ON ANISOTROPY OF ORTHOGONAL INVOLUTIONS

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Abstract. We show that an orthogonal involution of a central division algebra D (over a field of characteristic not 2) remains anisotropic over the generic splitting field of D. We also give a couple of other applications of the same technique.

Contents

0. The article structure

The main result of this article is Theorem 5.3. An introduction to its subject is given in section 5.

Theorem 5.3 is obtained as a consequence of a much more general Theorem 4.1 (giving also Corollary 4.2 as another interesting application) which, in its turn, follows from Theorem 2.1, a statement about the indexes of correspondences on certain Severi-Brauer varieties. The proof of Theorem 2.1 is, in the final analysis, based on the computation [20, §8] of the K-theory of Severi-Brauer varieties (used in the proof of Proposition 2.2).

In sections 6–9, we deal with projective quadrics (a reader, interested in Theorem 5.3 only, has no need to look at these sections) and prove Theorem 6.4, which is a precise analogue of Theorem 2.1, although the method of the proof is completely different. This theorem implies Proposition 7.1 in exactly

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the same way as Theorem 2.1 implies Theorem 4.1. Theorem 7.6 is a quite formal generalization of Proposition 7.1 to the arbitrary quadrics.

In section 8 we discuss consequences of Theorem 7.6 for the "pure" algebraic theory of quadratic forms. The main of them is Theorem 8.1. In particular, we recover a theorem of A. Vishik (see Corollary 8.2), proved originally in [24] by studying indecomposable summands in the motives of quadrics.

A consequence of Theorem 8.1 is a simplified characterization of the motivic equivalence of quadratic forms, obtained in section 9.

1. Indexes of correspondences on algebraic varieties

By a variety we always mean a complete smooth equidimensional algebraic variety over a field.

Let F be a field and let X, Y be some F-varieties. The Chow group $\mathrm{CH}(X \times Y)$ of algebraic cycles on the product $X \times Y$ modulo rational equivalence of cycles will be often considered as the group of correspondences $\mathrm{Corr}(X, Y)$. If Z is one more F-variety, $\alpha \in \mathrm{Corr}(X, Y)$, and $\beta \in \mathrm{Corr}(Y, Z)$, then the composite $\beta \circ \alpha \in \mathrm{Corr}(X, Z)$ is defined in the usual way (see [4, def. 16.1.1]).

For any $i \in \mathbb{Z}$,

$$\mathrm{Corr}^i(X, Y) := \mathrm{CH}^{\dim Y + i}(X \times Y) = \mathrm{CH}_{\dim X - i}(X \times Y)$$

is the group of correspondences of degree i (the upper index by Chow groups means the codimension, while the lower index means the dimension of the cycles). Note that the degrees of correspondences are added when the correspondences are composed (cf. [4, example 16.1.1]). In particular, the composite of degree 0 correspondences is a correspondence of degree 0 as well.

Definition 1.1 (cf. [4, example 16.1.4])**.** For an irreducible variety X and a variety Y, the *index* of a degree 0 correspondence $\alpha \in \mathrm{Corr}^0(X, Y)$ is the integer $i(\alpha)$ such that $(pr_X)_*(\alpha) = i(\alpha) \cdot [X]$, where $(pr_X)_*$ is the push-forward with respect to the projection $pr_X : X \times Y \to X$ and $[X]$ is the class of X in $\mathrm{CH}^0(X)$.

Example 1.2. Let $f : X \to Y$ be a rational morphism of varieties (where X is irreducible) and let Γ_f be the correspondence in $\mathrm{Corr}^0(X, Y)$ given by the closure of the graph of f. Then $i(\Gamma_f) = 1$.

Proof. Let $U \subset X$ be a non-empty open subset where f is defined. By the property [4, prop. 1.7] applied to the cartesian square

$$\begin{array}{ccc} U \times Y & \xrightarrow{in \times id_Y} & X \times Y \\ \downarrow{\scriptstyle pr_U} & & \downarrow{\scriptstyle pr_X} \\ U & \xrightarrow{in} & X \end{array}$$

one has

$$(*) \qquad in^* \circ (pr_X)_*(\Gamma_f) = (pr_U)_* \circ (in \times id_Y)^*(\Gamma_f)\ .$$

The pull-back $(in \times id_Y)^*(\Gamma_f)$ is the graph of the regular morphism $f\colon U \to Y$, i.e., the push-forward $(id_U, f)_*([U])$ of $[U] \in \mathrm{CH}^0(U)$ with respect to the graph morphism $(id_U, f)\colon U \to U \times Y$. Since $pr_U \circ (id_U, f) = id_U$, the right-hand side term of the equality $(*)$ equals $[U]$. On the other hand, the left-hand side term of $(*)$ is $in^*\big(i(\Gamma_f) \cdot [X]\big) = i(\Gamma_f) \cdot [U]$. Thus $i(\Gamma_f) = 1$. □

Corollary 1.3. *For the class $\Delta_X \in \mathrm{Corr}^0(X, X)$ of the diagonal of an irreducible variety X, one has $i(\Delta_X) = 1$.* □

Lemma 1.4. *Let $\theta_X \times id_Y : Y_{F(X)} \to X \times Y$ be the morphism of schemes obtained from the generic point morphism $\theta_X : \operatorname{Spec} F(X) \to X$ by the base change. Then for any $\alpha \in \mathrm{Corr}^0(X, Y)$ its pull-back $(\theta_X \times id_Y)^*(\alpha)$ is a 0-cycle on $Y_{F(X)}$ of the degree $i(\alpha)$.*

