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ALGEBRO-GEOMETRIC INVARIANTS OF QUADRATIC FORMS

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ABSTRACT. Let the characteristic of a field F be different from 2; let φ be a quadratic form of dimension n over the field F; and let X_φ be a quadric in the projective space $\mathbf{P}_F^{n-1}$ defined by the equation $\varphi = 0$. Two invariants of the quadratic form φ are investigated: the Chow ring $\mathrm{CH}^* X_\varphi$ of the quadric X_φ and the graded ring $G^* K_0(X_\varphi)$ associated with the topological filtration of the Grothendieck ring $K_0(X_\varphi)$. The ring $\mathrm{CH}^* X_\varphi$ for every quadratic form of dimension not greater than 6 and the ring $G^* K_0(X_\varphi)$ for forms of dimension ≤ 7 and for 8-dimensional forms with determinant 1 are computed. The components $\mathrm{CH}^2 X_\varphi$ and $G^{n-3} K_0(X_\varphi)$ are computed for a quadratic form of arbitrary dimension. An important role in obtaining these results is played by Swan's computation of the K-theory of quadrics (R. G. Swan, *K-theory of quadratic hypersurfaces*, Ann. of Math. (2) **122** (1985), no. 1, 113–154).

At the beginning of the 80's the American mathematician R. G. Swan stated two problems concerning quadratic hypersurfaces in projective space: the first is the computation of their K-theory, the second is the computation of the Chow ring CH^* (the graded ring of cycles modulo rational equivalence). The first of these closely related problems was solved by Swan himself in 1985 [1]. In the same paper he remarked that the second problem is still open. Meanwhile the study of the norm residue homomorphism [2], [3]—a rapidly developing trend in algebraic K-theory in recent times—required the execution of some computations in the Chow groups of quadrics. This strengthened interest in Swan's second problem and gave rise to the present paper.

Let X be a projective quadric over a field F $(\operatorname{char} F \neq 2)$, defined by a nondegenerate quadratic form φ. Along with $\mathrm{CH}^* X$ we will consider another invariant of the form φ, namely the graded ring $G^* K(X)$ associated with the topological filtration on the Grothendieck group $K(X) = K_0'(X) = K_0(X)$ of the quadric. The rings $\mathrm{CH}^* X$ and $G^* K(X)$ are very similar (see, for example, Corollary (4.5)), and we can consider the computation of the groups $G^p K(X)$ as the first step in computing the groups $\mathrm{CH}^p X$.

Among the main results of this paper concerning the Chow ring is the complete computation of this ring for any quadric of dimension not higher than 4 (§5) and the computation of the component CH^2 for an arbitrary quadric (§6). As for results about $G^* K(X)$, it is worth noticing theorems in §3, which give, in particular, a formula for the order of torsion in $G^* K(X)$; if $\varphi \notin I^2(F)$,

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this formula is most simple: the order of the torsion subgroup of $G^*K(X)$ is equal to $2^{s(\varphi)-i(\varphi)}$, where $s(\varphi)$ is a number connected with the Clifford algebra of the form (3.3) and $i(\varphi)$ denotes the Witt index. Moreover, in §8 the computation of $G^*K(X)$ is carried out for quadrics of dimension 5 and for a class of 6-dimensional quadrics; in §7 the component $G^{d-1}K(X)$, where $d = \dim X$, is computed for an arbitrary quadric.

We mention the main conventions and notation of the paper. For an algebraic variety X over an arbitrary field F we denote by $K_p(X)$ and $K'_p(X)$ Quillen's K-groups and K'-groups [4], $K_p(F) = K_p(\operatorname{Spec} F)$. By a prime cycle on X we mean an irreducible closed subset of X, and by a cycle we mean an element of the free abelian group generated by the prime cycles. For a point $x \in X$ we denote by $F(x)$ the residue class field; the field of rational functions on X we denote by $F(X)$. As a rule, we will work with irreducible and nonsingular varieties. But in the intermediate steps of some computations singular quadrics (§§2 and 4) and even reducible ones (2.1) will appear. Therefore the fundamental facts about K-cohomology in §1 are quoted for arbitrary varieties. Beginning with §2, the letter X stands, unless otherwise stated, for a projective quadric defined by a nondegenerate quadratic form φ. For a field F whose characteristic is assumed to be different from 2, $I(F)$ denotes the ideal of even-dimensional forms in the Witt ring $W(F)$ of quadratic forms over F; $\langle a_1, a_2, \dots, a_n\rangle$ stands for the quadratic form $a_1X_1^2 + a_2X_2^2 + \cdots + a_nX_n^2$; $D(\varphi)$ is the set of nonzero values of the form φ; $d_{\pm}\varphi = (-1)^{n(n-1)/2} \cdot \det\varphi$, where $n = \dim\varphi$, is the discriminant; $i(\varphi)$ denotes the Witt index; $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ is an n-fold Pfister form; for two quadratic forms φ_1 and φ_2 we write $\varphi_1 \sim \varphi_2$ and say "φ_1 is proportional to φ_2" if $\varphi_1 \cong c\varphi_2$ for some $c \in F^*$ (proportional quadratic forms define the same quadric). Requisite definitions and facts from the theory of quadratic forms can be found in [5].

