A-UPPER MOTIVES

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Abstract. For a given a reductive algebraic group G over a field F, let E/F be a minimal field extension over which G becomes of inner type. The extension E/F is finite Galois; let us assume that, for some prime number p, its Galois group is the product of a p-group and a group of p-coprime order. (The assumption holds with any prime p for all absolutely simple groups of type not ${}^6D_4$.) Working with coefficients $\mathbb{Z}/p\mathbb{Z}$, we define the *A-upper motives* of G. These are indecomposable Chow motives naturally related to indecomposable summands in the motives of spectra of intermediate fields in E/F. We show that motives of projective homogeneous varieties under G are isomorphic to direct sums of Tate shifts of A-upper motives. Based on that, we get a motivic classification of the varieties by means of their *higher Artin-Tate traces*. We also show how Tits indexes over suitable field extensions determine motivic equivalence classes for these algebraic groups.

Contents

1. Introduction

Envisioned by Alexander Grothendieck in the sixties, Chow motives provide powerful invariants to study arithmetic and geometry of smooth projective varieties over fields. The case of projective homogeneous varieties has received a lot of attention over the years and numerous breakthroughs and solutions to classical conjectures were obtained through the study of their motives. Most of these results are proved in the framework of semisimple

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algebraic groups of inner type, i.e. such that the $*$-action of the absolute Galois group of the base field on the associated Dynkin diagram is trivial. In this work, we initiate the study of motives and motivic decompositions for projective homogeneous varieties under arbitrary reductive groups.

The extensive study of motives of projective quadrics, which were essential to Voevodsky's proof of the Milnor conjecture [21], was carried out by Alexander Vishik in [19]. This milestone led notably to advances on the Kaplansky problem [20] and a proof of Hoffmann's conjecture [10]. Vishik provides on the way a qualitative description of motivic structure of projective quadrics through the motives of Čech simplicial schemes associated to orthogonal Grassmannians. Working now with $\mathbb{Z}/p\mathbb{Z}$ coefficients and motivated by the case of generalized Severi-Brauer varieties, the second author then obtains a description of indecomposable summands in the motives of projective G-homogeneous varieties for G a reductive group of inner type: the indecomposable summands are Tate shifts of *upper motives* of G [13]. (More generally, the description holds for *p-inner* reductive groups, see [11]). This result led to many applications, notably on the anisotropy of orthogonal involutions [12] and the classification of motivic decompositions for exceptional groups [9] as well as of motives of projective homogeneous varieties under p-inner groups [6].

We set $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ for the ring of coefficients, where p is a prime. Given a finite separable field extension L/F and a projective homogeneous variety X over L, we introduce in this work the A-upper F-motives of X (see Definition 7.10). An A-upper F-motive of X is indecomposable and naturally related with an indecomposable Artin motive which is a summand in the F-motive of the spectrum of L. This leads to the notion of A-upper motives of a reductive group G, satisfying certain conditions (see Definition 7.11). If G is of inner type or, more generally, p-inner type, the A-upper motives of G are the upper motives of G considered previously.

Now assume that G is an absolutely simple group of any Dynkin type different from ${}^6D_4$, and pick a projective homogeneous variety X under G. Theorem 8.1 provides a qualitative analysis of the motivic structure of X, stating that the motive of X decomposes (in a unique way) as a direct sum of Tate shifts of A-upper motives of G. The result holds more generally for *p-separately inner* reductive groups (see §7), that is reductive groups which become of inner type over a finite Galois field extension E/F such that $\mathrm{Gal}(E/F)$ is the product of a p-group and a group of p-coprime order. As a consequence of this structural result we obtain a complete classification of motives of projective homogeneous varieties under such reductive groups, through their *higher Artin-Tate traces* (Corollary 9.4). We then provide criteria of motivic equivalence, by means of combinatorial invariants derived from the classical Tits indices of reductive algebraic groups. These results expound how higher isotropy of reductive groups determines motives of projective homogeneous varieties.

2. Preliminaries

A *variety* is a separated scheme of finite type over a field. Let X be an F-variety. For a field extension L/F, X_L is the L-variety given by the product of the F-schemes X and $\operatorname{Spec} L$. For a finite field extension F/K, X^K is the K-variety given by the scheme X endowed with the composition $X \to \operatorname{Spec} F \to \operatorname{Spec} K$. In particular, $X = X_F = X^F$

and the notation with F can be employed as a way to evoke the base field of X. By default, the spectrum of a field is the variety over this very field; the correct notation for the K-variety given by the spectrum of F is $(\operatorname{Spec} F)^K$.

Throughout this paper, p is a prime number and $\mathbb{F}$ is the field (of coefficients) $\mathbb{Z}/p\mathbb{Z}$. We use the notation $\operatorname{Ch}(\cdot)$ for Chow groups with coefficients in $\mathbb{F}$.

3. The field of constants

Any smooth connected F-variety X determines a finite separable field extension L/F and a smooth geometrically connected L-variety Y with $Y^F = X$. The underlying scheme of the variety Y is just the scheme of X. The field L coincides with the algebraic closure of F inside the function field $F(X)$ of X and is called the *field of constants* of X. Let's fix a separable closure $\bar{F}$ of F containing L and write Γ for the absolute Galois group $\operatorname{Gal}(\bar{F}/F)$ of F. The finite Γ-set, which determines the étale F-algebra L in the sense of [14, §18.A], is the set of connected components of the $\bar{F}$-variety $\bar{X} := X_{\bar{F}}$.

Let us consider the *category of degree* 0 *correspondences* with coefficients in $\mathbb{F}$. The objects of this category are given by smooth projective F-varieties, the morphisms – by the degree 0 correspondences, where the degree of a correspondence is defined as in [8, §63]. (Note a difference with the definition of [16].) This is a full subcategory in the category of Chow motives so that the object, given by a smooth projective F-variety X, can already be called the motive of X and denoted $M(X)$. Associating to the motive $M(X)$ of a smooth projective F-variety X the F-variety $(\operatorname{Spec} L)^F$ given by its field of constants L, we get a functor $\mathbf{m}$ of the category of degree 0 correspondences to its full subcategory of 0-dimensional varieties: for one more pair X', L', any morphism of motives $M(X) \to M(X')$ yields a homomorphism $\operatorname{Ch}^0(\bar{X}') \to \operatorname{Ch}^0(\bar{X})$ of permutation Γ-modules, which yields a morphism $M(L')^F \to M(L)^F$ (cf. [3, §7]). Note that

$$\operatorname{Hom}(M(L')^F, M(L)^F) = \operatorname{Hom}(M(L)^F, M(L')^F)$$

so that we can define $\mathbf{m}$ to be a functor, not a cofunctor.

Passing to the idempotent completions, we extend this functor to the category of *effective Chow motives* $\operatorname{CM}_{\text{eff}}(F, \mathbb{F})$; the extension, for which we continue to write $\mathbf{m}$, takes values in the category of *Artin Chow motives* (see §4 for more details on this category). The restriction of $\mathbf{m}$ to the subcategory of Artin Chow motives is the identity, so, $\mathbf{m}$ is a "retraction" of the entire category of the effective Chow motives to its subcategory of Artin Chow motives.

Note that $\operatorname{CM}_{\text{eff}}(F, \mathbb{F})$ is a full subcategory in the entire category of Chow motives $\operatorname{CM}(F, \mathbb{F})$ of [8, §64] mainly used in this paper. The dual of $\operatorname{CM}_{\text{eff}}(F, \mathbb{F})$ is defined and studied in [16] without mentioning the word "effective" in the name (see [16, Remark of §8]). Since the subcategory $\operatorname{CM}_{\text{eff}}(F, \mathbb{F}) \subset \operatorname{CM}(F, \mathbb{F})$ is closed under taking direct summands, and we are investigating motivic decompositions of varieties in this paper, $\operatorname{CM}(F, \mathbb{F})$ can be replaced by $\operatorname{CM}_{\text{eff}}(F, \mathbb{F})$ everywhere below.

4. Artin motives

The letter "A" in the name of *A-upper motives*, introduced in the next section, indicates their relationship with the *Artin motives*. By definition (cf. [22, Definition 1.2]), an Artin motive (over F) is a direct summand in the Chow motive of the spectrum of an étale F-algebra. This includes the motive $\mathbb{F}$ of $\operatorname{Spec} F$, also called a Tate motive. Here is the simplest example of an Artin motive not isomorphic to $\mathbb{F}$:

Example 4.1 (cf. [11, Example 3.3])**.** Let p be an odd prime number. In the category of Chow motives with coefficients $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ over a base field F, the motive of the F-variety $(\operatorname{Spec} L)^F$, given by a separable quadratic field extension L/F, is a direct sum of two indecomposable motives. One of them is the Tate motive $\mathbb{F}$ – the motive of the base point $\operatorname{Spec} F$. The other one, let's call it A, becomes $\mathbb{F}$ over L but is not isomorphic to $\mathbb{F}$ over F. Moreover,

$$\operatorname{Hom}(\mathbb{F}, A) = 0 = \operatorname{Hom}(A, \mathbb{F}).$$

The shifts $\mathbb{F}\{i\}$ (with $i \in \mathbb{Z}$) of the Tate motive $\mathbb{F}$ are also called Tate motives. The shifts $A\{i\}$ of an Artin motive A (as well as finite direct sums of $A\{i\}$ with various A and i) are called the *Artin-Tate Chow motives* [22, Definition 1.3].[1]

Now let p be any prime number, possibly even. Let L/F be a finite separable field extension. Let $M(L)^F$ be the Chow motive with coefficients $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ of the F-variety $(\operatorname{Spec} L)^F$. For any finite Galois field extension E/F, an analysis of the full additive subcategory of Chow motives generated by all $M(L)^F$ with $F \subset L \subset E$ is made in [3, §7]. It is shown to be equivalent to the category of (finite-dimensional over $\mathbb{F}$) permutation modules over the group ring $\mathbb{F}[\Gamma]$, where Γ is the Galois group $\operatorname{Gal}(E/F)$. Below are some interesting examples of computations in these categories.