Proof. The cartesian square

$$\begin{array}{ccc} Y_{F(X)} & \xrightarrow{\theta_X \times id_Y} & X \times Y \\ \downarrow & & \downarrow{\scriptstyle pr_X} \\ \operatorname{Spec} F(X) & \xrightarrow{\theta_X} & X \end{array}$$

produces the commutative square ([4, prop. 1.7])

$$\begin{array}{ccc} \mathrm{CH}_0(Y_{F(X)}) & \xleftarrow{(\theta_X \times id_Y)^*} & \mathrm{Corr}^0(X, Y) \\ \downarrow{\scriptstyle \deg} & & \downarrow{\scriptstyle (pr_X)_*} \\ \mathrm{CH}_0(\operatorname{Spec} F(X)) = \mathbb{Z} = \mathrm{CH}^0(\operatorname{Spec} F(X)) & \xleftarrow{(\theta_X)^*} & \mathrm{CH}^0(X) \end{array}$$

Since $\theta_X^*\big((pr_X)_*(\alpha)\big) = \theta_X^*\big(i(\alpha) \cdot [X]\big) = i(\alpha)$, we are done. □

Corollary 1.5. *Let X, Y be F-varieties (where X is irreducible) and let d be an integer. The following two statements are equivalent:*

1) *on $Y_{F(X)}$, there exists a 0-cycle of degree d;*
2) *in $\mathrm{Corr}^0(X, Y)$, there exists a correspondence of index d.*

Proof. If $\alpha \in \mathrm{Corr}^0(X, Y)$ is a correspondence of index d, then the image of α under

$$(\theta_X \times id_Y)^*\colon \mathrm{Corr}^0(X, Y) \to \mathrm{CH}_0(Y_{F(X)})$$

is a 0-cycle on $Y_{F(X)}$ of the degree d by Lemma 1.4, i.e., 2) implies 1). Since the homomorphism $(\theta_X \times id_Y)^*$ is evidently surjective (cf. [9, §5]), the inverse implication also follows. □

Lemma 1.6. *Let X, Y, Z be F-varieties and let $\alpha \in \mathrm{Corr}(X, Y)$ and $\beta \in \mathrm{Corr}(Y, Z)$ be some correspondences (of an arbitrary degree or even not homogeneous). Then $(pr_X)_*(\beta \circ \alpha) = (pr_X)_*\Big(\big(X \times (pr_Y)_*(\beta)\big) \cdot \alpha\Big)$.*

Proof. By the definition of the composite, we have

$$\beta \circ \alpha = (pr_{XZ})_*\big((X \times \beta) \cdot (\alpha \times Z)\big)\ .$$

Since $(pr_X)_* \circ (pr_{XZ})_* = (pr_X)_* \circ (pr_{XY})_*$, we get

$$(pr_X)_*(\beta \circ \alpha) = (pr_X)_* \circ (pr_{XY})_*\big((X \times \beta) \cdot (\alpha \times Z)\big) \overset{1)}{=}$$
$$(pr_X)_*\big((pr_{XY})_*(X \times \beta) \cdot \alpha\big) \overset{2)}{=} (pr_X)_*\Big(\big(X \times (pr_Y)_*(\beta)\big) \cdot \alpha\Big)\ ,$$

where 1) is the projection formula for pr_{XY} (note that $\alpha \times Z = pr^*_{XY}(\alpha)$), while 2) is the property [4, prop. 1.7] of the cartesian square

$$\begin{array}{ccc} X \times Y \times Z & \longrightarrow & Y \times Z \\ \big\downarrow{\scriptstyle pr_{XY}} & & \big\downarrow{\scriptstyle pr_Y} \\ X \times Y & \longrightarrow & Y\ . \end{array}$$

□

Corollary 1.7. *In the conditions of Lemma 1.6, assume that the correspondences α and β are homogeneous of degree 0 and that the varieties X and Y are irreducible (in order we may speak of the indexes of the correspondences). Then $i(\beta \circ \alpha) = i(\beta) \cdot i(\alpha)$.*

Proof. We have $i(\beta \circ \alpha) \cdot [X] = (pr_X)_*(\beta \circ \alpha)$ by the definition of $i(\beta \circ \alpha)$. Applying Lemma 1.6, we get

$$(pr_X)_*(\beta \circ \alpha) = (pr_X)_*\Big(\big(X \times (pr_Y)_*(\beta)\big) \cdot \alpha\Big) \overset{1)}{=}$$
$$(pr_X)_*\big(i(\beta) \cdot \alpha\big) \overset{2)}{=} i(\beta) \cdot i(\alpha) \cdot [X]\ ,$$

where 1) holds by the definition of $i(\beta)$, while 2) holds by the definition of $i(\alpha)$. Therefore $i(\beta \circ \alpha) = i(\beta) \cdot i(\alpha)$. □

Corollary 1.8. *In the conditions of Lemma 1.6, assume that the correspondence β is homogeneous of some negative degree. Then $(pr_X)_*(\beta \circ \alpha) = 0$.*

Proof. If $\beta \in \mathrm{Corr}^i(Y, Z)$, then $(pr_Y)_*(\beta) \in \mathrm{CH}^i(Y)$. If $i < 0$, then $\mathrm{CH}^i(Y) = 0$, whereby $(pr_Y)_*(\beta) = 0$. According to Lemma 1.6, it follows that $(pr_X)_*(\beta \circ \alpha) = 0$. □

Corollary 1.9. *Let X, Y be irreducible varieties, $\alpha \in \mathrm{Corr}^0(X, Y)$, and $\beta \in \mathrm{Corr}^0(Y, X)$. Set $\gamma := \beta \circ \alpha \in \mathrm{Corr}^0(X, X)$. Then $i(\gamma) = i(\beta) \cdot i(\alpha)$; moreover, if $\dim Y < \dim X$, then $i(\gamma^t) = 0$, where γ^t is the transpose of γ.*

Proof. The statement on $i(\gamma)$ is a particular case of Corollary 1.7. To prove the statement on $i(\gamma^t)$, note that $\gamma^t = \alpha^t \circ \beta^t$ and $\alpha^t \in \mathrm{CH}^{\dim Y}(Y \times X) = \mathrm{Corr}^{\dim Y - \dim X}(Y, X)$. So, if $\dim Y - \dim X < 0$, then $i(\gamma^t) = 0$ according to Corollary 1.8. □

2. Indexes of correspondences on Severi-Brauer varieties

Let D be a central division F-algebra of degree a power of a prime p and let X be the Severi-Brauer variety of D (see [1] or [12, §10]).

