. Math. Soc., II. Ser. 95*, 2 (2017), 567–585.
[6] De Clercq, C., and Quéguiner-Mathieu, A. Higher Tate traces of Chow motives. *J. Reine Angew. Math. 815* (2024), 1–22.
[7] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2008.
[8] Garibaldi, S., Petrov, V., and Semenov, N. Shells of twisted flag varieties and the Rost invariant. *Duke Math. J. 165*, 2 (2016), 285–339.
[9] Hovey, M., Palmieri, J. H., and Strickland, N. P. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc. 128*, 610 (1997), x+114.
[10] Karpenko, N. A. Weil transfer of algebraic cycles. *Indag. Math. (N.S.) 11*, 1 (2000), 73–86.
[11] Karpenko, N. A. On the first Witt index of quadratic forms. *Invent. Math. 153*, 2 (2003), 455–462.
[12] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.
[13] Karpenko, N. A. Isotropy of orthogonal involutions. *Amer. J. Math. 135*, 1 (2013), 1–15. With an appendix by Jean-Pierre Tignol.
[14] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.
[15] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.
[16] Manin, J. I. Correspondences, motifs and monoidal transformations. *Mat. Sb. (N.S.) 77 (119)* (1968), 475–507.
[17] Scheiderer, C. *Real and étale cohomology*, vol. 1588 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1994.
[18] Tits, J. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965).* Amer. Math. Soc., Providence, R.I., 1966, pp. 33–62.
[19] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.
[20] Vishik, A. Fields of u-invariant $2^r + 1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.
[21] Voevodsky, V. On motivic cohomology with $\mathbb{Z}/l$-coefficients. *Ann. of Math. (2) 174*, 1 (2011), 401–438.
[22] Wildeshaus, J. Notes on Artin-Tate motives. In *Autour des motifs—École d'été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres/Asian-French Summer School on Algebraic*

Geometry and Number Theory. Vol. III, vol. 49 of *Panor. Synthèses*. Soc. Math. France, Paris, 2016, pp. 101–131.

Université Sorbonne Paris Nord, Intitut Galilée, Laboratoire Analyse, Géométrie et Applications, Villetaneuse, FRANCE
Email address: declercq@math.univ-paris13.fr
URL: www.math.univ-paris13.fr/~declercq

Mathematical & Statistical Sciences, University of Alberta, Edmonton, CANADA
Email address: karpenko@ualberta.ca
URL: www.ualberta.ca/~karpenko

Université Sorbonne Paris Nord, Intitut Galilée, Laboratoire Analyse, Géométrie et Applications, Villetaneuse, FRANCE
Email address: queguin@math.univ-paris13.fr
URL: www.math.univ-paris13.fr/~queguin