Example 4.2. Let L/F be a cubic Galois field extension and $p = 2$. The $\mathbb{F}$-algebra $\operatorname{End} M(L)^F$ is generated by a single element x subject to the relation $x^3 = 1$. It follows that the ring $\operatorname{End} M(L)^F$ is the direct product $\mathbb{F}_2 \times \mathbb{F}_4$, where $\mathbb{F}_2 = \mathbb{F}$ is the field of 2 elements and

$$\mathbb{F}_4 = \mathbb{F}[x]/(x^2 + x + 1)$$

is the field of 4 elements. One gets a complete decomposition $M(L)^F \simeq \mathbb{F} \oplus A$, where the indecomposable summand A satisfies $\operatorname{Hom}(\mathbb{F}, A) = 0 = \operatorname{Hom}(A, \mathbb{F})$ over F and becomes isomorphic to $\mathbb{F} \oplus \mathbb{F}$ over L.

Example 4.3. Let L/F be a finite Galois field extension and let Γ be its Galois group. Let us consider the group $\mathbb{F}$-algebra $\mathbb{F}[\Gamma]$. As explained in [3, §7], $\operatorname{End} M(L)^F$ is the ring of endomorphisms of the left $\mathbb{F}[\Gamma]$-module $\mathbb{F}[\Gamma]$. Associating to any element $\sigma \in \Gamma$ the right multiplication by σ^{-1}, we get an identification $\mathbb{F}[\Gamma] = \operatorname{End} M(L)^F$.

Example 4.4. Let q be a prime number different from p and let L/F be a finite Galois field extension of degree q. The $\mathbb{F}$-algebra $\operatorname{End} M(L)^F$ is generated by a single element x subject to the relation $x^q = 1$. It follows that the $\mathbb{F}$-algebra $\operatorname{End} M(L)^F$ is the direct product $\mathbb{F} \times B$ of the $\mathbb{F}$-algebra $\mathbb{F}$ and the $\mathbb{F}$-algebra

$$B := \mathbb{F}[x]/(x^{q-1} + x^{q-2} + \cdots + 1).$$

The Tate motive $\mathbb{F}$ splits off as a direct summand in $M(L)^F$. (The complementary summand can be but is not necessarily indecomposable.)

[1]Thanks to Stefan Gille and Alexander Vishik for suggestion to consider the Artin-Tate motives in this context.

Example 4.5. In general, let us embed a finite separable field extension L/F in a finite Galois field extension E/F. Then $\operatorname{End} M(L)^F$ is the ring of endomorphisms of the $\mathbb{F}[\Gamma]$-module $\mathbb{F}[\Gamma/\Gamma']$, where Γ is the Galois group of E/F, $\Gamma' \subset \Gamma$ is the subgroup of elements fixing the elements of L, and Γ/Γ' is the set of left cosets on which the group Γ acts by left multiplication.

Example 4.6. Direct sum decompositions of the motive $M(L)^F$ are given by the direct sum decompositions of the Γ-module $\mathbb{F}[\Gamma/\Gamma']$. In Example 4.1, where p is odd, $\Gamma = \{1, \sigma\}$, and $\Gamma' = \{1\}$, the Γ-module $\mathbb{F}[\Gamma]$ is a direct sum of the submodule generated by $1+\sigma$ (which corresponds to the Tate summand of $M(L)^F$) and the submodule generated by $1-\sigma$ (corresponding to A). Note that the generator $1+\sigma$ is Γ-invariant whereas $1-\sigma$ is not (although the $\mathbb{F}$-subspace it generates is Γ-invariant).

5. A-upper motives for Galois extensions

Let L/F be a finite separable field extension and let Y be a projective homogeneous variety over L. Let us consider its upper L-motive $U(Y)$ defined as a (unique up to an isomorphism) indecomposable summand of the L-motive of Y satisfying the condition

$$\operatorname{Ch}^0(U(Y)) := \operatorname{Hom}(U(Y), \mathbb{F}) \neq 0$$

(or, equivalently, $\operatorname{Ch}^0(U(Y)) = \operatorname{Ch}^0(Y)$) on its codimension 0 Chow group.

We are going to consider the L/F-*corestriction* $U(Y)^F := \operatorname{cor}_{L/F} U(Y)$ of $U(Y)$ – the F-motive defined as in [11, §3]. In general, in contrast to the L-motive $U(Y)$, the F-motive $U(Y)^F$ is not indecomposable anymore. Next we are going to investigate its complete decomposition.

Let $M(Y)^F$ be the F-motive given by the L/F-corestriction of the motive $M(Y)$ of Y, i.e., $M(Y)^F := M(Y^F)$. Since L is the field of constants of the F-variety Y^F, the functor $\mathbf{m}$ from §3 yields a ring homomorphism

$$m\colon \operatorname{End} M(Y)^F \to \operatorname{End} M(L)^F. \tag{5.1}$$

Example 5.2. Over L, the homomorphism

$$m\colon \operatorname{End} M(Y) \to \operatorname{End} M(L) = \operatorname{End} \mathbb{F} = \mathbb{F}$$

takes an element of $\operatorname{End} M(Y)$, viewed as a correspondence $Y \rightsquigarrow Y$, to its *multiplicity*, defined as in [8, §75].

Let $p \in \operatorname{End} M(Y)$ be the projector defining the upper motive $U(Y)$. By definition of the corestriction of motives (cf. [11, §3]), the F-motive $U(Y)^F$ is the direct summand of $M(Y)^F$ given by the projector p^F which is the image of p under the push-forward homomorphism

$$\operatorname{End}(M(Y)) = \operatorname{Ch}_d(Y \times Y) \to \operatorname{Ch}_d(Y^F \times Y^F) = \operatorname{End} M(Y)^F$$

with respect to the closed embedding $Y \times Y \hookrightarrow Y^F \times Y^F$, where $d = \dim Y$. Since $m(p^F)$ is the identity in $\operatorname{End} M(L)^F$, the functor $\mathbf{m}$ also yields a homomorphism

$$m\colon \operatorname{End} U(Y)^F \to \operatorname{End} M(L)^F. \tag{5.3}$$

Note that the additive group of $\operatorname{End} U(Y)^F$ is a direct summand of $\operatorname{End} M(Y)^F$. The homomorphism (5.3) is the composition of the embedding $\operatorname{End} U(Y)^F \hookrightarrow \operatorname{End} M(Y)^F$

followed by (5.1). Besides, the homomorphism (5.1) is the composition of the projection $\operatorname{End} M(Y)^F \twoheadrightarrow \operatorname{End} U(Y)^F$ followed by (5.3).

Example 5.4. The kernel of the homomorphism

$$m\colon \operatorname{End} U(Y) \to \mathbb{F}$$

from Example 5.2 consists of nilpotents. Indeed, by [13, Corollary 2.2], any endomorphism of $U(Y)$ raised to an appropriate power becomes idempotent. Since the motive $U(Y)$ is indecomposable, the idempotent is 1 or 0. If the endomorphism vanishes under m, the idempotent has to be 0, i.e., the endomorphism has to be nilpotent.

Now we turn our attention to the case where the finite separable field extension L/F is Galois. For $\sigma \in \Gamma := \operatorname{Gal}(L/F)$ and any L-variety X, let us write X_σ for the L-variety obtained from X by the base change via σ. Thus X_σ is the scheme X viewed as an L-variety via the composition $X \to \operatorname{Spec} L \xrightarrow{\sigma^{-1}} \operatorname{Spec} L$, i.e., $X_\sigma = X^{\sigma^{-1}}$. We use the similar notation for the motives. Since the base change by σ is invertible (namely, $(M_\sigma)_{\sigma^{-1}} = M$ for any L-motive M), the motive M_σ is indecomposable provided that M is. By a similar reason, the variety X_σ has a rational point if and only if X has one.

Proposition 5.5. *For a finite Galois field extension L/F, the ring homomorphism (5.3) is surjective; its kernel consists of nilpotents.*

Proof. The étale F-algebra $L \otimes_F L$ decomposes into the direct product

$$L \otimes_F L = \prod_{\sigma \in \Gamma} L$$

of $[L : F]$ copies of L indexed by $\sigma \in \Gamma$, where we identify the two algebras by sending $l \otimes 1$ to the diagonal image of l in $\prod_{\sigma\in\Gamma} L$ and $1 \otimes l$ to the tuple $(\sigma(l))_{\sigma\in\Gamma}$. It follows that

$$\operatorname{End} M(Y)^F = \operatorname{Ch}_d(Y^F \times Y^F) = \operatorname{Ch}_d(Y \times \operatorname{Spec}(L \otimes_F L) \times Y) = \bigoplus_\sigma \operatorname{Ch}_d(Y \times Y_\sigma),$$

where $d := \dim Y$.

In terms of this direct sum decomposition, the homomorphism (5.1)

$$m\colon \operatorname{End} M(Y)^F \to \operatorname{End} M(L)^F = \bigoplus_\sigma \mathbb{F}$$

is the direct sum of the multiplicity homomorphisms $\operatorname{Ch}_d(Y \times Y_\sigma) \to \mathbb{F}$. The variety Y has a rational point if and only if the variety Y_σ has one. Therefore there exists a multiplicity 1 correspondence $Y \rightsquigarrow Y_\sigma$. This proves that the homomorphism (5.1) is surjective. It follows that the homomorphism (5.3) is also surjective.

To prove the statement on the kernel, let us take some $f \in \operatorname{End} U(Y)^F$ vanishing under (5.3). By [13, Corollary 2.2], some power of f is a projector; this projector determines a summand M of $U(Y)^F$ satisfying $\mathbf{m}(M) = 0$. To show that f is nilpotent, it is enough to show that $M = 0$.

The L-variety $(Y^F)_L$ is the disjoint union of the L-varieties Y_σ, $\sigma \in \Gamma$, and the motive $(U(Y)^F)_L$ is the direct sum of the indecomposable motives $U(Y)_\sigma = U(Y_\sigma)$, $\sigma \in \Gamma$. By the Krull-Schmidt property, M_L decomposes in a direct sum of some of them. Since $\mathbf{m}(U(Y_\sigma)) \neq 0$ for every $\sigma \in \Gamma$, we conclude that $M_L = 0$. Consequently, $M = 0$ by the nilpotence principle [11, Theorem 2.1]. $\square$

Corollary 5.6. *For a finite Galois field extension L/F, the following holds:*

1. *Every summand in $M(L)^F$ is isomorphic to the image under $\mathbf{m}$ of a summand in $U(Y)^F$.*
2. *Two summands in $U(Y)^F$ with isomorphic images under $\mathbf{m}$ are isomorphic.*
3. *A summand in $U(Y)^F$ is indecomposable if and only if its image under $\mathbf{m}$ is so.*

Proof. (1) By Proposition 5.5, the projector defining a given summand in $M(L)^F$ lifts to an element of $\operatorname{End} U(Y)^F$. By [13, Corollary 2.2], an appropriate power of this element is a projector.