Theorem 2.1. *For any* $\alpha \in \operatorname{Corr}^0(X, X)$*, one has*

$$i(\alpha) \equiv i(\alpha^t) \pmod p .$$

Let us fix a finite field extension E/F of degree $\deg D := \sqrt{\dim_F D}$ such that the E-algebra D_E is split. Since the E-variety X_E is isomorphic to a projective space, the group $\operatorname{CH}^i(X_E)$ is identified with $\mathbb{Z}$ for any i with $0 \le i \le \dim X$.

The proof of Theorem 2.1 is based on the following

Proposition 2.2 ([11, prop. 2.1.1])**.** *For any cycle* $\alpha \in \operatorname{CH}^i(X)$ *of a positive codimension* $i > 0$*, one has* $\alpha_E \neq 1$*.*

Corollary 2.3. *For any* $\alpha \in \operatorname{CH}^i(X)$ *with* $i > 0$*, the element* $\alpha_E \in \operatorname{CH}^i(X_E)$ *is divisible by* p*.*

Proof. By the transfer argument, there exists a cycle $\beta \in \operatorname{CH}^i(X)$ such that $\beta_E = [E : F] = \deg D$. Recall that $\deg D$ is a power of p. Now, if we assume that α_E is not divisible by p, then some linear combination (with integral coefficients) of α and β is a cycle $\gamma \in \operatorname{CH}^i(X)$ such that $\gamma_E = 1$, which is in contradiction with Proposition 2.2. □

Proof of Theorem 2.1. As a scheme over X via, say, pr_1, the variety $X \times X$ is isomorphic to a projective X-bundle ([18, prop. 4.7]). Therefore,

$$\operatorname{CH}^n(X \times X) = \sum_{i=0}^{n} \operatorname{CH}^i(X) \cdot \xi^{n-i}$$

for certain $\xi \in \operatorname{CH}^1(X \times X)$ ([5, app. A, §2, A11]).

Consider the homomorphism

$$f \colon \operatorname{CH}^n(X \times X) \to \mathbb{Z}/p \oplus \mathbb{Z}/p$$

given by the formula $f(\alpha) = (i(\alpha), i(\alpha^t))$ modulo p. By Corollary 2.3, $f(\alpha) = 0$ if $\alpha \in \operatorname{CH}^i(X) \cdot \xi^{n-i}$ with positive i (because $f(\alpha_E) = f(\alpha)$ and α_E is divisible by p). Therefore, the image of f is a cyclic subgroup of $\mathbb{Z}/p \oplus \mathbb{Z}/p$ (generated by $f(\xi^n)$). Since $f(\Delta_X) = (1, 1)$ for the diagonal class $\Delta_X \in \operatorname{CH}^n(X \times X)$ (see Corollary 1.3), the image of f is generated by $(1, 1)$. □

3. Stably birational and quasi-birational equivalence of algebraic varieties

We recall that two irreducible F-varieties X and Y are called *stably birationally equivalent* (*sb-equivalent* for short), if for some positive integers n and m the varieties $X \times \mathbb{P}^n$ and $Y \times \mathbb{P}^m$ are birationally equivalent. Sometimes we will write $X \overset{\text{sb}}{\sim} Y$ to notify that X and Y are sb-equivalent.

Instead of the stably birational equivalence, it will be natural for us to use certain similar but different equivalence relations of algebraic varieties:

Definition 3.1. Two irreducible F-varieties X and Y are called *quasi-birationally equivalent* (*qb-equivalent* for short), if

$$\deg \operatorname{CH}_0(Y_{F(X)}) = \mathbb{Z} = \deg \operatorname{CH}_0(X_{F(Y)}) ,$$

i.e., if each of $Y_{F(X)}$ and $X_{F(Y)}$ possesses a 0-cycle of degree 1.

Let p be a prime number. Two irreducible F-varieties X and Y are called *quasi-birationally p-equivalent* (*qb_p-equivalent* for short), if

$$\deg \mathrm{CH}_0(Y_{F(X)}) + p\mathbb{Z} = \mathbb{Z} = \deg \mathrm{CH}_0(X_{F(Y)}) + p\mathbb{Z}\ ,$$

i.e., if each of $Y_{F(X)}$ and $X_{F(Y)}$ possesses a 0-cycle of some degree prime to p.

Sometimes we will write $X \overset{\mathrm{qb}}{\sim} Y$ and $X \overset{\mathrm{qb}_p}{\sim} Y$ for the qb-equivalence and the qb_p-equivalence of X and Y.

We are going to clarify a little bit the relation between the sb- and qb-equivalences, although it is not essential for our main purposes.