1. K-cohomology

In this section, algebraic varieties over a fixed field F are considered. By a variety one means a reduced separated scheme of finite type over F which is *equidimensional* (the dimension of every irreducible component is the same). We point out some more or less known facts about K-cohomology.

(1.1). The Brown-Gersten-Quillen spectral sequence (spectral BGQ-sequence). For an arbitrary variety X a canonical cohomology-type spectral sequence is constructed in [4], called the spectral BGQ-sequence,

$$E_1^{p,q}(X) = \coprod_{x\in X^p} K_{-p-q}(F(x)) \Rightarrow K'_{-p-q}(X),$$

where by X^p is denoted the set of those points of the variety X which have codimension p (by the dimension of a point one means the dimension of its closure). The associated filtration on $K'_*(X)$ coincides with the filtration by the codimension of the support (also called the topological filtration). We point out that the spectral BGQ-sequence is contravariant with respect to flat morphisms. Moreover, if $X = \varprojlim X_i$, where $i \mapsto X_i$ is a directed projective system of varieties together with flat affine morphisms, then the spectral sequence for X is the inductive limit of the spectral sequences for X_i.

For an arbitrary variety X we denote by $H^p(X, K_q)$ the group

$$H\left(\coprod_{x\in X^{p-1}} K_{q-p+1}(F(x)) \to \coprod_{x\in X^p} K_{q-p}(F(x)) \to \coprod_{x\in X^{p+1}} K_{q-p-1}(F(x))\right) = E_2^{p-q}(X).$$

Let $d = \dim X$. We will call the group $H^p(X, K_p)$ the Chow group of codimension p (of dimension $d-p$) and we will denote it by $\mathrm{CH}^p X$ (or $\mathrm{CH}_{d-p}X$); $\mathrm{CH}^* X = \coprod_{p=0}^{d} \mathrm{CH}^p X$ is the graded Chow group.

(1.2). Functorial properties. Any flat morphism $f: X \to Y$ induces a morphism of spectral BGQ-sequences, and consequently a group homomorphism $f^*: H^p(Y, K_q) \to H^p(X, K_q)$. Now let $f: X \to Y$ be a proper morphism. We obtain a homomorphism of complexes (where $n = \dim Y - \dim X$)

$$\begin{array}{ccccc} \cdots \to & \coprod_{x\in X^p} K_{q-p}(F(x)) & \to & \coprod_{x\in X^{p+1}} K_{q-p-1}(F(x)) & \to \cdots \\ & \downarrow & & \downarrow & \\ \cdots \to & \coprod_{y\in Y^{p+n}} K_{q-p}(F(y)) & \to & \coprod_{y\in Y^{p+n+1}} K_{q-p-1}(F(y)) & \to \cdots \end{array}$$

which sends elements of the group $K_{q-p}(F(x))$ into $K_{q-p}(F(f(x))$ by the norm function $N_{F(x)/F(f(x))}$ if $\dim x = \dim f(x)$, and to zero otherwise. This homomorphism of complexes defines the group homomorphism $f_*: H^p(X, K_q) \to H^{p+n}(X, K_{q+n})$.

In particular, let E/F be an arbitrary extension, $X_E = X\otimes_F E$ and $f: X_E \to X$ be the projection. The morphism f is flat; the induced homomorphism f^* is denoted by $\mathrm{res}_{E/F}$. If the extension E/F is finite, then the morphism f is proper (more precisely, it is finite); the homomorphism f_* that arises is denoted by $N_{E/F}$. The composition $N_{E/F} \circ \mathrm{res}_{E/F}$ coincides with multiplication by $[E:F]$.

(1.3). Two exact sequences, which will be constructed now, are the main tools of computation.