(2) Let M_1 and M_2 be summands of $U(Y)^F$. Any morphism between $\mathbf{m}(M_1)$ and $\mathbf{m}(M_2)$ is given by an endomorphism of $M(L)^F$ and therefore, by Proposition 5.5, can be lifted to a morphism between M_1 and M_2. In particular, if $\mathbf{m}(M_1)$ and $\mathbf{m}(M_2)$ are isomorphic, mutually inverse isomorphisms lift to some morphisms $f\colon M_1 \to M_2$ and $g\colon M_2 \to M_1$. By Proposition 5.5 once again, each of the compositions $g \circ f$ and $f \circ g$ has the form $\mathrm{id} + \varepsilon$ with some nilpotent ε and so is an isomorphism (with the inverse given by the finite sum $\mathrm{id} - \varepsilon + \varepsilon^2 - \dots$).

(3) This is a consequence of (1) and (2). □

Example 5.7. We describe a counter-example to the conclusion of Proposition 5.5 and Corollary 5.6 with a *non-Galois* finite separable field extension L/F. We take $p = 2$, i.e., we use the coefficient ring $\mathbb{F} := \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ here.

Let F be the field $\mathbb{Q}$ of rational numbers. Consider the cubic field extension L/F obtained by adjoining the real cubic root $\sqrt[3]{2} \in \mathbb{R}$ of 2. Let E/F be the normal closure of L/F inside the field of complex numbers $\mathbb{C}$. The Galois group of E/F is the symmetric group S_3 of order 6. The field extension E/L is quadratic and is obtained by adjoining a primitive 3d complex root of unity $\omega \in \mathbb{C}$.

Let Y' be an anisotropic conic over $\mathbb{Q}$ which splits over $\mathbb{Q}(\omega)$. We set $Y := Y'_L$. Since the L-motive $M(Y)$ is indecomposable, $U(Y) = M(Y)$. Since $L \otimes_F L = L \times E$, we have

$$\operatorname{End} U(Y)^F = \operatorname{End} M(Y)^F = \mathrm{Ch}_1(Y \times Y) \oplus \mathrm{Ch}_1(Y_E \times Y_E).$$

Since the conic Y_E is split, the ring $\operatorname{End} M(Y_E) = \mathrm{Ch}_1(Y_E \times Y_E)$ contains a nonzero projector which yields a complementary to $U(Y_E) = \mathbb{F}$ summand $\mathbb{F}\{1\}$ in $M(Y_E)$. This nonzero projector vanishes under the homomorphism $m\colon \operatorname{End} U(Y)^F \to \operatorname{End} M(L)^F$; the corresponding (non-trivial) summand of $U(Y)^F$ vanishes under the functor $\mathbf{m}$.

6. A-upper motives for p-coprime extensions

A finite separable field extension is *p-coprime* if so is the degree of its normal closure.

Let G be a reductive algebraic group over a field F; L/F a p-coprime extension; Y a projective G_L-homogeneous variety over L. We would like to understand the complete decomposition of the motive $U(Y)^F$.

Note that the Tate motive $\mathbb{F}$ is a direct summand in $M(L)^F$.

Lemma 6.1. *Assume that the p-coprime extension L/F is Galois. Then $U(Y^F) \simeq U_{\mathbb{F}}(Y)$.*

Proof. We need L/F to be Galois for the motive $U_{\mathbb{F}}(Y)$ to be defined (in §5). By definition, this motive is an indecomposable summand in $M(Y)^F$ with $\mathbf{m}(U_{\mathbb{F}}(Y)) = \mathbb{F}$. In particular, $\mathrm{Ch}^0 U_{\mathbb{F}}(Y) = \mathrm{Ch}^0 \mathbb{F} = \mathbb{F}$. So, $U(Y^F) \simeq U_{\mathbb{F}}(Y)$ by definition of $U(Y^F)$. □

Lemma 6.2. $U(Y) \simeq U(Y^F)_L$.

Proof. For any finite separable field extension L/F, the indecomposable motive $U(Y)$ is a direct summand in $U(Y^F)_L$. Therefore $U(Y) \simeq U(Y^F)_L$ provided that the latter is also indecomposable.

If L/F is p-coprime and Galois, then $U(Y)^F_L = \bigoplus_{\sigma\in \mathrm{Gal}(L/F)} U(Y)_\sigma$ and $U_{\mathbb{F}}(Y)_L$ is the summand $U(Y) = U(Y)_1$. It follows by Lemma 6.1 that the motive $U(Y^F)_L$ is indecomposable.

For arbitrary p-coprime L/F, let E/F be its normal closure. Note that $U(Y^L_E) \simeq U(Y)$ by the criterion [13, Corollary 2.15] because the degree $[E : L]$ is p-coprime. So, by the preceding case we know that the motive $U(Y^F)_E$ is indecomposable. It follows that $U(Y^F)_L$ is also indecomposable. □

Lemma 6.3. $U(Y)^F \simeq U(Y^F) \otimes M(L)^F$.

Proof. The second isomorphism in the chain

$$U(Y)^F \simeq \big(U(Y^F)_L \otimes M(L)\big)^F \simeq U(Y^F) \otimes M(L)^F$$

is a particular case of the following general formula that holds for any finite separable field extension L/F, an F-motive M, and an L-motive N: $(M_L \otimes N)^F \simeq M \otimes N^F$. □

Let A an indecomposable direct summand in $M(L)^F$. We set $U_A(Y) := U(Y^F) \otimes A$. Note that $\mathbf{m}(U_A(Y)) = A$.

The following proposition proves Corollary 7.9 for p-coprime L/F:

Proposition 6.4. *The motive $U_A(Y)$ is indecomposable.*

Proof. Let E/F be the normal closure of L/F. The motive $U_A(Y)_E$ is the direct sum of $\mathrm{rk}(A)$ copies of the indecomposable motive $U(Y^F)_E$. To prove indecomposability of $U_A(Y)$, we take its nonzero direct summand U and check that U_E is still the sum of $\mathrm{rk}(A)$ copies of $U(Y^F)_E$. Since $\dim \mathrm{Ch}^0(U(Y^F)_E) = 1$, it suffices to show that $\dim \mathrm{Ch}^0(U_E) \geq \mathrm{rk}\, A$. In these formulas, $\mathrm{rk}\, A$ stands for the rank of A defined as the number of (Tate) summands in the complete decomposition of A over its splitting field (e.g., the field E).

Let X be the F-variety of Borel subgroup in G. The motive $U_A(Y)_{F(X)}$ is a direct sum of the Artin motive $A_{F(X)}$ and some positive shifts of some effective motives. It follows that the (indecomposable by Lemma 7.6 and Corollary 8.8) motive $A_{F(X)}$ is a summand of $U_{F(X)}$. Therefore

$$\dim \mathrm{Ch}^0(U_E) \geq \dim \mathrm{Ch}^0(A_{E(X)}) = \mathrm{rk}\, A. \qquad \square$$

7. A-upper motives for p-separated extensions

Let G be a reductive algebraic group over a field F and let E/F be the finite Galois field extension corresponding to the kernel of the $*$-action of the absolute Galois group of F on the Dynkin diagram of G. Recall from [11] that G is *p-inner*, if $\Gamma := \mathrm{Gal}(E/F)$ is a p-group. We say that G is *p-separately inner*, if the finite group Γ is *p-separated*, i.e., decomposes into the direct product

$$\Gamma = \Gamma_p \times \Gamma_c \tag{7.1}$$

of a p-group Γ_p and a group Γ_c of p-coprime order. Note that every absolutely simple group G of any Dynkin type different from ${}^6D_4$ is p-separately inner; moreover, the minimal field extension E/F with G_E of inner type is always quadratic except from ${}^3D_4$, where it is cubic.

Note that Γ_p and Γ_c as above are uniquely determined subgroups in Γ. Namely, they respectively are: the subgroup of the elements of p-primary orders and the subgroup of the elements of p-coprime orders.

Example 7.2. Assume that every subextension in a given finite Galois field extension E/F is also Galois. This means that the Galois group of E/F is *Dedekind*, i.e., all its subgroups are normal. According to [7], a given finite group is Dedekind if and only if it is abelian or the direct product of the (non-abelian order 8) quaternion group, an (abelian) group of exponent 2, and an abelian group of odd order. It follows that the group $\mathrm{Gal}(E/F)$ is p-separated.

Lemma 7.3. *Any subgroup of a p-separated group is p-separated. A homomorphic image of a p-separated group is also p-separated. Direct product of two p-separated groups is p-separated.*

Proof. In terms of the decomposition (7.1), any subgroup $H \subset \Gamma$ is the direct product $H_p \times H_c$, where H_p (resp., H_c) is the image of the projection of H to Γ_p (resp., to Γ_c). (The group H_p (resp., H_c) is also the intersection of H with Γ_p (resp., with Γ_c).) It follows that H is p-separated.

The subgroup $H \subset \Gamma$ is normal if and only if H_p is normal in Γ_p and H_c is normal in Γ_c. In that case, the quotient $\Gamma/H = (\Gamma_p/H_p) \times (\Gamma_c/H_c)$ is also p-separated.

The last statement of Lemma 7.3 holds because the direct product of two p-coprime groups (resp., p-groups) is a p-coprime group (resp., p-group). $\square$

A *p-separated Galois extension* is a finite Galois field extension whose Galois group is p-separated. A *p-separated (separable) extension* is a subextension of a p-separated Galois extension. In the extreme case, where the order of the Galois group is p-coprime (resp., a p-power), we get the notion of a p-coprime extension (resp., a p-extension).

Corollary 7.4. *Let G be a reductive group over a field F which acquires inner type over a p-separated extension of F. Then G is p-separately inner. Moreover, for any field extension L/F, the group G_L is also p-separately inner. Two p-separately inner groups acquire inner type over a common p-separated extension.*

Proof. If G acquires inner type over a p-separated extension, then it does it over a (larger) p-separated Galois extension E/F. The Galois group of the minimal field extension (which can be found inside E/F) is a homomorphic image of $\mathrm{Gal}(E/F)$ and so a p-separated group by Lemma 7.3.