Lemma 3.2. *Sb-equivalent varieties are qb-equivalent (and, therefore, qb_p-equivalent for any p).*

Proof. Assuming that $X \overset{\mathrm{sb}}{\sim} Y$, we have two birational isomorphisms: $X \times \mathbb{P}^n \to Y \times \mathbb{P}^m$ and $Y \times \mathbb{P}^m \to X \times \mathbb{P}^n$ with some m, n. Let us forget that these are mutually inverse birational isomorphisms and only keep in mind that these are rational morphisms. The existence of the first one tells us that the set

$$(Y \times \mathbb{P}^m)(F(X \times \mathbb{P}^n)) = Y(F(X \times \mathbb{P}^n)) \times \mathbb{P}^m(F(X \times \mathbb{P}^n))$$

is not empty. In particular, the set $Y(F(X \times \mathbb{P}^n))$ is not empty and consequently $\deg \mathrm{CH}_0(Y_{F(X \times \mathbb{P}^n)}) = \mathbb{Z}$. Since the field extension $F(X \times \mathbb{P}^n)/F(X)$ is purely transcendental, the restriction homomorphism

$$\mathrm{CH}_0(Y_{F(X)}) \to \mathrm{CH}_0(Y_{F(X \times \mathbb{P}^n)})$$

is bijective (see, for example, [10, lemma 1.4 a)]; we only need the surjectivity here). Therefore $\deg \mathrm{CH}_0(Y_{F(X)}) = \mathbb{Z}$. Using the second rational morphism, we similarly get $\deg \mathrm{CH}_0(X_{F(Y)}) = \mathbb{Z}$. Thus, the varieties X and Y are qb-equivalent. □

The inverse implication holds for some particular types of varieties only. To try to go the other way round, let us introduce an aditional notion, which is almost intermediate in a sense with respect to the sb- and qb-equivalences: let us say that two irreducible varieties X and Y are *strictly* qb-equivalent, if the sets $Y(F(X))$ and $X(F(Y))$ are non-empty (in the case of infinite F, the sb-equivalence implies in fact the strict qb-equivalence; two strictly qb-equivalent varieties are clearly always qb-equivalent).

Lemma 3.3. *Strictly qb-equivalent projective homogeneous varieties are sb-equivalent.*

Proof. Assume that two projective homogeneous varieties X and Y are strictly qb-equivalent. Since Y over $F(X)$ has a rational point, the field $F(Y)(X) = F(Y \times X)$ is purely transcendental over $F(X)$ (here we use the assumption that Y is homogeneous and a theorem that a projective homogeneous variety with a rational point is rational, [14, thm. 3.10]), whereby X is sb-equivalent to $Y \times X$. Similarly, since $X(F(Y)) \neq \emptyset$, the variety Y is sb-equivalent with $Y \times X$. By transitivity, X is sb-equivalent with Y. □

Corollary 3.4. *Two projective quadrics or Severi-Brauer varieties X and Y are qb-equivalent if and only if they are sb-equivalent.*

Proof. A projective quadric or a Severi-Brauer variety possesses a degree 1 zero-cycle only if it possesses a rational point (for projective quadrics this is the Springer theorem [17, thm. 7.2.3]). Therefore, qb-equivalent quadrics or Severi-Brauer varieties are strictly qb-equivalent. The rest is served by Lemmas 3.3 and 3.2. □

To show that the relations $\overset{\mathrm{qb}}{\sim}$ and $\overset{\mathrm{qb}_p}{\sim}$ are transitive, one may use their following characterization, which is an immediate consequence of Corollary 1.5:

Lemma 3.5. *Two irreducible varieties X and Y are qb-equivalent if and only if there exist correspondences $\alpha \in \mathrm{Corr}^0(X,Y)$ and $\alpha' \in \mathrm{Corr}^0(Y,X)$ with $i(\alpha) = 1 = i(\alpha')$. Two varieties X and Y are qb_p-equivalent if and only if there exist a correspondence $\alpha \in \mathrm{Corr}^0(X,Y)$ with $i(\alpha)$ prime to p and a correspondence $\alpha' \in \mathrm{Corr}^0(Y,X)$ with $i(\alpha')$ prime to p.* □

Corollary 3.6. *If $X \overset{qb}{\sim} Y$ and $Y \overset{qb}{\sim} Z$, then $X \overset{qb}{\sim} Z$. If $X \overset{qb_p}{\sim} Y$ and $Y \overset{qb_p}{\sim} Z$ for some p, then $X \overset{qb_p}{\sim} Z$.*

Proof. Use Corollary 1.7. □

4. Quasi-birational equivalence with Severi-Brauer varieties

Theorem 4.1. *Let D be a central division F-algebra of degree a power of a prime p, let X be the Severi-Brauer variety of D, and let Y be a (smooth complete irreducible) F-variety. If $X \overset{qb_p}{\sim} Y$, then $\dim Y \geq \dim X$.*

Proof. Assume that $X \overset{\mathrm{qb}_p}{\sim} Y$, while $\dim Y < \dim X$. According to Lemma 3.5, there exist correspondences $\alpha \in \mathrm{Corr}^0(X,Y)$ and $\alpha' \in \mathrm{Corr}^0(Y,X)$ with $i(\alpha)$ and $i(\alpha')$ coprime to p. We set $\gamma := \alpha' \circ \alpha$. This is a degree 0 correspondence on X with $i(\gamma) = i(\alpha') \cdot i(\alpha)$ coprime to p and $i(\gamma^t) = 0$ (Corollary 1.9). We are in contradiction with Theorem 2.1. □

By now, we have two applications of Theorem 4.1. The one being a new statement is described in the next section. The other one is originally proved in [6, §9]. An alternative proof is given in [8, prop. 7.6]. Here we give a third proof which is particularly short.

Corollary 4.2. *Let ϕ be an 8-dimensional quadratic F-form (where $\mathrm{char}\, F \neq 2$) with the trivial discriminant and let D be a division algebra Brauer-equivalent to the Clifford algebra of ϕ. If the degree of D is 8, then the form $\phi_{F(X)}$ over the function field $F(X)$ of the Severi-Brauer variety X of D is anisotropic.*

Proof. Let Y be a connected component of the variety of the maximal totally isotropic subspaces of ϕ (see [12, §13]). This is a projective homogenous F-variety of dimension 6 such that the form ϕ becomes hyperbolic over the function field $F(Y)$ (to see that $\dim Y = 6$, one may use that in the case of hyperbolic ϕ, the variety Y contains an open subset isomorphic to the affine space on the vector space of the skew-symmetric bilinear forms on a fixed maximal, i.e., 4-dimensional totally isotropic subspace of ϕ, and the dimension of this vector space equals $4 \cdot (4-1)/2 = 6$, see [12, §13]). In particular, the algebra $D_{F(Y)}$ is split, i.e., the set $X(F(Y))$ is non-empty.