(1.3.1). *Excision.* Let $i: Y \hookrightarrow X$ be a closed imbedding of codimension n and let $j: U = X\setminus Y \hookrightarrow X$. For any p, the set X^p is a disjoint union of the sets U^p and Y^{p-n}. The exact sequence of complexes

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \coprod_{x\in Y^{p-n}} K_{q-p}(F(x)) & \to & \coprod_{x\in X^p} K_{q-p}(F(x)) & \to & \coprod_{x\in U^p} K_{q-p}(F(x)) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & \vdots & & \vdots & & \vdots & \end{array}$$

that arises induces the exact sequence of cohomology groups

$$\cdots \to H^{p-n}(Y, K_{q-n}) \xrightarrow{i_*} H^p(X, K_q) \xrightarrow{j^*} H^p(U, K_q) \to H^{p-n+1}(Y, K_{q-n}) \xrightarrow{i_*} \cdots.$$

In particular, for the Chow groups we get $\mathrm{CH}^{p-n} Y \xrightarrow{i_*} \mathrm{CH}^p X \xrightarrow{j^*} \mathrm{CH}^p U \to 0$.

(1.3.2). *Fibering over a curve.* Let $\pi: X \to Y$ be a flat morphism of varieties with $\dim Y = 1$. Let us denote by θ the generic point of the curve Y. For any p the set X^p is a disjoint union of the sets $(X_\theta)^p$ and $\bigcup_{y\in Y^1}(X_y)^{p-1}$ (X_y denotes the fiber of π over the point y). The resulting short exact sequence of complexes

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & \coprod_{y\in Y^1}\left(\coprod_{x\in(X_y)^{p-1}} K_{q-p}(F(x))\right) & \to & \coprod_{x\in X^p} K_{q-p}(F(x)) & \to & \coprod_{x\in(X_\theta)^p} K_{q-p}(F(x)) & \to 0, \\ & \downarrow & & \downarrow & & \downarrow & \\ & \vdots & & \vdots & & \vdots & \end{array}$$

where the left term is a direct sum of complexes, gives the exact cohomology sequence

$$\cdots \to \coprod_{y\in Y^1} H^{p-1}(X_y, K_{q-1}) \xrightarrow{\sum(i_y)_*} H^p(X, K_q) \xrightarrow{j^*} H^p(X_\theta, K_q)$$
$$\to \coprod_{y\in Y^1} H^p(X_y, K_{q-1}) \to \cdots,$$

where i_y denotes the closed imbedding $X_y \hookrightarrow X$ and $j\colon X_\theta \to X$ denotes the natural morphism from the fiber over the generic point, a flat morphism. In particular, for the Chow groups we obtain an exact sequence

$$\coprod_{y\in Y^1} \mathrm{CH}^{p-1} X_y \to \mathrm{CH}^p X \to \mathrm{CH}^p X_\theta \to 0.$$

(1.4). Homotopy invariance. Let $\pi\colon X \to Y$ be a vector bundle. Then the mapping $\pi^*\colon H^p(Y, K_q) \to H^p(X, K_q)$ is an isomorphism. In particular, for the affine space $\mathbf{A}_F^n$ we obtain

$$H^p(\mathbf{A}_F^n, K_q) \cong H^p(\operatorname{Spec} F, K_q) = \begin{cases} 0 & \text{if } p > 0, \\ K_q(F) & \text{if } p = 0. \end{cases}$$

(1.5). Smooth varieties. Let $\mathscr{K}_q = \mathscr{K}_q(X)$ be the sheaf on the variety X associated with the presheaf $U \mapsto K_q(U)$. If X is nonsingular, then for any p the cohomology group $H^p(X, \mathscr{K}_q)$ of the sheaf $\mathscr{K}_q$ is canonically isomorphic to the group $H^p(X, K_q)$.

(1.5.1). *Multiplicative structure.* The product operation in K-theory defines a pairing of the sheaves $\mathscr{K}_q \times \mathscr{K}_{q'} \to \mathscr{K}_{q+q'}$ and consequently a pairing of the cohomology groups $H^p(X, \mathscr{K}_q) \times H^{p'}(X, \mathscr{K}_{q'}) \to H^{p+p'}(X, \mathscr{K}_{q+q'})$. This pairing defines the structure of a bigraded (commutative and associative) ring on the group $\coprod_{p,q\geq 0} H^p(X, \mathscr{K}_q)$. In particular, the Chow group CH^*X becomes a graded ring.

(1.5.2). *The inverse image homomorphism* constructed in (1.2) for a flat morphism can be defined for an arbitrary morphism $f\colon X \to Y$ in the case of smooth varieties $f^*\colon H^p(Y, \mathscr{K}_q) \to H^p(X, \mathscr{K}_q)$ is induced by the natural morphism of sheaves $\mathscr{K}_q(Y) \to f_*(\mathscr{K}_q(X))$. In addition, $f^*\colon H^*(Y, \mathscr{K}_*) \to H^*(X, \mathscr{K}_*)$ is a ring homomorphism. If the morphism f is proper, then f_* and f^* are linked by the projection formula.