For any given field extension L/F, we can find a larger field extension containing L/F and E/F. The group G_L acquires inner type over the composite $L \cdot E$. The field extension $(L \cdot E)/L$ is finite Galois. Its Galois group is isomorphic to a subgroup in $\mathrm{Gal}(E/F)$ and so is a p-separated group by Lemma 7.3.

Concerning the last statement of Corollary 7.4, it suffices to notice that the composite of two p-separated (Galois) extensions of a field F (placed, say, inside the same separable closure of F) is p-separated (Galois). $\square$

Let us consider a p-separated Galois extension E/F and let us write Γ for its Galois group decomposed as in (7.1).

Let L/F be a subextension in E/F and let $H \subset \Gamma$ be the corresponding subgroup. We have the following diagram of subgroups (below on the left) and diagram of subfields (below on the right), where $E_p \subset E$ (resp., $E_c \subset E$) is the subfield of elements invariant under $\Gamma_p \subset \Gamma$ (resp., $\Gamma_c \subset \Gamma$), and where the bars stand for inclusions (represented upside down in the case of groups):

(7.5)
$$\begin{tikzcd}[column sep=small]
 & & 1 \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \Gamma_p \cap H \arrow[dl, dash] \arrow[dr, dash] & & H \cap \Gamma_c \arrow[dl, dash] \arrow[dr, dash] & \\
\Gamma_p \arrow[dr, dash] & & H \arrow[dl, dash] \arrow[dr, dash] & & \Gamma_c \arrow[dl, dash] \\
 & \Gamma_p \cdot H \arrow[dr, dash] & & H \cdot \Gamma_c \arrow[dl, dash] & \\
 & & \Gamma & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & E \arrow[dl, dash] \arrow[dr, dash] & & \\
 & E_p \cdot L \arrow[dl, dash] \arrow[dr, dash] & & L \cdot E_c \arrow[dl, dash] \arrow[dr, dash] & \\
E_p \arrow[dr, dash] & & L \arrow[dl, dash] \arrow[dr, dash] & & E_c \arrow[dl, dash] \\
 & E_p \cap L \arrow[dr, dash] & & L \cap E_c \arrow[dl, dash] & \\
 & & F & &
\end{tikzcd}$$

We set $L_p := E_p \cap L$ and $L_c := L \cap E_c$. For the field extension L/L_c, let A and A' be some of its Artin motives, i.e., direct summands in the motive $M(L)^{L_c}$.

Lemma 7.6. *The Artin L_c-motive A is indecomposable if and only if the Artin F-motive A^F is indecomposable. Besides, $A' \simeq A$ if and only if $A'^F \simeq A^F$.*

Proof. The Artin motive A corresponds to a direct summand of the $\mathbb{F}[H \cdot \Gamma_c]$-module $\mathbb{F}[(H \cdot \Gamma_c)/H]$; the motive is indecomposable if and only if the corresponding module is. The Artin motive A^F is given by the induced summand $B := A \otimes_{\mathbb{F}[H \cdot \Gamma_c]} \mathbb{F}[\Gamma]$ of the $\mathbb{F}[\Gamma]$-module $\mathbb{F}[\Gamma/H]$. We need and are going to show that the $\mathbb{F}[\Gamma]$-module B is indecomposable if and only if the module A is.

Let I be the augmentation ideal of the group ring $\mathbb{F}[\Gamma_p]$. By [15, Corollary 1.11.11], the ideal $J := I \otimes \mathbb{F}[\Gamma_c]$ of the ring $\mathbb{F}[\Gamma] = \mathbb{F}[\Gamma_p] \otimes \mathbb{F}[\Gamma_c]$ is its Jacobson radical. By Nakayama's Lemma [15, Theorem 1.10.4)], the $\mathbb{F}[\Gamma]$-module B is indecomposable if and only if the $\mathbb{F}[\Gamma]/J$-module B/JB is indecomposable. Note that $\mathbb{F}[\Gamma]/J = \mathbb{F}[\Gamma_c]$ and the quotient ring homomorphism $\mathbb{F}[\Gamma] \to \mathbb{F}[\Gamma]/J$ is the homomorphism $\mathbb{F}[\Gamma_p] \times \mathbb{F}[\Gamma_c] \to \mathbb{F}[\Gamma_c]$ induced by the projection $\Gamma_p \times \Gamma_c \to \Gamma_c$.

Viewing now A as a Γ-module, note that it is still indecomposable. Since the subgroup $H \subset \Gamma$ acts on A trivially, A is also an indecomposable $\mathbb{F}[\Gamma_c]$-module. The $\mathbb{F}[\Gamma_c]$-module B/JB is computed as

$$B/JB = A \otimes_{\mathbb{F}[\Gamma_c]} \mathbb{F}[\Gamma_c] = A.$$

This gives the first statement of Lemma 7.6. Since the above formula reconstructs A from B, the second statement of Lemma 7.6 also follows. □

Remark 7.7. Lemma 7.6 and the Krull-Schmidt property [11, Corollary 2.2] imply that the association $A \mapsto A^F$ induces a bijection of the set of isomorphism classes of indecomposable Artin motives given by the p-separated extension L/F with the analogues set given by the p-coprime extension L/L_c. Besides, the association $B \mapsto B_{L_c}$ yields a bijection of the analogues sets for L_p/F and L/L_c. In particular, every indecomposable summand in $M(L)^F$ is isomorphic to $B^F_{L_c}$ for a uniquely (up to an isomorphism) determined indecomposable summand B in $M(L_p)^F$.

Let G be a reductive group over F. For L and A as in Lemma 7.6, let us consider the motive $U_A(Y)$ given by A and a projective G_L-homogeneous L-variety Y. This motive (which lives over the field $L_c = L \cap E_c$) has been defined just before Proposition 6.4; it is indecomposable by Proposition 6.4.

Proposition 7.8. *The F-motive $U_A(Y)^F$ is indecomposable.*

Proof. Since E_c/F is a Galois p-extension and L_c/F is its subextension, there is a chain

$$L_c = F_n \supset F_{n-1} \supset \cdots \supset F_1 \supset F_0 = F$$

of degree p Galois field extensions. Employing induction, we may assume that the motive $U := U_A(Y)^{F_1}$ is indecomposable. We need to check that the motive $U^F = U_A(Y)^F$ is also indecomposable.

The F_1-motive $U^F_{F_1} = (U^F)_{F_1}$ is the direct sum of the indecomposable motives U_σ with σ running over the Galois group $\mathrm{Gal}(F_1/F)$, where U_σ is the base change of U via $\sigma\colon F_1 \to F_1$. To prove indecomposability of U^F, we take its nonzero direct summand U' and check that U'_{F_1} is still the sum of all U_σ. What we know a priori is that U'_{F_1} is the sum of some nonzero number of U_σ. Since $\dim \mathrm{Ch}^0(U_\sigma)_E = [F_n : F_1] \cdot \mathrm{rk}\, A$, to show that all U_σ are involved in the sum, it suffices to show that $\dim \mathrm{Ch}^0(U'_E) \geq [F_n : F] \cdot \mathrm{rk}\, A$. In the above formulas, $\mathrm{rk}\, A$ is the rank of A, i.e., the number of (Tate) summands in the complete decomposition of A over its splitting field.

Let X be the F-variety of Borel subgroup in G. The motive $U_A(Y)_{F_n(X)}$ is a direct sum of the Artin motive $A_{F_n(X)}$ and some positive shifts of some A-upper motives. It follows that the (indecomposable by Lemma 7.6 and Corollary 8.8) motive $A^F_{F(X)}$ is a summand of U'. Therefore

$$\dim \mathrm{Ch}^0(U'_E) \geq \dim \mathrm{Ch}^0(A^F_{E(X)}) = [F_n : F] \cdot \mathrm{rk}\, A. \qquad \square$$

As a consequence, we get an analogue of Corollary 5.6, where a finite Galois field extension L/F is replaced by a p-separated extension L/F:

Corollary 7.9. *For any L/F and Y as in Proposition 7.8 and the additive functor $\mathbf{m}$ of §3, the following holds:*

1. *Every summand in $M(L)^F$ is isomorphic to the image under $\mathbf{m}$ of a summand in $U(Y)^F$.*
2. *Two summands in $U(Y)^F$ with isomorphic images under $\mathbf{m}$ are isomorphic.*
3. *A summand in $U(Y)^F$ is indecomposable if and only if its image under $\mathbf{m}$ is so.* $\square$

Definition 7.10. Given a finite separable field extension L/F and a projective homogeneous L-variety Y, indecomposable summands of $U(Y)^F$, whose images under $\mathbf{m}$ are also indecomposable, will be called *A-upper* F-motives of Y, where "A" honors Emil Artin and Artin motives.

Assuming that L/F is p-separated, that Y is G_L-homogeneous for a reductive F-group G, and given an indecomposable summand A of $M(L)^F$, we will write $U_A(Y)$ for the corresponding (defined up to an isomorphism) A-upper motive of Y. (The base field F of the motive $U_A(Y)$ does not show up in the notation because it is concealed in the motive A.)

Definition 7.11. Let G be a p-separately inner algebraic group over F and let E/F be a minimal field extension such that G_E is of inner type. A motive M over an intermediate field L of E/F is an *A-upper motive of* G if there is an indecomposable direct summand A of $M(L)^F$, and a projective G_L-homogeneous L-variety Y such that M is isomorphic to $U_A(Y)$.

Note that for a given G, the field extension E/F in the above definition is uniquely determined up to an isomorphism so that its choice does not influence the notion of A-upper motives of G.

8. Motivic decompositions

The following result generalizes [13, Theorem 3.5] (dealing with the case of inner type G) as well as [11, Theorem 1.1] (dealing with the p-inner case):

Theorem 8.1. *Let G be a p-separately inner algebraic group. Every summand in the complete decomposition of the Chow motive with coefficients in $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ of any projective G-homogeneous variety X is a Tate shift of an A-upper motive of G.*

Proof. We modify the proof of [11, Theorem 1.1]. Since the center of G acts on X trivially, we may assume that G is semisimple and adjoint.

