GRASSMANNIANS FOR ISOTROPIC FORMS

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Corollary 0.2 below has been used during step (ii) in the proof of [2, Theorem 5.4].

Let F be a field and let V be a vector F-space of odd dimension $2n+1$. Let $\varphi\colon V \to F$ be a non-degenerate isotropic quadratic form over F. We fix an isotropic line $L \subset V$ and let φ' be the quadratic form on $V' := L^\perp/L$ induced by φ. Thus φ' is a Witt-equivalent to φ non-degenerate quadratic form of dimension $2n-1$ over F.

For $m \in \{0, \dots, n\}$, let X_m be the mth grassmannian of φ. In particular, $X_0 = \operatorname{Spec} F$ and X_1 is the projective quadric of φ.

For $m \in \{1, \dots, n\}$, let $X_m^1 \subset X_m$ be the closed subvariety of m-dimensional totally isotropic subspaces in $L^\perp$ and let $X_m^2 \subset X_m^1$ be the closed subvariety of those of them that contain L.

The variety X_m^2 is identified with the $(m-1)$st grassmannian X'_{m-1} of φ' via $U \mapsto U/L$. The difference $X_m^1 \backslash X_m^2$ is an affine bundle over X'_m via $U \mapsto (U+L)/L$. Finally, the difference $X_m \backslash X_m^1$ is an affine bundle over X'_{m-1} via $U \mapsto (U \cap L^\perp + L)/L$, c.f. [1, Lemma 86.1].

It follows by [1, Corollary 66.4] that the integral Chow motive $M(X_m)$ contains the motive $M(X'_{m-1})$ as a direct summand. In particular, we have a split monomorphism of graded groups $\mathrm{CH}^*(X'_{m-1}) \to \mathrm{CH}^*(X_m)$ given by the pull-back with respect to the closed embedding

$$i\colon X'_{m-1} = X_m^2 \hookrightarrow X_m,$$

or, in slightly different terms, induced by the correspondence $X'_{m-1} \rightsquigarrow X_m$ given by the transpose ι^t of the graph ι of i.

The incidence correspondence $\alpha\colon X'_1 \rightsquigarrow X_1$ is defined as the variety of pairs $(U'/L, U)$ of totally isotropic lines in V' and V with $U \subset U'$. By [1, Lemma 72.3], the homomorphism

$$\alpha^*\colon\ \mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}'_1)$$

given by α^t maps l_i to l'_{i-1} for all i.

The incidence correspondence $\gamma\colon X_1 \rightsquigarrow X_m$ is defined as the variety of pairs (U_1, U_m) with $U_1 \subset U_m$. Recall that the homomorphism

$$\gamma_*\colon\ \mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}_m)$$

maps the elements $l_{n-1}, \dots, l_0$ respectively to $e_{n-m+1}, \dots, e_{2n-m}$. Similarly, the homomorphism

$$\gamma'_*\colon\ \mathrm{CH}(\bar{X}'_1) \to \mathrm{CH}(\bar{X}'_{m-1})$$

induced by the incidence correspondence $\gamma'\colon X_1 \rightsquigarrow X_m$ maps the elements $l'_{n-2}, \dots, l'_0$ respectively to $e'_{n-m+1}, \dots, e'_{2n-m-1}$.

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Lemma 0.1 (c.f. [1, Lemma 86.7])**.** *For any $m \in \{1, \dots, n\}$, the square of correspondences*

$$\begin{array}{ccc} X_1 & \overset{\alpha^t}{\rightsquigarrow} & X'_1 \\ {\scriptstyle\gamma}\wr\downarrow & & \wr\downarrow{\scriptstyle\gamma'} \\ X_m & \overset{i^t}{\rightsquigarrow} & X'_{m-1} \end{array}$$

is commutative.

Corollary 0.2. *For any $m \in \{2, \dots, n\}$, the homomorphism $i^* \colon \operatorname{CH}^*(\bar{X}_m) \to \operatorname{CH}^*(\bar{X}'_{m-1})$ maps the elements $e_{n-m+1}, \dots, e_{2n-m-1}$ respectively to $e'_{n-m+1}, \dots, e'_{2n-m-1}$ (whereas e_{2n-m} is mapped to 0).* □

References

[1] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[2] Karpenko, N. A. Fields of u-invariant 11. Preprint (21 Apr 2026, 15 pages). Available on author's web page.

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