(1.6). K-cohomology of the projective space $\mathbb{P}^n$ can easily be computed using the exact sequence (1.3.1) for the triple $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n \hookleftarrow \mathbb{A}^n$. We present a description of the Chow ring: the ring homomorphism $\mathbb{Z}[h]/(h^{n+1}) \to \mathrm{CH}^{\bullet}\mathbb{P}^n$ sending h to the class of a hyperplane is an isomorphism.

2. Elementary information about Chow groups of projective quadrics

Henceforth we will consider only fields of characteristic different from 2. Let φ be a nondegenerate quadratic form of dimension $d+2$ $(d \geq 1)$ over a field F, and let m be the integral part of the number $(d+1)/2$. In the sequel (unless otherwise stipulated) the letter X will always denote a quadric in the projective space $\mathbb{P}_F^{d+1}$, determined by a form φ (in particular, $\dim X = d$). The variety X is nonsingular, since φ is nondegenerate; $\mathrm{CH}^0 X = \mathbb{Z} \cdot [X]$ because X is irreducible. Let $h \in \mathrm{CH}^1 X$ be the class of a hyperplane section (more precisely, h be the inverse image of a hyperplane with respect to the imbedding $X \hookrightarrow \mathbb{P}^{d+1}$).

(2.1). Computation of the Chow groups in case of a maximally splitting form. Let $\varphi = X_0X_1 + \cdots$, and Y be the section of X by the hyperplane $X_0 = 0$. Then Y is a singular quadric of dimension $d-1$ and $U = X \setminus Y \cong \mathbb{A}^d$. The exact excision sequence reduces the question of the Chow groups of X to the question of the Chow groups of Y. Continuing in this manner, we reduce the problem to the computation of Chow groups of a projective space (see (1.6)). We present the final result. First suppose d is odd; then $\varphi = X_0X_1 + \cdots + X_{d-1}X_d + X_{d+1}^2$. The quadric X contains the $(m-1)$-dimensional projective space $\mathbb{P}^{m-1}$ (which is defined, say, by equations $X_0 = X_2 = X_4 = \cdots = X_{d+1} = 0\}$; consequently it contains projective spaces of every smaller dimension. Let $l_p \in \mathrm{CH}_p X$ be the class of $\mathbb{P}^p \subset X$; then $\mathrm{CH}^p X = \mathbb{Z} \cdot h^p$ and $\mathrm{CH}_p X = \mathbb{Z} \cdot l_p$ for $p < m$. But if d is even and hence $\varphi = X_0X_1 + \cdots + X_{d-2}X_{d-1} + X_dX_{d+1}$, then the cycles $X_0 = X_2 = X_4 = \cdots = X_d = 0$ and $X_1 = X_2 = X_4 = \cdots = X_d = 0$ are not equivalent; let $l_m, l'_m \in \mathrm{CH}_m X$ denote the classes of these cycles. Then $\mathrm{CH}_m X = \mathbb{Z} \cdot l_m \oplus \mathbb{Z} \cdot l'_m$ and for $p < m$ we have as before $\mathrm{CH}^p X = \mathbb{Z} \cdot h^p$ and $\mathrm{CH}_p X = \mathbb{Z} \cdot l_p$. Multiplication in $\mathrm{CH}^{\bullet} X$ is described by the formulas

$$h l_p = l_{p+1}, \qquad h^m = \begin{cases} l_m + l'_m & \text{if } d \text{ is even}, \\ 2l_{m-1} & \text{if } d \text{ is odd}. \end{cases}$$

(2.2). Reduction to the anisotropic case. Computation of the Chow groups for arbitrary quadrics reduces to the anisotropic case in the following way. Let $\varphi = \psi \perp n\mathbb{H}$, where ψ is the anisotropic part of φ and n is the Witt index of φ; let Y denote the nonsingular quadric corresponding to the form ψ $(\dim Y = d - 2n)$. We will denote the quadric $X \otimes_F E$ by $\overline{X}$, where E/F is a field extension which completely splits φ. Then $\mathrm{CH}^p X \overset{\text{res}}{\underset{\sim}{\to}} \mathrm{CH}^p \overline{X}$ and $\mathrm{CH}_p X \overset{\text{res}}{\underset{\sim}{\to}} \mathrm{CH}_p \overline{X}$ for $p < n$; $\mathrm{CH}^p X \cong \mathrm{CH}^{p-n} Y$ for $p = n, n+1, \ldots, d-n$. We develop the construction of the last isomorphism. Let $\varphi = \psi(Y_0, \ldots, Y_{d-2n+1}) + X_1X'_1 + \cdots + X_nX'_n$, and $i \colon Z \hookrightarrow X$ be the closed subvariety defined by the equations $X'_1 = X'_2 = \cdots = X'_n = 0$. The singular quadric Z is a cone over Y with vertex $\mathbb{P}^{n-1}$. If we throw the vertex out, we obtain a vector bundle $\pi\colon Z\backslash\mathbb{P}^{n-1} \to Y$. The chain of morphisms $Y \overset{\pi}{\leftarrow} Z\backslash\mathbb{P}^{n-1} \overset{j}{\hookrightarrow} Z \overset{i}{\hookrightarrow} X$ gives a chain of isomorphisms