We write D_G (or simply D) for the set of vertices of the Dynkin diagram of G. We write F for the base field of G and let E/F be a Dedekind field extension with inner G_E. The Galois group $\Gamma = \mathrm{Gal}(E/F)$ of the field extension E/F acts on D. For a field L with $F \subset L \subset E$, the set D_{G_L} is identified with $D = D_G$. Any $\mathrm{Gal}(E/L)$-stable subset τ in D determines a projective G_L-homogeneous variety $Y_{G_L,\tau}$ the way described in [13, §3] (which is opposite to the original convention of [17, §1.6]). For instance, $Y_{G_L,D}$ is the variety of Borel subgroups of G_L, and $Y_{G_L,\emptyset} = \operatorname{Spec} L$. Any projective G_L-homogeneous variety is isomorphic to $Y_{G_L,\tau}$ for some $\mathrm{Gal}(E/L)$-stable $\tau \subset D$. Given an indecomposable summand A of the motive $M(L)^F$, we write $U_{G_L,\tau,A}$ for the corresponding (in the sense of Corollary 5.6) indecomposable summand $U_A(Y_{G_L,\tau})$ of $U(Y_{G_L,\tau})^F$.

We prove Theorem 8.1 simultaneously for all F, G, X using induction on $n := \dim X$. The base of the induction is $n = 0$ where $X = \operatorname{Spec} F$ and the statement is trivial.

From now on we are assuming that $n \geq 1$ and that Theorem 8.1 is already proven for varieties of dimension $< n$.

For any field extension L/F, we write $\tilde{L}$ for the function field $L(X)$ (note that any projective homogeneous variety and, in particular, X is geometrically integral). Let G' be the semisimple group over the field $\tilde{F} = F(X)$ given by the semisimple anisotropic kernel of the group $G_{\tilde{F}}$. We note that the group $G'_{\tilde{E}}$ is of inner type. The field extension $\tilde{E}/\tilde{F}$ is Galois with the Galois group

$$\Gamma = \mathrm{Gal}(\tilde{E}/\tilde{F}) = \mathrm{Gal}(E/F)$$

(see Lemma 8.7). In particular, any its intermediate field is of the form $\tilde{L}$ for some intermediate field L of the extension E/F; moreover, the indecomposable summands of the motive $M(L)^F$ are in one-to-one correspondence with the indecomposable summands of $M(\tilde{L})^{\tilde{F}}$ (see Corollary 8.8). The set $D_{G'}$ is identified with a Γ-invariant subset in D_G; the complement $D_G \setminus D_{G'}$ contains the subset in D_G corresponding to X.

Let M be an indecomposable summand of the motive of X. We are going to show that M is isomorphic to a shift of $U_{G_L,\tau,A}$ for some intermediate field L of E/F, some $\mathrm{Gal}(E/L)$-stable subset $\tau \subset D_G$ containing the complement of $D_{G'}$, and some A. This will prove Theorem 8.1.

According to [1, Theorem 4.2] (an enhancement of [2, Theorem 7.5]), the motive of $X_{\tilde{F}}$ decomposes into a sum of shifts of motives of projective $G'_{\tilde{L}}$-homogeneous (where L runs over intermediate fields of the extension E/F) varieties Y, satisfying $\dim Y < \dim X = n$. It follows by the induction hypothesis that each summand of the complete motivic decomposition of $X_{\tilde{F}}$ is a shift of $U_{G'_{\tilde{L}},\tau',A'}$ for some L, some $\tau' \subset D_{G'}$, and some A' – an indecomposable summand in $M(\tilde{L})^{\tilde{F}}$. By the Krull-Schmidt property [11, Corollary 2.2], the complete decomposition of $M_{\tilde{F}}$ consists of shifts of some of these $U_{G'_{\tilde{L}},\tau',A'}$.

In the complete decomposition of $M_{\tilde{F}}$, let us choose a summand $N' := U_{G'_{\tilde{L}},\tau',A'}\{i\}$ with minimal i. We set $\tau := \tau' \cup (D_G \setminus D_{G'}) \subset D_G$. We will show that

$$M \simeq N := U_{G_L,\tau,A}\{i\}$$

for these L, τ, and i, where A is the summand in $M(L)^F$ from Corollary 8.8 satisfying $A_{\tilde{F}} = A'$. Since M is indecomposable, it suffices to construct morphisms

$$\alpha : N \to M \quad \text{and} \quad \beta : M \to N$$

satisfying the condition $\beta \circ \alpha = \mathrm{id}_N$. Since N is indecomposable, the condition on the composition is satisfied if (and only if) over some extension of the base field a power of the composition is a nonzero projector. We recall that by [13, Corollary 2.2], an appropriate power of any endomorphism of N (over any field extension of the base) is a projector; the point of the formulated condition is the non-vanishing of the projector.

We first construct predecessors $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β defined over the field $\tilde{F}$. Note that N' is a summand of $N_{\tilde{F}}$ as well as of $M_{\tilde{F}}$. Using projections to and inclusions of the direct summand, we define $\tilde{\alpha}$ and $\tilde{\beta}$ as the compositions

$$\tilde{\alpha}\colon N_{\tilde{F}} \to N' \to M_{\tilde{F}} \quad \text{and} \quad \tilde{\beta}\colon M_{\tilde{F}} \to N' \to N_{\tilde{F}}.$$

The composition $\tilde{\beta} \circ \tilde{\alpha}$ is the (nonzero) projector which yields the summand N' of $N_{\tilde{F}}$.

Recall that the F-motive N is a shift of the summand $U_A(Y)$ of $U(Y)^F$, where Y is the projective homogeneous L-variety $Y := Y_{G_L,\tau}$. Therefore $\mathrm{End}\, N \subset \mathrm{End}\, U(Y)^F$ and the homomorphism m of (5.3) is defined on $\mathrm{End}\, N$. Extending the base field F to $\tilde{F}$, let us consider a restriction of the ring homomorphism of (5.1)

$$\mathrm{End}\, N_{\tilde{F}} \hookrightarrow \mathrm{End}\, M(Y^F)_{\tilde{F}} = \mathrm{End}\, M(Y_{\tilde{L}})^{\tilde{F}} \to \mathrm{End}\, M(\tilde{L})^{\tilde{F}} = \mathrm{End}\, M(L)^F. \tag{8.2}$$

The image under it of the composition $\tilde{\beta} \circ \tilde{\alpha}$ is the (nonzero) projector corresponding to the Artin motive A.

Now we construct α and β starting with α. Note that $\tilde{\alpha}$ is an element of the Chow group $\mathrm{Ch}(Y^F \times X)_{\tilde{F}}$ over $\tilde{F}$. We take for α an element of the Chow group $\mathrm{Ch}(Y^F \times X)$

over F such that its image under the surjective ring homomorphism

$$\mathrm{Ch}(Y^F \times X) \to \mathrm{Ch}(X_{F(Y^F)})$$

(from [8, Corollary 57.11]) followed by the change of field homomorphism for the field extension $\tilde{F}(Y^F)/F(Y^F)$, coincides with the image of $\tilde{\alpha}$ under the surjective ring homomorphism

$$\mathrm{Ch}(Y^F \times X)_{\tilde{F}} \to \mathrm{Ch}(X_{\tilde{F}(Y^F)}).$$

Such α exists because the field extension $\tilde{F}(Y^F)/F(Y^F)$ is purely transcendental and therefore the change of field homomorphism $\mathrm{Ch}(X_{F(Y^F)}) \to \mathrm{Ch}(X_{\tilde{F}(Y^F)})$ is surjective as follows from the homotopy invariance of Chow groups (see [8, Theorem 57.13] or [8, Corollary 52.11]) and [8, Corollary 57.11].

In order to define β, we note that $\tilde{\beta}$ is an element of $\mathrm{Ch}(X \times Y^F)_{\tilde{F}}$ and let β' be an element of $\mathrm{Ch}(X \times X \times Y^F)$ mapped to $\tilde{\beta}$ under the surjection (from [8, Corollary 57.11])

$$\mathrm{Ch}(X \times X \times Y^F) \to \mathrm{Ch}(X \times Y^F)_{\tilde{F}}$$

given by the generic point of the *second* factor in the product $X \times X \times Y^F$. We consider β' as a correspondence $X \rightsquigarrow X \times Y^F$ and let β'' be the composition of correspondences $\beta' \circ \mu$, where $\mu \in \mathrm{Ch}(X \times X)$ is the projector which yields the motivic summand M of X. Finally, we define β as the pullback of β'' with respect to the closed embedding

$$X \times Y^F \hookrightarrow X \times X \times Y^F, \quad (x, y) \mapsto (x, x, y)$$

given by the diagonal of X.

By construction, the image under (8.2) of $(\beta \circ \alpha)_{\tilde{F}}$ coincides with the image of $\tilde{\beta} \circ \tilde{\alpha}$: the detailed verification made in [13, End of Proof of Theorem 3.5] for $L = F$ carries over the general case. Therefore a power of $\beta \circ \alpha$ is a nonzero projector. □

Remark 8.3. Instead of [1, Theorem 4.2], the weaker result [2, Theorem 7.5] can be used in the proof of Theorem 8.1. To do so, it suffices to take for G' the semisimple part of the parabolic subgroup defining $X_{\tilde{F}}$.

Remark 8.4. The A-upper motives, whose Tate shifts are direct summands of $M(X)$ in Theorem 8.1, are associated with varieties *dominating* X in the sense of [4] (see also [6, Lemma 2.2]). This can be seen directly using [6, Lemma 2.2] or deduced from the proof of Theorem 8.1.

Remark 8.5. Let G be a reductive group over a field F. Assume that G becomes quasisplit over some finite field extension of F of p-coprime degree. In this case, for any field extension L/F, the upper motive of any projective G_L-homogeneous variety over L is the Tate motive $\mathbb{F} = M(L)$. If G is also p-separately inner, it follows by Theorem 8.1 that every summand in the complete motivic decomposition of any projective G-homogeneous variety over F is a shift of an Artin motive given by an intermediate field of a minimal field extension over which G acquires inner type. In fact, since no A-upper motives aside from the classical Artin motives show up here, the proof of Theorem 8.1 goes through and the above statement holds without the assumption that G is p-separately inner.

Remark 8.6. Recall that any projective homogeneous variety is so under an adjoint semisimple group G. Let D be the Dynkin diagram of such G and let G_0 be the corresponding split adjoint semisimple group. Write P for the set of torsion primes of G_0 (these are the prime divisors of the torsion index of G_0 determined in [18]) together with the prime divisors of the order $|\operatorname{Aut} D|$. Then for any prime p outside of P, the group G splits over some finite field extension of F of p-coprime degree; in particular, the situation of Remark 8.5 occurs.