On the other hand, the Clifford algebra of the 8-dimensional quadratic form $\phi_{F(X)}$ is split. Therefore, if the form $\phi_{F(X)}$ is isotropic, then it is hyperbolic (by the Arason-Pfister Hauptsatz [17, thm. 10.3.1]), i.e., the set $Y(F(X))$ is non-empty, too.

Thus, if the form $\phi_{F(X)}$ is isotropic, then the varieties X and Y are strictly qb-equivalent. In particular, they are then qb$_2$-equivalent. We fall into a contradiction with Theorem 4.1 because $\dim Y = 6 < \dim X = 7$. □

5. Anisotropy of orthogonal involutions

In this section we assume that char $F \neq 2$.

Recall that the classical Springer-Satz [17, 7.2.3] in the theory of quadratic forms asserts that an anisotropic quadratic F-form ϕ remains anisotropic over any odd degree field extension of F.

The "Springer-Satz for orthogonal involutions" is the following

Conjecture 5.1 (cf. [3, question after prop. 4.1])**.** *Let D be a central simple F-algebra equipped with an anisotropic orthogonal involution σ. Then for any odd degree field extension E/F, the involution σ_E on the E-algebra D_E is anisotropic as well.*

We refer to [16] for the notion of an orthogonal involution and its anisotropy.

Note that a weaker version of Conjecture 5.1 — the so called "weak form of the Springer-Satz for orthogonal involutions" — is settled (see [2, prop. 1.2]).

Since the algebra D admits an involution of the first type, the Schur index $\operatorname{ind} D$ of D is a power of 2. In the case of split D (i.e., of $\operatorname{ind} D = 1$), σ is adjoint to some quadratic F-form, and Conjecture 5.1 coincides with the classical Springer-Satz. The case of $\operatorname{ind} D = 2$ is settled in [19]. For $\operatorname{ind} D > 2$, Conjecture 5.1 is open. It can however be easily deduced (with a help of the classical Springer-Satz) from the following other

Conjecture 5.2. *Let D be a central simple F-algebra equipped with an anisotropic orthogonal involution σ and let X be the Severi-Brauer variety of D. Then the involution $\sigma_{F(X)}$ on the split $F(X)$-algebra $D_{F(X)}$ is anisotropic.*

For $\operatorname{ind} D = 1$, the field extension $F(X)/F$ is purely transcendental; by this reason, Conjecture 5.2 is trivial in the case of $\operatorname{ind} D = 1$.

The case of $\operatorname{ind} D = 2$ is settled in [19, cor 3.4]. Our main result is the opposite border case of this conjecture, namely

Theorem 5.3. *Let D be a central division F-algebra equipped with an orthogonal involution σ. Then the involution $\sigma_{F(X)}$, where X is the Severi-Brauer variety of D, is anisotropic.*

Remark 5.4. We do not have to assume here that σ itself is anisotropic: it is automatically anisotropic by the reason that D is a division algebra.

Proof of Theorem 5.3. Let Y be the involution variety of σ ([22] or [12, §15]). We recall that Y is a closed subvariety of X of codimension 1. In particular, the set $X(F(Y))$ is not empty.

Assume that the involution $\sigma_{F(X)}$ is isotropic. By the definition of Y, this means that the set $Y(F(X))$ is not empty. Thus, the varieties X and Y are strictly qb-equivalent. In particular, they are qb_2-equivalent, and we are in contradiction with Theorem 4.1. □

6. Indexes of correspondences on quadrics

For a variety X, consider the diagonal morphism $\Delta_X\colon X \to X\times X$ and define a homomorphism $\delta = \delta_X\colon \mathrm{Corr}^0(X,X) \to \mathbb{Z}$ by the formula $\delta(\alpha) = \deg \Delta_X^*(\alpha)$, where $\Delta_X^* : \mathrm{Corr}^0(X,X) \to \mathrm{CH}_0(X)$ is the pull-back homomorphism with respect to Δ_X.

From now on we assume that $\operatorname{char} F \neq 2$. By the Springer theorem [17, 7.2.3] we evidently have

Lemma 6.1. *If X be a projective quadric with no rational points, then $\delta(\alpha) \in 2\mathbb{Z}$ for any $\alpha \in \mathrm{Corr}^0(X,X)$.* □

Proposition 6.2. *Let $\phi = \phi'\perp\mathbb{H}$, where ϕ' is a quadratic form of dimension ≥ 2 and $\mathbb{H}$ is the hyperbolic plane. Let X and X' be the projective quadrics given by the quadratic forms ϕ and ϕ' respectively. For any $\alpha \in \mathrm{Corr}^0(X,X)$, there exists some $\alpha' \in \mathrm{Corr}^0(X',X')$ such that*

$$\delta(\alpha) - i(\alpha) - i(\alpha^t) = \delta(\alpha')\ .$$

Proof. As a polynomial, the form ϕ is written down as $\phi = x_1x_2 + \phi'$. Let X_1 and X_2 are hypersurfaces in X determined by the equations $x_1 = 0$ and $x_2 = 0$ respectively. The subvarieties X_1 and X_2 have a transversal intersection which is equal to X', so that there is a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\text{imbedding}} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta_X} \\ X_1\times X_2 & \xrightarrow{\text{imbedding}} & X\times X \end{array}$$