$$\mathrm{CH}^{p-n} Y \underset{\sim}{\overset{\pi^*}{\to}} \mathrm{CH}^{p-n}(Z\backslash\mathbb{P}^{n-1}) \underset{\sim}{\overset{j^*}{\leftarrow}} \mathrm{CH}^{p-n} Z \underset{\sim}{\overset{i_*}{\to}} \mathrm{CH}^p X.$$

The isomorphism thus constructed sends a prime cycle on Y, defined, say, by equations $f_\alpha(Y_i) = 0$, to a prime cycle on X, namely, the cycle defined by equations $f_\alpha(Y_i) = 0$, $X'_1 = X'_2 = \cdots = X'_n = 0$.

For the computation of the divisor class group $\mathrm{CH}^1 X$ we need

(2.3). LEMMA. *Let E/F be a Galois extension with Galois group G. There exists an exact sequence $0 \to \mathrm{CH}^1 X \to (\mathrm{CH}^1 X_E)^G \to \mathrm{Br}(E/F)$, where $\mathrm{Br}(E/F)$ denotes the relative Brauer group.*

PROOF. We have two exact sequences of G-modules

$$0 \to E^* \to E(X_E)^* \to \mathrm{P.Div}\, X_E \to 0, \tag{1}$$

$$0 \to \mathrm{P.Div}\, X_E \to \mathrm{Div}\, X_E \to \mathrm{CH}^1 X_E \to 0, \tag{2}$$

where Div and P.Div denote the divisor group and the principal divisor group, respectively. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & E^{*G} & \to & E(X_E)^{*G} & \to & (\mathrm{P.Div}\, X_E)^G & \to & H^1(G, E^*) \\ & & \| & & \| & & & & \\ 0 & \to & F^* & \to & F(X)^* & \to & \mathrm{P.Div}\, X & \to & 0 \end{array}$$

The top row of the diagram is the beginning of the long exact sequence of Galois cohomology groups for (1). By Hilbert's Theorem 90 $H^1(G, E^*) = 0$. Consequently, $(\mathrm{P.Div}\, X_E)^G = \mathrm{P.Div}\, X$. From (2) we obtain

$$\begin{array}{ccccccccc} 0 & \to & (\mathrm{P.Div}\, X_E)^G & \to & (\mathrm{Div}\, X_E)^G & \to & (\mathrm{CH}^1 X_E)^G & \to & H^1(G, \mathrm{P.Div}\, X_E) \\ & & \| & & \| & & & & \\ 0 & \to & \mathrm{P.Div}\, X & \to & \mathrm{Div}\, X & \to & \mathrm{CH}^1 X & \to & 0 \end{array}$$

and this gives us the exact sequence

$$0 \to \mathrm{CH}^1 X \to (\mathrm{CH}^1 X_E)^G \to H^1(G, \mathrm{P.Div}\, X_E).$$

It remains to use the injection $H^1(G, \mathrm{P.Div}\, X_E) \hookrightarrow H^2(G, E^*) = \mathrm{Br}(E/F)$, which arises from (1), since $H^1(G, E(X_E)^*) = 0$. ●

According to (2.2) it is sufficient to describe $\mathrm{CH}^1 X$ in the anisotropic case only.

(2.4). PROPOSITION. *Let φ be anisotropic. The divisor class group $\mathrm{CH}^1 X$ is equal to $\mathbb{Z} \cdot h$ for every quadric, unless it is two-dimensional and defined by a form of determinant 1. In that exceptional case φ has, up to scalar multiple, the form $\langle\langle a, b\rangle\rangle$, and the homomorphism* $\mathrm{res}\colon \mathrm{CH}^1 X \to \mathrm{CH}^1 X_E = \mathbb{Z} \oplus \mathbb{Z}$ *identifies $\mathrm{CH}^1 X$ with the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by the elements $(1, 1) = h$ and $(2, 0)$. If, say, $\varphi = X_0^2 - aX_1^2 - bX_2^2 + abX_3^2$, then $(2, 0) = [Z]$, where $Z\colon X_0^2 - aX_1^2 = 0$, $X_0X_2 - aX_1X_3 = 0$.*