Here is the constitution of the set P for every absolutely simple G depending on its type: 2 and prime divisors of $n+1$ for A_n, $n \geq 1$; just 2 for B_n and C_n with $n \geq 2$ as well as for G_2 and D_n with $n \geq 5$; 2 and 3 for D_4, F_4, and E_6; 2, 3, 5 for E_8. (The group $\operatorname{Aut} D$ is non-trivial here for A_n, D_4, and E_6 only.)

The following lemma and corollary have been applied in the proof of Theorem 8.1 and earlier – in Proposition 7.8:

Lemma 8.7. *Let X be a geometrically integral variety over a field F and let E/F be a finite Galois field extension. Then $E(X)/F(X)$ is also a finite Galois field extension and its Galois group $\tilde{\Gamma}$ is isomorphic to $\Gamma := \operatorname{Gal}(E/F)$.*

Proof. The extension $E(X)/F(X)$ is algebraic, normal, and separable; therefore it is Galois. Since E is algebraically closed in $E(X)$, any element of $\tilde{\Gamma}$ maps E to E. Since the subfields E and $F(X)$ both together generate the field $E(X)$, the group homomorphism $\tilde{\Gamma} \to \Gamma$, $\sigma \mapsto \sigma|_E$ is injective. Since any element of E, which is stable under the image of $\tilde{\Gamma}$, belongs to $E \cap F(X) = F$, the image of $\tilde{\Gamma}$ is the entirety of Γ. □

Corollary 8.8. *Let X be a geometrically integral F-variety. Let L/F be a subextension of a finite Galois field extension E/F. For any direct summand $\tilde{A}$ of the motive $M(L(X))^{F(X)}$, there is one and only one direct summand A of $M(L)^F$ satisfying $A_{F(X)} = \tilde{A}$. The motive $\tilde{A}$ is indecomposable if and only if A is. Direct summands A and A' of $M(L)^F$ with isomorphic $A_{F(X)}$ and $A'_{F(X)}$ are isomorphic.* □

9. Criterion of isomorphism for A-upper motives

Let L/F and L'/F be p-separated extensions, let Y be a projective G_L-homogeneous variety over L and Y' a projective $G_{L'}$-homogeneous variety over L', where G and G' are reductive algebraic groups over F. Let A be an Artin motive isomorphic to an indecomposable direct summand of $M(L)^F$ and let A' be an Artin motive isomorphic to an indecomposable direct summand of $M(L')^F$.

We are going to formulate a criterion of isomorphism for the A-upper F-motives $U_A(Y)$ and $U_{A'}(Y')$. We start with

Proposition 9.1. *If $U_A(Y) \simeq U_{A'}(Y')$, then $A \simeq A'$.*

Proof. Applying the functor $\mathbf{m}$ to an isomorphism $U_A(Y) \to U_{A'}(Y')$, we get an isomorphism $A \to A'$. □

Recall from [6, §2] that the variety Y^F *dominates* Y'^F if there is a multiplicity 1 correspondence $Y^F \rightsquigarrow Y'^F$. The varieties Y^F and Y'^F are *equivalent*, $Y^F \approx Y'^F$, if each of them dominates the other.

Theorem 9.2. *The motives $U_A(Y)$ and $U_{A'}(Y')$ are isomorphic if and only if $A \simeq A'$ and $Y^F \approx Y'^F$.*

Proof. By Proposition 9.1, we may assume that $A \simeq A'$.

The Artin motive $A_{F(Y^F)}$ is a direct summand in $U_A(Y)_{F(Y^F)}$. Assuming $U_A(Y) \simeq U_{A'}(Y')$, we conclude that the Artin motive $A_{F(Y^F)} \simeq A'_{F(Y^F)}$ is also a direct summand in $U_{A'}(Y')_{F(Y^F)}$. This implies that the Tate motive $\mathbb{F}$ is a direct summand in $U(Y'^F)_{F(Y^F)}$ and so the variety $(Y'^F)_{F(Y^F)}$ is isotropic (i.e., has a 0-cycle of degree $1 \in \mathbb{F}$), which means that Y^F dominates Y'^F. Similarly, Y'^F dominates Y^F and we conclude that $Y^F \approx Y'^F$.

Conversely, assume that

$$Y^F \approx Y'^F. \tag{9.3}$$

Let L_p and L_c be the intermediate fields in L/F as in §7; let L'_p and L'_c be the similar intermediate fields in L'/F. Note that there is a common p-separated Galois extension containing both L/F and L'/F.

By Remark 7.7, we can find an indecomposable direct summand B in $M(L_p)^F$ with $B^F_{L_c} \simeq A$. Similarly, we can find an indecomposable direct summand B' in $M(L'_p)^F$ with $B'^F_{L'_c} \simeq A'$. Note that $B^F_{L_c} = B \otimes M(L_c)^F$ by the general formula mentioned in the proof of Lemma 6.3; similarly, $B'^F_{L'_c} = B' \otimes M(L'_c)^F$.

We claim that $B' \simeq B$. To prove the claim, note that both L_c and L'_c are contained in a common Galois p-extension E/F. Since the natural map $\mathrm{Gal}((L_p \cdot E)/E) \to \mathrm{Gal}(L_p/F)$ (resp., $\mathrm{Gal}((L'_p \cdot E)/E) \to \mathrm{Gal}(L'_p/F)$) is an isomorphism, the change of field from F to E yields a bijection of the set of isomorphism classes of indecomposable direct summands in $M(L_p)^F$ (resp., $M(L'_p)^F$) with the corresponding set for $M(L_p \cdot E)^E$ (resp., $M(L'_p \cdot E)^E$). In particular, the Artin E-motives B_E and B'_E are still indecomposable. Since the motive $M(K)^F_E$ (resp., $M(K')^F_E$) is split, the motive A_E (resp., A'_E) is a direct sum of several copies of B_E (resp., B'_E). Since $A \simeq A'$, we have an isomorphism $A_E \simeq A'_E$ implying that there is an isomorphism $B_E \simeq B'_E$ and so an isomorphism $B \simeq B'$ of the claim.

It follows that

$$\begin{aligned} U_A(Y) \simeq U_{B_{L_c}}(Y)^F \simeq \big(U(Y^{L_c}) \otimes B_{L_c}\big)^F \simeq U(Y^{L_c})^F \otimes B \simeq U(Y^F) \otimes B \simeq \\ \simeq U(Y'^F) \otimes B' \simeq U(Y'^{L'_c})^F \otimes B' \simeq \big(U(Y'^{L'_c}) \otimes B'_{L'_c}\big)^F \simeq U_{B'_{L'_c}}(Y')^F \simeq U_{A'}(Y'). \end{aligned}$$

The third and the seventh isomorphisms here are particular cases of the general formula mentioned in the proof of Lemma 6.3. □

Let M and M' be F-motives which are finite direct sums, where each summand N is a shift of the motive $U_A(Y)$ for some reductive algebraic group G over F, some projective G_L-homogeneous variety Y over a p-separated extension L/F, and for an indecomposable summand A in $M(L)^F$ (where G, L, Y, A may vary with N). For any such A, let M_A be the sum of the summands in M involving an Artin motive isomorphic to A. We say that M and M' have *isomorphic higher Artin-Tate traces*, if for every (isomorphism class of) A the motives M_A and M'_A have *isomorphic higher Tate traces* as defined in [6, Remark 3.16].

Corollary 9.4. *The motives M and M' are isomorphic if and only if they have isomorphic higher Artin-Tate traces.*

Proof. If M and M' are isomorphic, then by Proposition 9.1 and the Krull-Schmidt property, the motives M_A and M'_A are isomorphic and have isomorphic higher Tate traces, for any indecomposable Artin motive A.

Conversely, assume that M and M' have isomorphic higher Artin-Tate traces. Given an indecomposable Artin motive A, we prove that M_A and M'_A are isomorphic by induction on the maximum of the numbers of summands in their complete motivic decompositions. If this maximum is zero, both M_A and M'_A are trivial. Else, write

$$M_A = U_A(X_1)\{n_1\} \oplus .. \oplus U_A(X_k)\{n_k\} \text{ and } M'_A = U_A(Y_1)\{m_1\} \oplus ... \oplus U_A(Y_s)\{m_s\}.$$

We may assume that $n = \min_{1\leq i\leq k} n_i$ is not higher than $m = \min_{1\leq j\leq s} m_j$. Pick an integer $1 \leq \alpha \leq k$ such that X^F_α is minimal for the domination relation among the X^F_i's such that $U_A(X_i)\{n\}$ is a direct summand in the above decomposition of M_A. By assumption on the higher Tate traces of M_A and M'_A, the latter contains a Tate motive $\mathbb{F}\{n\}$ over the function field of X^F_α. It follows that $n = m$ and that M'_A contains a direct summand isomorphic to $U_A(Y_\beta)\{n\}$, for some $1 \leq \beta \leq s$, such that X^F_α dominates Y^F_β. The same reasoning over the function field of Y_β implies that a direct summand of M_A is isomorphic to $U_A(X_\gamma)\{n\}$ for some $1 \leq \gamma \leq k$, where X^F_γ is dominated by Y^F_β.

The varieties X^F_α and Y^F_β are equivalent, by minimality of X^F_α. The A-upper motives $U_A(X_\alpha)$ and $U_A(Y_\beta)$ are then isomorphic by Theorem 9.2. Induction, applied to the summands $\tilde{M}_A$ and $\tilde{M}'_A$ given by the decompositions $M_A = U_A(X_\alpha)\{n\} \oplus \tilde{M}_A$ and $M'_A = U_A(Y_\beta)\{n\} \oplus \tilde{M}'_A$, proves that M_A and M'_A are isomorphic. □

The previous result shows that for the p-separately inner groups, isomorphism classes of direct summands of motives of projective homogeneous varieties are determined by their higher Artin-Tate traces. The following example shows further that the higher Tate traces of [6] are already not sufficient to distinguish between non-isomorphic Artin motives arising from finite Galois field extensions of prime degree. Namely, the Artin motives A and B (each of which is a twisted form of the Tate motive $\mathbb{F}$), constructed in the example, are not isomorphic and though have isomorphic higher Tate traces.