where the left vertical arrow is given by the composite of the diagonal morphism $\Delta_{X'}\colon X' \to X'\times X'$ and the imbedding $X'\times X' \hookrightarrow X_1\times X_2$. Therefore, one may draw the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & & & \mathbb{Z} & & \\
 & & & \nearrow^{\deg} & \uparrow^{\deg} & & \\
 & & \mathrm{CH}_0(X') & \to & \mathrm{CH}_0(X) & & \\
 & \nearrow^{\Delta^*_{X'}} & \uparrow & & \uparrow^{\Delta^*_X} & & \\
\mathrm{CH}^{n-2}(X' \times X') & \leftarrow & \mathrm{CH}^{n-2}(X_1 \times X_2) & \to & \mathrm{CH}^n(X \times X) & \xrightarrow{f} & \mathbb{Z} \oplus \mathbb{Z} \;,
\end{array}
$$

where $n := \dim X$ and $f : \alpha \mapsto (i(\alpha), i(\alpha^t))$. Note that since the varieties X_1 and X_2 are singular, one should be careful with the definition of the pull-back homomorphisms from $\mathrm{CH}^{n-2}(X_1 \times X_2)$. They can be defined as the Gysin homomorphisms with respect to the regular imbeddings ([4, §6.2]). They can be also defined simply as the composite of a flat pull-back and a non-singular one, because X' is contained in $X_1 \setminus y_1$ and in $X_2 \setminus y_2$, where y_1 and y_2 are the only singular (closed) points of X_1 and X_2. Anyway, the square of the diagram is commutative by [4, thm. 6.2], while the left triangle is commutative by the functorial property [4, thm. 6.5] of the Gysin homomorphsims (the upper triangle is commutative by the functorial property of the push-forward, or, more specially, of the degree homomorphism).

Let us check that the lower row is exact in $\mathrm{CH}^n(X \times X)$. A part of the localization exact sequence for the closed subvariety $X_1 \times X_2 \subset X \times X$ looks as

$$\mathrm{CH}^{n-2}(X_1 \times X_2) \to \mathrm{CH}^n(X \times X) \to \mathrm{CH}^n\left((X \times X) \setminus (X_1 \times X_2)\right) .$$

A part of the Mayer-Vietoris exact sequence for the open covering $U_1 \times X \cup X \times U_2$ of the difference $(X \times X) \setminus (X_1 \times X_2)$, where $U_i := X \setminus X_i$, looks as

$$0 \to \mathrm{CH}^n\left((X \times X) \setminus (X_1 \times X_2)\right) \to$$
$$\to \mathrm{CH}^n(U_1 \times X) \oplus \mathrm{CH}^n(X \times U_2) \to \mathrm{CH}^n(U_1 \times U_2) .$$

Since U_1 and U_2 are isomorphic to affine spaces, we have

$$\mathrm{CH}^n(U_1 \times X) = \mathrm{CH}^n(X) = \mathrm{CH}_0(X) , \quad \mathrm{CH}^n(X \times U_2) = \mathrm{CH}^n(X) = \mathrm{CH}_0(X) ,$$

and $\mathrm{CH}^n(U_1 \times U_2) = 0$. Since the quadric X has a rational point x, we have $\mathrm{CH}_0(X) = \mathbb{Z} \cdot [x]$. Thus, $\mathrm{CH}^n\left((X \times X) \setminus (X_1 \times X_2)\right) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and it is easy to check that the obtained isomorphism coincides with f.

To finish the proof of Proposition 6.1, take any $\alpha \in \mathrm{Corr}^0(X, X) = \mathrm{CH}^n(X \times X)$. Since $i([X \times x]) = 1$, the cycle

$$\tilde{\alpha} := \alpha - i(\alpha)[X \times x] - i(\alpha^t)[x \times X]$$

vanishes under f. Therefore, $\tilde{\alpha}$ is the image of some $\tilde{\alpha}' \in \mathrm{CH}^{n-2}(X_1 \times X_2)$. Passing along the left-looking arrow, we get the required cycle $\alpha' \in \mathrm{CH}^{n-2}(X' \times X')$. □

Remark 6.3. Using the motivic decomposition [21, prop. 2] of X, one may give a "shorter" proof of Proposition 6.2: this motivic decomposition produces the decomposition $\mathrm{Corr}^0(X, X) \simeq \mathbb{Z} \times \mathrm{Corr}^0(X', X') \times \mathbb{Z}$, and the projection $\mathrm{Corr}^0(X, X) \to \mathrm{Corr}^0(X', X')$ gives the map $\alpha \mapsto \alpha'$ with the required property.

Let ϕ be a quadratic F-form. We write $i(\phi) = i_{\mathrm{W}}$ for the Witt index of ϕ. In the case of anisotropic ϕ, we write $i_1(\phi)$ (and say "the first Witt index of ϕ") for $\min\{i(\phi_E)\}$, where E runs over all field extension of F such that the form ϕ_E is isotropic. Clearly, $i_1(\phi) = i(\phi_{F(\phi)})$, where $F(\phi)$ is the function field of the quadratic form ϕ, i.e., the function field of the quadric given by ϕ.

Theorem 6.4. *Let X be the projective quadric given by an anisotropic quadratic form ϕ of dimension ≥ 3 with $i_1(\phi) = 1$. Then $i(\alpha) \simeq i(\alpha^t) \pmod 2$ for any $\alpha \in \mathrm{Corr}^0(X, X)$.*

Proof. Take an arbitrary $\alpha \in \mathrm{Corr}^0(X, X)$. First of all, since X has no rational points, the number $\delta(\alpha)$ is even (Lemma 6.1). If X is a conic, then $\delta(\alpha) = i(\alpha) + i(\alpha^t)$ and we are already done.