PROOF. We apply Lemma (2.3) to a quadratic extension E/F splitting φ. If $d \geq 3$ or if $d = 2$, but $\det\varphi \neq 1$, then $\mathrm{CH}^1 X_E = \mathbb{Z} \cdot h$, and it follows immediately that $\mathrm{CH}^1 X = \mathbb{Z}\cdot h$. If $\varphi \sim \langle\langle a, b\rangle\rangle$, then $\mathrm{CH}^1 X_E = \mathbb{Z}\oplus\mathbb{Z}$, G acts trivially on $\mathrm{CH}^1 X_E$, the images of the elements $(1, 0)$ and $(0, 1)$ in $\mathrm{Br}(E/F)$ coincide and are equal to the class of the quaternion algebra $\left(\frac{a,b}{F}\right)$. Since a quaternion algebra has exponent 2 we conclude that $\mathrm{CH}^1 X$ is the subgroup of elements of $\mathbb{Z}\oplus\mathbb{Z}$ with even sum of coordinates. Finally, if $d = 2$ and $\varphi \sim \langle 1, -a, -b\rangle$, then $\mathrm{CH}^1 X_E = (\mathrm{CH}^1 X_E)^G - \mathbb{Z}\cdot \mathfrak{l}_0$ and the image of $\mathfrak{l}_0$ in $\mathrm{Br}(E/F)$ is again equal to $[\left(\frac{a,b}{F}\right)]$. Consequently, $\mathrm{CH}^1 X$ is generated by the element $2\mathfrak{l}_0$, which is equal to h. •

In the sequel a fact about Chow groups of an arbitrary variety will be needed.

(2.5). Lemma. *Let X be an arbitrary variety over F, Z be a prime cycle on X, and E/F be a finite field extension with $E \subset F(Z)$. Then $[Z] = N_{E/F}([T])$ for some prime cycle T on X_E.*

Proof. First let $X = \operatorname{Spec} A$. Let us denote by $\mathfrak{p}$ the prime ideal of A corresponding to Z. The morphism $X_E \to X$ is induced by the injection $A \hookrightarrow B = A\otimes_F E$. B is finitely generated as an A-module; in particular, $A \hookrightarrow B$ is an integral ring extension, and therefore $\mathfrak{p} = \mathfrak{q}\cap A$ for some prime ideal $\mathfrak{q} \subset B$. Let T be a prime cycle on X_E corresponding to $\mathfrak{q}$. Then $N([T]) = [E(T) : F(Z)]\cdot[Z]$. The ring B is generated over A by elements of E; therefore $E(T)$ is also generated over $F(Z)$ by elements of E, but here $E \subset F(Z)$. Consequently, $E(T) = F(Z)$ and $N([T]) = [Z]$. In the general case let U be an open affine set intersecting Z and let T' be a prime cycle on U_E such that $N([T]) = [Z\cap U]$, where T is the closure of T' in X_E. Then $N([T]) = [Z]$. •

Again let X be a projective quadric defined by a nondegenerate quadratic form φ. We give a computation of the zero-dimensional Chow group.

(2.6). Proposition. *The degree homomorphism* $\deg\colon \mathrm{CH}_0 X \to \mathbb{Z}$ *is injective. Consequently, if φ is isotropic, then $\mathrm{CH}_0 X = \mathbb{Z}\cdot\mathfrak{l}_0 = \mathbb{Z}\cdot[x]$, where x is an arbitrary closed rational point; if φ is anisotropic, then $\mathrm{CH}_0 X = \mathbb{Z}\cdot h^d = \mathbb{Z}\cdot[x]$, where x is an arbitrary closed point of degree 2.*

Proof is required in the anisotropic case only. We develop it first under the assumption that the field F has no extension of odd degree. For $d = 1$ the statement is a particular case of (2.4). Let $d \geq 2$. We take two closed points $x, y \in X$ of degree two and show that $[x] = [y]$. Any two closed points of degree 2 in $\mathbb{P}^{d+1}$ lie in the same 3-dimensional linear subvariety $\mathbb{P}^3 \subset \mathbb{P}^{d+1}$. Taking the intersection of X with this $\mathbb{P}^3$ we reduce the problem to the case $d = 2$. Let us pass a plane through x and a plane through y. As planes in $\mathbb{P}^3$, they intersect in a line. This line cuts out from the quadric a point z of degree 2. Consequently, the two planes cut out from the quadric two conics intersecting at the point z; moreover, one of the conics contains x and the other contains y. Hence $[x] = [z] = [y]$. Now let the degree of y be arbitrary, say, $2k$. We prove that $[y] = k[x]$. Since $[F(y) : F]$ is a power of two (see the assumption on F), there exists an intermediate extension $F \subset E \subset F(y)$ such that $[F(y) : E] = 2$. Let y' be a point of X_E for which $N_{E/F}([y']) = [y]$ (by Lemma (2.5) such a point exists) and let $\alpha = \operatorname{res}_{E/F}([x])$. Then $\deg y' = 2 = \deg\alpha$; therefore $[y'] = \alpha$ and hence $[y] = N_{E/F}([y']) = N_{E/F}(\alpha) = [E : F]\cdot[x] = k\cdot[x]$.