Example 9.5. Let $p = 7$ and let E/F be a cubic Galois field extension. Recall that the $\mathbb{F}$-algebra $\operatorname{End} M(E)^F$ is identified with the group algebra $\mathbb{F}[\Gamma]$, where $\Gamma := \mathrm{Gal}(E/F)$, and therefore with the $\mathbb{F}$-algebra $\mathbb{F}[x]/(x^3 - 1)$. Since the polynomial $x^3 - 1 \in \mathbb{F}[x]$ splits into the product of three linear factors

$$x^3 - 1 = (x - 1)(x - 2)(x + 3),$$

the $\mathbb{F}$-algebra $\operatorname{End} M(E)^F$ is isomorphic to the product $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$ of three copies of $\mathbb{F}$, and the motive $M(E)^F$ splits as $\mathbb{F} \oplus A \oplus B$ for some Artin motives A and B.

For any field extension K/F, the K-algebra $K \otimes_F E$ is either still a cubic Galois field extension or the split étale K-algebra $K \times K \times K$. In the latter case, we have $A_K = \mathbb{F} = B_K$. We will verify that in the former case, the motives $\mathbb{F}$, A, B over F are pairwise non-isomorphic. We may assume that $K = F$ for this verification.

A rank 1 direct summand of the motive $M(E)^F$ (we recall that the rank is the number of (Tate) summands in the complete decomposition of the motive over an algebraic closure of its base field) is given by a Γ-invariant dimension 1 ideal in the group algebra $\mathbb{F}[\Gamma]$. Let σ be a generator of the group Γ. The elements

$$u := 1 + \sigma + \sigma^2,\ v := 1 + 2\sigma - 3\sigma^2,\ w := 1 - 3\sigma + 2\sigma^2\ \in \mathbb{F}[\Gamma]$$

satisfy

$$\sigma u = u,\ \sigma v = -3v,\ \sigma w = 2w \tag{9.6}$$

and therefore each of them does generate a Γ-invariant 1-dimensional ideal. Note that u, v, w are linearly independent over $\mathbb{F}$ so that the $\mathbb{F}[\Gamma]$-module $\mathbb{F}[\Gamma]$ decomposes as

$$\mathbb{F}[\Gamma] = \mathbb{F}u \oplus \mathbb{F}v \oplus \mathbb{F}w.$$

Moreover, since multiplication by σ yields multiplication by three different constants in the three formulas of (9.6), the three ideals are pairwise non-isomorphic (as $\mathbb{F}[\Gamma]$-modules). By the Krull-Schmidt property, these three ideals correspond to the motives $\mathbb{F}$, A, B. In particular, the three motives are also pairwise non-isomorphic.

10. Motivic equivalence

In this section we produce a criterion of motivic equivalence for p-separately inner reductive algebraic groups which are inner forms of each other. We remind that absolutely simple algebraic groups of any type other than ${}^6D_4$ are p-separately inner.

Recall that a projective homogeneous variety is *isotropic* (with coefficients in $\mathbb{F}$) if it possesses a 0-cycle of degree coprime to p.

Proposition 10.1. *Let K be the function field of a projective homogeneous F-variety X.*

i) If L/F is a p-separated extension and A is an indecomposable direct summand in the F-motive $M(L)^F$, then the K-motive A_K is indecomposable.

ii) Let G and G' be reductive algebraic groups over F, let L/F, L'/F be two p-separated extensions, and let Y, Y' be projective homogeneous varieties over L and L' under G_L and $G'_{L'}$, respectively, each of which dominates X. If the A-upper K-motives $U_{A_K}(Y_K)$ and $U_{A'_K}(Y'_K)$ are isomorphic, then the F-motives $U_A(Y)$ and $U_{A'}(Y')$ are isomorphic as well.

Proof. Given an indecomposable summand A of $M(L)^F$, the K-motive A_K is indecomposable by Corollary 8.8, proving *i*).

We prove *ii*) using Theorem 9.2. First, by Corollary 8.8 once again, if the Artin K-motives A_K and A'_K are isomorphic, then the F-motives A and A' are isomorphic.

Assume that $U_{A_K}(Y_K)$ and $U_{A'_K}(Y'_K)$ are isomorphic. By Theorem 9.2, the K-varieties $(Y^F)_K$ and $(Y'^F)_K$ are equivalent and $A_K \simeq A'_K$, hence $A \simeq A'$. By [4, Proof of Proposition 9], Y and Y' are equivalent and so $U_A(Y) \simeq U_{A'}(Y')$ by Theorem 9.2. $\square$

Let G be a reductive group over F. Recall that we write D_G for its Dynkin diagram, which can be canonically attached to G using the generic point of the variety of pairs $T \subset B$ with T a maximal torus and B a Borel subgroup. Sometimes, depending on the context, D_G stands for the set of vertices of the Dynkin diagram.

Any $\mathrm{Gal}(\bar{F}/F)$-invariant subset of D_G, where $\bar{F}$ is a separable closure of F, yields a projective G-homogeneous variety (we keep the same convention as in the proof of Theorem 8.1). This induces a bijection between the $\mathrm{Gal}(\bar{F}/F)$-invariant subsets of D_G

and the isomorphism classes of projective G-homogeneous varieties. An invariant subset $\tau \subset D_G$ is *p-distinguished*, if the associated projective G-homogeneous variety $X_{G,\tau}$ is isotropic. The union of all p-distinguished orbits yields the largest p-distinguished subset, denoted D_G^p (see [5]).

We are going to consider two reductive groups G and G' each of which is an inner form of the other. In such a situation, the Dynkin diagrams D_G and $D_{G'}$ are $\mathrm{Gal}(\bar{F}/F)$-equivariantly isomorphic and we will be fixing one of the possible isomorphisms.

Proposition 10.2. *Let G and G' be p-separately inner reductive groups over F, inner forms of each other. Fix an equivariant isomorphism of their Dynkin diagrams $\varphi\colon D_G \longrightarrow D_{G'}$ and an invariant subset τ_0 of D_G. The following conditions on G, G', τ_0, and φ are equivalent:*

- *i) for any field extension K/F, one has $\tau_0 \subset D^p_{G_K}$ (i.e., τ_0 is p-distinguished over K) if and only if $\varphi(\tau_0) \subset D^p_{G'_K}$; moreover, $\varphi(D^p_{G_K}) = D^p_{G'_K}$ in this case;*
- *ii) for any minimal field extension E/F such that G_E (and G'_E) are of inner type, any field extensions L/F contained in E, any indecomposable summand A of the motive $M(L)^F$, and any $\mathrm{Gal}(E/L)$-invariant subset $\tau \subset D_G$ containing τ_0, the A-upper motives $U_A(X_{G_L,\tau})$ and $U_A(X_{G'_L,\varphi(\tau)})$ are isomorphic.*

Proof. *i)* $\Rightarrow$ *ii)* Assuming *i)*, fix a field extension L/F contained in E, an Artin motive A, and a subset $\tau \supset \tau_0$ as in *ii)*. The subset τ_0 is p-distinguished for G over the function field $\tilde{L}$ of the variety $X_{G_L,\tau}$. It follows from *i)* that the subset $\varphi(\tau) \subset D_{G'}$ is p-distinguished over $\tilde{L}$. The L-variety $X_{G_L,\tau}$ thus dominates $X_{G'_L,\varphi(\tau)}$. The same reasoning with $\varphi(\tau)$ and the inverse of φ implies that the L-varieties $X_{G_L,\tau}$ and $X_{G'_L,\varphi(\tau)}$ are equivalent. It follows that the F-varieties $X^F_{G_L,\tau}$ and $X^F_{G'_L,\varphi(\tau)}$ are equivalent and hence the A-upper motives $U_A(X_{G_L,\tau})$ and $U_A(X_{G'_L,\varphi(\tau)})$ are isomorphic by Theorem 9.2.

ii) $\Rightarrow$ *i)* First, given a field extension K/F, the variety X_{G_K,τ_0} is isotropic if and only if $X_{G_K,\varphi(\tau_0)}$ is isotropic as well, since by assumption *ii)* (with $L = F$) the upper motives $U(X_{G,\tau_0})$ and $U(X_{G',\varphi(\tau_0)})$ are isomorphic. This means that τ_0 is p-distinguished over K if and only if $\varphi(\tau_0)$ is.

Now fix a field extension K/F such that τ_0 is p-distinguished over K. Fix a minimal subextension L/F of K such that $D^p_{G_K} \subset D_G$ is $\mathrm{Gal}(E/L)$-invariant, for some minimal field extension E/F over which G (and G') become of inner type. By assumption *ii)*, the A-upper motives $U_A(X_{G_L,D^p_{G_K}})$ and $U_A(X_{G'_L,\varphi(D^p_{G_K})})$ are isomorphic for any A, thus, by Theorem 9.2, the L-varieties $X_{G_L,D^p_{G_K}}$ and $X_{G'_L,\varphi(D^p_{G_K})}$ are equivalent. As L is contained in K, it follows that the K-varieties $X_{G_K,D^p_{G_K}}$ and $X_{G'_K,\varphi(D^p_{G_K})}$ are also equivalent.

Since the first of the two equivalent K-varieties is isotropic, the second one is also isotropic (see, e.g., [6, Lemma 2.2] and [13, Corollary 2.15]) which means that the subset $\varphi(D^p_{G_K})$ is p-distinguished for G' over K. The same reasoning with G replaced by G', τ_0 by $\varphi(\tau_0)$, and φ by its inverse, gives that $\varphi^{-1}(D^p_{G'_K}) \subset D^p_{G_K}$. Hence $\varphi(D^p_{G_K}) = D^p_{G'_K}$. $\square$

Let G be a reductive algebraic group over a field F. Recall that the classical Tits index of G is its Dynkin diagram D_G, endowed with the action of the absolute Galois group of F, together with the subset D^0_G of *distinguished vertices*. A vertex of D_G is distinguished if it is contained in a Galois orbit τ such that the projective homogeneous variety $X_{G,\tau}$ has a rational point.

For any subset τ of D_G, let us consider the minimal subextension F_τ/F in $\bar{F}/F$ such that τ is $\mathrm{Gal}(\bar{F}/F_\tau)$-invariant. The F-motive $M_{G,\tau} := M(X_{G_{F_\tau},\tau})^F$ is called the *standard motive of G of type τ*. Up to isomorphism, the motive $M_{G,\tau}$ does not depend on the choice of the separable closure $\bar{F}/F$. If τ is $\mathrm{Gal}(\bar{F}/F)$-invariant, it is simply the motive of the projective G-homogeneous variety $X_{G,\tau}$.