Assume that $\dim X > 1$. Let E/F be a field extension such that $i(\phi_E) = 1$. We have $\phi_E \simeq \mathbb{H} \perp \phi'$, where ϕ' is an anisotropic quadratic form with $\dim \phi' \geq 2$. Therefore, by Proposition 6.2, $\delta(\alpha_E) - i(\alpha_E) - i(\alpha_E^t) = \delta(\alpha')$ for some $\alpha' \in \mathrm{Corr}^0(X', X')$, where X' is the projective quadric given by ϕ'. Since $\delta(\alpha') \in 2\mathbb{Z}$ while $\delta(\alpha_E) = \delta(\alpha) \in 2\mathbb{Z}$, $i(\alpha_E) = i(\alpha)$, and $i(\alpha_E^t) = i(\alpha^t)$, it follows that $i(\alpha) + i(\alpha^t) \in 2\mathbb{Z}$. □

7. Quasi-birational equivalence with quadrics

In this section, F is a field of characteristic not 2 and X is the projective quadric over F given by an anisotropic quadratic F-form ϕ with $\dim \phi \geq 3$.

An immediate consequence of Theorem 6.4 is the following

Proposition 7.1. *Suppose that $i_1(\phi) = 1$. If the quadric X is qb$_2$-equivalent to a (smooth complete irreducible) variety Y, then $\dim Y \geq \dim X$.*

Proof. The same as for Theorem 4.1 by using Theorem 6.4 instead of Theorem 2.1. □

We are going to generalize this result to the case of arbitrary $i_1(\phi)$.

We say that two quadratic forms are sb-equivalent, it their quadrics are. Note that the analogously defined notion of the qb-equivalent quadratic form is not a different one (Corollary 3.4) and it is natural to choose only one name for use. We choose the name "sb-equivalence" because this name has been already used in the literature.

Lemma 7.2. *If $i_1(\phi) > 1$, then for any 1-codimensional subform ϕ' of ϕ one has:*

1. $i_1(\phi') \geq i_1(\phi) - 1$;
2. $\phi' \overset{sb}{\sim} \phi$.

Proof. 1. Let E/F be a field extension such that the form ϕ'_E is isotropic. Then the form ϕ_E is isotropic as well and therefore $i(\phi_E) \geq i := i_1(\phi)$, i.e., the form ϕ_E contains a totally isotropic subspace of dimension i. The intersection of this subspace with ϕ'_E is a totally isotropic subspace of ϕ'_E of dimension $i-1$. Consequently, $i(\phi'_E) \geq i-1$. Thus $i_1(\phi'_E) \geq i-1$.

2. Let E/F be a field extension. If the form ϕ'_E is isotropic, then the form ϕ_E is isotropic as well. The other way round, if the form ϕ_E is isotropic, then $i(\phi_E) > 1$ and so the form ϕ'_E is isotropic (see the proof of Item 1). □

Lemma 7.3. *Suppose that $i_1(\phi) > 1$. Then there exist a purely transcendental field extension $\tilde{F}/F$ and a 1-codimensional subform $\phi' \subset \phi_{\tilde{F}}$ such that $i_1(\phi') = i_1(\phi) - 1$.*

Remark 7.4. A posteriori (see Corollary 8.3), one may simply take $\tilde{F} = F$ and an arbitrary 1-codimensional subform $\phi' \subset \phi$. In the proof given below we take as ϕ' a generic 1-codimensional subform of ϕ.

Proof of Lemma 7.3. We put $\tilde{F} := F(\mathbb{A}(V))$, where $\mathbb{A}(V)$ is the affine space of the vector space V of definition of ϕ. Let x be the generic point of $\mathbb{A}(V)$. If we consider ϕ as a function on $\mathbb{A}(V)$, we get an element $\phi(x) \in \tilde{F}$. Since this element is a value of the quadratic $\tilde{F}$-form $\phi_{\tilde{F}}$, one has $\phi_{\tilde{F}} = \langle \phi(x) \rangle \perp \phi'$ for some 1-codimensional subform $\phi' \subset \phi_{\tilde{F}}$.

According to Lemma 7.2, we have $i_1(\phi') \geq i_1(\phi_{\tilde{F}}) - 1 = i_1(\phi) - 1$. Therefore, to finish the proof, it suffices to produce a field extension of $\tilde{F}$ over which the form ϕ' has the Witt index $i_1(\phi) - 1$.

According to Lemma A.1, for any field extension E/F with $i(\phi_E) \geq 1$, one has $i(\phi'_{\tilde{E}}) = i(\phi_E) - 1$, where $\tilde{E} := E(\mathbb{A}(V))$. If E/F is an extension with $i(\phi_E) = i_1(\phi)$, then $\tilde{E}/\tilde{F}$ is the extension required. □

Corollary 7.5. *Let ϕ be an arbitrary quadratic form over F. Changing eventually the base field F to some purely transcendental extension, one may always find a subform $\phi' \subset \phi$ of dimension $\dim \phi - i_1(\phi) + 1$ with $i_1(\phi') = 1$. Hereat, $\phi' \overset{sb}{\sim} \phi$.* □

The following result is a generalization of Proposition 7.1 to the case of arbitrary $i_1(\phi)$:

Theorem 7.6. *Let X be the projective quadric given by a quadratic F-form ϕ and let Y be a (smooth complete irreducible) F-variety. If $X \overset{qb_2}{\sim} Y$, then $\dim Y \geq \dim X - i_1(\phi) + 1$.*

Proof. This is an immediate consequence of Proposition 7.1 and Corollary 7.5. □

8. Stably birational equivalence of quadratic forms

Applying Theorem 7.6 to the case where the variety Y is also a projective quadric, we get

Theorem 8.1. *Let ϕ and ψ be quadratic forms over F of dimensions ≥ 3. If $\phi \overset{sb}{\sim} \psi$, then $\dim\phi - i_1(\phi) = \dim\psi - i_1(\psi)$.*

Proof. Since the condition is symmetric in ϕ and ψ, it suffices to check the inequality $\dim\psi - i_1(\psi) \geq \dim\phi - i_1(\phi)$. Take a subform $\psi' \subset \psi$ of the dimension $\dim\psi' = \dim\psi - i_1(\psi) + 1$. By Lemma 7.2, $\psi' \overset{\mathrm{sb}}{\sim} \psi$, thereafter $\psi' \overset{\mathrm{sb}}{\sim} \phi$. Applying Theorem 7.6 to the projective quadrics X and Y given respectively by ϕ and ψ', we get the inequality required. □