In the case of an arbitrary base field, let F' be the compositum of all finite extensions E/F of odd degree. Every homomorphism $\mathrm{res}_{E/F}\colon \mathrm{CH}_0 X \to \mathrm{CH}_0 X_E$ is injective, since the composition $N_{E/F} \circ \mathrm{res}_{E/F}$ is multiplication by an odd number; $\mathrm{CH}_0 X_{F'} = \varinjlim_{E/F} \mathrm{CH}_0 X_E$, by (1.1). Therefore $\mathrm{res}_{F'/F}\colon \mathrm{CH}_0 X \to \mathrm{CH}_0 X_{F'}$ is also a monomorphism, which completes the proof. ●

(2.7). Some remarks about Chow groups of arbitrary codimension in the case of an anisotropic quadric. For any p let $\mathrm{TCH}^p \subset \mathrm{CH}^p$ be the torsion subgroup, $\overline{\mathrm{CH}}^p = \mathrm{CH}^p/\mathrm{TCH}^p$. For $p \neq d/2$ the computation of $\overline{\mathrm{CH}}^p$ is trivial: $\overline{\mathrm{CH}}^p = \mathbb{Z}\cdot h^p$, in other words,

$$\overline{\mathrm{CH}}^p = \begin{cases} \mathbb{Z} & \text{if } p < d/2, \\ 2\mathbb{Z} & \text{if } p > d/2, \end{cases}$$

where we identify $\overline{\mathrm{CH}}^p X$ with a subgroup of $\mathrm{CH}^p \overline{X}$. The group $\overline{\mathrm{CH}}^{d/2} X \subset \mathrm{CH}^{d/2}\overline{X} = \mathbb{Z}\oplus\mathbb{Z}$ (where d is even) is generated by the element $(1, 1) = h^{d/2}$, if $\varphi \notin I^2(F)$. In the contrary case $\overline{\mathrm{CH}}^{d/2} X$ has the two generators $(1, 1)$ and $(2^r, 0)$ for some $r \geq 1$, and a method of computing $r = r(\varphi)$ is unknown; it is clear only that 2^r divides $[E : F]$, where E/F is an arbitrary extension which completely splits φ, in particular, $r \leq m$. The group TCH^p is a 2-torsion group. For the proof, it suffices to take an extension E/F which completely splits φ, such that $[E : F] = 2^K$. The composition $\mathrm{CH}^p X \overset{\mathrm{res}}{\to} \mathrm{CH}^p\overline{X} \overset{N}{\to} \mathrm{CH}^p X$ coincides with multiplication by 2^K, and $T\mathrm{CH}^p\overline{X} = 0$. In the rest, the structure of the group TCH^* is very enigmatic; one may give an example of a quadric for which this group is infinite.

3. The topological filtration on the Grothendieck group

Let $K(X)$ be the Grothendieck group of the quadric X. The variety X is nonsingular and therefore $K(X) = K_0'(X) = K_0(X)$. The tensor product of locally free sheaves induces a ring structure on $K_0(X)$, and the group $K_0'(X)$ admits so-called topological filtration [4]. Therefore $K_0(X)$ is a filtered ring and the two structures are compatible. We introduce notation for the filtration: $0 = K^{(d+1)} \subset K^{(d)} \subset \cdots \subset K^{(0)} = K$, $K_{(p)} = K^{(d-p)}$. Moreover, we put $G^p K = G_{d-p} K = K^{(p)}/K^{(p+1)}$, and let $TG^p K \subset G^p K$ be the torsion subgroup, with $\overline{G}^p K = G^p K/TG^p K$.