We now introduce a set of integers describing motivic decompositions. Let G be a p-separately inner reductive group, E/F a Galois p-extension such that G_E is of inner type, X a projective G-homogeneous variety, and M a direct summand in $M(X)$. For any A-upper F-motive $U_A(Y)$ and any integer n, we write $l_{A,Y,n}(M)$ for the number of indecomposable summands isomorphic to $U_A(Y)\{n\}$ in a complete decomposition of M.

Theorem 10.3. *Let G and G' be p-separately inner reductive groups over a field F which are inner forms of each other. Let τ_0 be a invariant subset in D_G. The equivalent conditions of Proposition 10.2 are satisfied if and only if for any subset $\tau \subset D_G$ containing τ_0, the motives $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic.*

Proof. The "if" part is clear: if the motives $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic, then for any intermediate field L of a minimal field extension E/F such that G_E and G'_E are of inner type, the varieties $X^F_{G_L,\tau}$ and $X^F_{G'_L,\varphi(\tau)}$ are equivalent. Hence, by Theorem 9.2, G and G' satisfy condition $ii)$ of Proposition 10.2.

We prove the opposite implication by induction on the (common) semisimple rank of G and G'. More concretely, assuming the conditions of Proposition 10.2, we will prove that for every $\tau \supset \tau_0$ the motives $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic. For $\tau = \emptyset$ the isomorphism trivially holds. This covers the rank zero case, which is the base of the induction. Below we assume that $\tau \neq \emptyset$.

We first show that $M_{G,\tau}$ and $M_{G',\varphi(\tau)}$ are isomorphic if τ and $\varphi(\tau)$ are $\mathrm{Gal}(E/F)$-invariant and the associated varieties both have a rational point (hence the reductive algebraic groups G and G' are isotropic).

Let $\tilde{G}$ be the semisimple part of a parabolic subgroup in G of type τ. The Dynkin diagram $D_{\tilde{G}}$ of $\tilde{G}$ is obtained by removing the subset τ from D_G, and $\tilde{G}_E$ is of inner type. By [1, Theorem 4.2], there is a motivic decomposition

$$M_{G,\tau} \simeq \bigoplus_{i\in\mathcal{I}} M^F_{\tilde{G}_{L_i},\tau_i}\{n_i\}$$

with some field extensions L_i/F contained in E and some $\mathrm{Gal}(E/L_i)$-invariant $\tau_i \subset D_{\tilde{G}}$. Note that the fields L_i, the projective $\tilde{G}_{L_i}$-homogeneous varieties $X_{\tilde{G}_{L_i},\tau_i}$, and the shifting numbers n_i in this decomposition are completely determined by the underlying combinatorics of G. The isomorphism $\varphi : D_G \longrightarrow D_{G'}$ from Proposition 10.2 yields an analogous decomposition of $M_{G',\varphi(\tau)}$ with respect to its semisimple part $\tilde{G}'$ of a parabolic subgroup in G' of type $\varphi(\tau)$ with the same $\mathcal{I}$, L_i, τ_i, and n_i:

$$M_{G',\varphi(\tau)} \simeq \bigoplus_{i\in\mathcal{I}} M^F_{\tilde{G}'_{L_i},\varphi(\tau_i)}\{n_i\}$$

Since G and G' are inner forms of each other and satisfy condition $i)$ of Proposition 10.2, so do $\tilde{G}$ and $\tilde{G}'$. Indeed, for any field extension K/F, we have disjoint union decompositions

$$D^p_{G_K} = D^p_{\tilde{G}_K} \sqcup \tau \quad \text{and} \quad D^p_{G'_K} = D^p_{\tilde{G}'_K} \sqcup \varphi(\tau).$$

Condition $i)$ of Proposition 10.2 for G and G' gives that $D^p_{G'_K} = \varphi(D^p_{G_K})$ and hence $D^p_{\tilde{G}'_K} = \varphi(D^p_{\tilde{G}_K})$. It follows that for any any $i \in \mathcal{I}$ and any field extension L_i/F, the reductive groups $\tilde{G}_{L_i}$ and $\tilde{G}'_{L_i}$ satisfy condition $i)$ of Proposition 10.2 with respect to the restriction of φ and the subset $\tau_0 = \emptyset$. By induction, the motives $M_{\tilde{G}_{L_i},\tau_i}$ and $M_{\tilde{G}'_{L_i},\varphi(\tau_i)}$ are thus isomorphic. Therefore, the motives $M^F_{\tilde{G}_{L_i},\tau_i}$ and $M^F_{\tilde{G}'_{L_i},\varphi(\tau_i)}$ are isomorphic as well and so $M_{G,\tau} \simeq M_{G',\varphi(\tau)}$.

We now treat the case of arbitrary $\mathrm{Gal}(E/F)$-invariant subsets τ and $\varphi(\tau)$. Assume that the motives of $X_{G,\tau}$ and $X_{G',\varphi(\tau)}$ are not isomorphic. By Theorem 8.1, since G and G' satisfy conditions of Proposition 10.2, this means that $l_{A,Y,n}(M_{G,\tau}) \neq l_{A,Y,n}(M_{G',\varphi(\tau)})$ for some indecomposable Artin motive A and some projective homogeneous variety Y defined over a field extension contained in E. Consider the minimal integer n for which such a non-equality occurs.

Over the function field K/F of the product $X_{G,\tau} \times X_{G',\varphi(\tau)}$ both $X_{G,\tau}$ and $X_{G',\varphi(\tau)}$ have a rational point. The motive A_K is indecomposable (see Corollary 8.8) and so we can investigate the integer $l_{A_K,Y_K,n}(M_{G_K,\tau})$. If $U_{A_K}(Y_K)\{n\}$ is a direct summand of $M_{G_K,\tau}$, then by the Krull-Schmidt property and Theorem 8.1, it is a direct summand in the K/F-restriction $(U_B(Z))_K$ of an A-upper motive $U_B(Z)$ of G, shifted by some $k \leq n$.

Note that $(U_B(Z))_K \simeq U_{B_K}(Z_K) \oplus N$, where N is a direct sum of A-upper motives with Tate shifts at least 1. Since $X_{G,\tau}$ and $X_{G',\varphi(\tau)}$ are equivalent, any projective homogeneous variety which dominates $X_{G,\tau}$ (or $X_{G',\varphi(\tau)}$) dominates their product. In particular, Proposition 10.1 implies that a direct summand $U_{A_K}(Y_K)\{k\}$ of $M(X_{G_K,\tau})$ may only arise from a K/F-restriction $(U_B(Z)\{k\})_K$ (with the same shift) if $B \simeq A$ and $Z \approx Y$, that is from the A-upper motive $U_A(Y)\{k\}$ (see Theorem 9.2).

Write M for the direct summand of $M_{G,\tau}$ given by the sum of all its indecomposable summands with shifts strictly lower than n (in a fixed complete decomposition). Separating the summands $U_{A_K}(Y_K)\{n\}$ of $M_{G_K,\tau}$ which arise from M_K, we get thanks to the previous discussion the equality

$$l_{A_K,Y_K,n}(M_{G_K,\tau}) = l_{A,Y,n}(M_{G,\tau}) + l_{A_K,Y_K,n}(M_K).$$

Since by assumption the A-upper motives of G and G' are pairwise isomorphic, the same reasoning with $X_{G',\varphi(\tau)}$ ensures that

$$l_{A_K,Y_K,n}(M_{G'_K,\varphi(\tau)}) = l_{A,Y,n}(M_{G',\varphi(\tau)}) + l_{A_K,Y_K,n}(M'_K),$$

where M' is the direct sum of the summands in a complete motivic decomposition of $X_{G',\varphi(\tau)}$ with shifts strictly lower than n. By minimality of n, the motives M and M' are isomorphic, hence M_K and M'_K are isomorphic as well and $l_{A_K,Y_K,n}(M_K) = l_{A_K,Y_K,n}(M'_K)$. As by assumption $l_{A,Y,n}(M_{G,\tau}) \neq l_{A,Y,n}(M_{G',\varphi(\tau)})$, it follows that $l_{A_K,Y_K,n}(M_{G_K,\tau})$ and $l_{A_K,Y_K,n}(M_{G'_K,\varphi(\tau)})$ are not equal, a contradiction to the fact that the motives of $X_{G_K,\tau}$ and of $X_{G'_K,\varphi(\tau)}$ are isomorphic (recall that both of these varieties have a rational point).

We can now conclude: let τ be an arbitrary subset of D_G. The reductive groups G_{F_τ} and G'_{F_τ} satisfy condition $i)$ of Proposition 10.2. It follows from the Galois-invariant case that $M_{G_{F_\tau},\tau}$ and $M_{G'_{F_\tau},\varphi(\tau)}$ are isomorphic, hence so are $M_{G,\tau} = M^F_{G_{F_\tau},\tau}$ and $M_{G',\varphi(\tau)} = M^F_{G'_{F_\tau},\varphi(\tau)}$. □

A field is called *p-special* if every its finite extension has a p-power degree. Let G and G' be two reductive algebraic groups, inner forms of each other. Similarly to [4, Definition 1], we say that G and G' are *motivic equivalent* (with coefficients in $\mathbb{F}$) with respect to a Galois-equivariant isomorphism $\varphi : D_G \longrightarrow D_{G'}$, if for any subset τ of D_G, the motives $M_{\tau,G}$ and $M_{\varphi(\tau),G'}$ are isomorphic.

Corollary 10.4. *Let G and G' be reductive algebraic groups over F, inner forms of each other, becoming of inner type over a Galois p-separated extension E/F. Let φ be a* $\mathrm{Gal}(E/F)$*-equivariant isomorphism of their Dynkin diagrams. The groups G and G' are motivic equivalent with respect to φ if and only if for any p-special field extension K/F, φ identifies the Tits indexes of G_K and G'_K.*

Proof. Theorem 10.3 with $\tau_0 = \emptyset$ states that G and G' are motivic equivalent with respect to φ if and only if for any field extension K/F, φ identifies the subsets of p-distinguished vertices of G_K and G'_K. Over p-special field K, this expresses as $\varphi(D^0_{G_K}) = D^0_{G'_K}$ (through classical Tits indexes), since a variety is isotropic if and only it has a rational point over a p-special closure of its base field [6, Proof of Lemma 4.11]. □

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