In particular, we recover [24, cor. 3]:

Corollary 8.2 (A. Vishik)**.** *Let ϕ be an anisotropic quadratic form and let ψ be a subform of ϕ. If $\operatorname{codim}_\phi\psi \geq i_1(\phi)$, then the form $\psi_{F(\phi)}$ is anisotropic.* □

Corollary 8.3 (A. Vishik)**.** *Let ϕ be an anisotropic quadratic F-form and let $\psi \subset \phi$ be a subform of codimension $< i_1(\phi)$. Then $i_1(\psi) = i_1(\phi) - \operatorname{codim}_\phi\psi$.*

Proof. By Lemma 7.2, the difference $d := i_1(\psi) - \big(i_1(\phi) - \operatorname{codim}_\phi\psi\big)$ is non-negative. If $d > 0$, then an arbitrary subform $\psi' \subset \psi$ of dimension $\dim\psi' = \dim\phi - i_1(\phi)$ is isotropic over $F(\psi)$. Since $\psi \overset{\mathrm{sb}}{\sim} \phi$ by Lemma 7.2, it follows that ψ' is isotropic over $F(\phi)$ as well, which is in contradiction with Corollary 8.2. □

9. Motivic equivalence of quadratic forms

In this section, we give an application of Theorem 8.1 to the question of characterization of the motivic equivalence of quadratic forms.

Recall from [15, def. 3.2], that a field extension E/F is called a *generic zero field* of an anisotropic quadratic F-form ϕ, if the form ϕ_E is isotropic and for any other field extension E'/F such that $\phi_{E'}$ is isotropic there exists a place $E \to E' \cup \infty$ over F.

A tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n$ is called *generic splitting tower* of a quadratic forms ϕ (cf. [15, §5]), if for any $i = 1, \ldots, n$ the field F_i is a generic zero field of the anisotropic part of the form $\phi_{F_{i-1}}$, while the form ϕ_{F_n} is maximally split.

The *splitting pattern* of an anisotropic quadratic form ϕ is the sequence of integers $(i(\phi_{F_j}))_{j=1}^n$, where $F = F_0 \subset F_1 \subset \cdots \subset F_n$ is a generic splitting tower of ϕ.

Recall from [13], that two quadratic forms ϕ and ψ with $\dim\phi = \dim\psi$ are called *motivic equivalent* (*m-equivalent* for short), if the Chow-motives of the projective quadrics determined by ϕ and ψ are isomorphic. A characterization of the motivic equivalence of quadratic forms, given in [23] and [13], can be reformulated as follows: two anisotropic quadratic forms ϕ and ψ with $\dim\phi =$

$\dim \psi$ are m-equivalent, if and only if they have a common generic splitting tower and a common splitting pattern. We are going to show that the condition on the splitting pattern is superfluous:

Proposition 9.1. *Two anisotropic quadratic forms ϕ and ψ with $\dim \phi = \dim \psi$ are m-equivalent, if and only if they possess a common generic splitting tower.*

Proof. Assume that two anisotropic quadratic forms ϕ and ψ with $\dim \phi = \dim \psi$ possess a common generic splitting tower $F = F_0 \subset F_1 \subset \cdots \subset F_n$. To prove the proposition, it suffices to show that the splitting patterns of ϕ and of ψ coincide. This is however an easy consequence of Theorem 8.1: since ϕ and ψ possess a common generic zero field F_1, they are sb-equivalent, whereby $\dim \phi - i_1(\phi) = \dim \psi - i_1(\psi)$, i.e., $i_1(\phi) = i_1(\psi)$. It follows that

$$\dim(\phi_{F_1})_{\mathrm{an}} = \dim \phi - 2i_1(\phi) = \dim \psi - 2i_1(\psi) = \dim(\psi_{F_1})_{\mathrm{an}} .$$

Proceeding in this way, we get what we want. □

Appendix. A lemma of Izhboldin

The following lemma, needed in the proof of Lemma 7.3, was in a previous version of [7], disappeared from the current version of the preprint, and is published here with a kind permission of its author.

Lemma A.1 (O. T. Izhboldin)**.** *In the notation of the proof of Lemma 7.3, for any field extension E/F, satisfying the condition $i(\phi_E) \geq 1$, one has $i(\phi'_{\tilde{E}}) = i(\phi_E) - 1$, where $\tilde{E} := E(\mathbb{A}(V))$.*

Proof. It is clear that $i(\phi'_{\tilde{E}})$ is equal to $i(\phi_E) - 1$ or to $i(\phi_E)$. Suppose that $i(\phi'_{\tilde{E}}) = i(\phi_E) = m \geq 1$. Since $i(\phi'_{\tilde{E}}) = m$, there exists an anisotropic $\tilde{E}$-form γ such that $\phi'_{\tilde{E}} = \gamma \perp m\mathbb{H}$. Since $i(\phi_E) = m$, there exists an anisotropic E-form ξ such that $\phi_E \simeq \xi \perp m\mathbb{H}$. Therefore,

$$\xi_{\tilde{E}} \perp m\mathbb{H} \simeq \phi_{\tilde{E}} \simeq \langle \phi(x) \rangle \perp \gamma \perp m\mathbb{H} .$$

By the Witt cancellation theorem, we have $\xi_{\tilde{E}} \simeq \langle \phi(x) \rangle \perp \gamma$. Therefore, the element $\phi(x) \in \tilde{E}^*$ is represented by the form $\xi_{\tilde{E}}$. By the Pfister representation theorem [17, thm. 9.2.8], it follows that $\phi_E \subset \xi$, which is a contradiction because $\dim \phi = \dim \xi + 2m > \dim \xi$. □

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