(3.1). Connection with the Chow groups. The natural epimorphism $\mathrm{CH}^p X \twoheadrightarrow G^p K(X)$, $[Z] \mapsto [\mathcal{O}_Z]$, coincides with the edge homomorphism of the spectral BGQ-structure $E_2^{p,q}(X) \Rightarrow K'_{p+q}(X)$. The kernel of this epimorphism is contained in $\mathrm{TCH}^p X$ for each p and is equal to zero if $p = 0, 1, 2, 3, d$ (for $p = 0, 1$ there is in the spectral sequence no differential ending in $\mathrm{CH}^p X$; for $p = 2$ the only differential from $E_2^{0,-1} = H^0(X, K_1) = F^*$ to $\mathrm{CH}^2 X$ is obviously the zero differential; the case $p = 3$ is examined in §4; finally, the group $\mathrm{CH}^d X$ is torsion-free by (2.6)). Note that the mapping $\mathrm{CH}^* X \to G^* K(X)$ is a graded ring homomorphism. In this section we obtain some information about $G^* K(X)$ with the help of the paper [1], devoted to the K-theory of quadrics.

(3.2). The case of completely splitting form. If the form φ, which defines the quadric, is completely splitting, then $\mathrm{TCH}^p X = 0$ by (2.1), whence $\mathrm{CH}^p X \to G^p K(X)$ is an isomorphism for every p. Consequently, if d is odd, then $K(X)$

is a free abelian group of rank $d+1$ with generators $1, h, h^2, \dots, h^{m-1}, l_{m-1}, l_{m-2}, \dots, l_0$ (where h denotes the class in $K(X)$ of the structure sheaf of a hyperplane section and l_p denotes the class of the structure sheaf of the space $\mathbb{P}^p \subset X$). But if d is even, then $K(X)$ is a free abelian group of rank $d+2$: the generators l_m and l'_m must be added. The filtration on $K(X)$ is natural in both cases: the group $K^{(p)}(X)$ is generated by those generators whose codimension is not less than p (codim $l_p = d-p$). Note that

$$h l_p = l_{p+1}, \qquad h^m = \begin{cases} l_m + l'_m - l_{m-1} & \text{for even } d, \\ 2l_{m-1} - l_{m-2} & \text{for odd } d. \end{cases}$$

(3.3). The Clifford algebra. Now let the Witt index of φ be arbitrary, and $C_0(\varphi)$ be the even component of the Clifford algebra $C(\varphi)$. Note that $\dim_F C_0(\varphi) = 2^{d+1}$. We define the number $s = s(\varphi)$ in the following way. If $\varphi \notin I^2(F)$, then $C_0(\varphi)$ is a simple algebra; hence by the Wedderburn therem $C_0(\varphi) \cong M_{2^s}(D)$ for some skew field D and a nonnegative integer s. But if $\varphi \in I^2(F)$, then $C_0(\varphi) \cong A \times A$, where A is a simple algebra (it can be proved that A is determined by the condition $C(\varphi) \cong M_2(A)$); in this case let s be a number such that $A \cong M_{2^s}(D)$. The invariant s satisfies the inequality $i(\varphi) \le s(\varphi) \le m$, where $i(\varphi)$ denotes the Witt index.

(3.4). As always let $\overline{X} = X \otimes_F E$, where E/F is an extension which completely splits φ. The group $K(X)$ is torsion-free [1]; therefore the ring homomorphism res: $K(X) \to K(\overline{X})$ is injective. Let us identify $K(X)$ with a subring of $K(\overline{X})$. Note that $K^{(p)}(X) \subset K^{(p)}(\overline{X})$. Let $H = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot h \oplus \cdots \oplus \mathbb{Z}\cdot h^d \subset K(X)$ be the subring generated by h. The equality $h = 1 - \xi$, where $\xi = [\mathscr{O}_X(-1)]$, shows that H coincides with the subgroup generated by all $[\mathscr{O}_X(n)]$ $(n \in \mathbb{Z})$. Observe that the two filtrations induced from $K(X)$ and $K(\overline{X})$ coincide: H^p is generated by the element $h^p, h^{p+1}, \dots, h^d$.

(3.5). PROPOSITION. *Suppose $\varphi \notin I^2(F)$. Then*

$$K(X) = H + \mathbb{Z}\cdot l_{s-1} = H + \mathbb{Z}\cdot 2^{m-s} l_{m-1}$$

($s = s(\varphi)$ is the splitting index; it is assumed that $l_{-1} = 0$); in particular, if $s = 0$, then $K(X) = H$. But if $\varphi \in I^2(F)$, then

$$K(X) = H + \mathbb{Z}\cdot 2^{m-s} l_m + \mathbb{Z}\cdot 2^{m-s} l'_m.$$

(3.6). LEMMA. *Let $\mathscr{U}$ denote the Swan sheaf of the quadric X [1]. Then $[\mathscr{U}(d)] = h^d + 2h^{d-1} + \cdots + 2^{d-1}h + 2^d$ in the group $K(X)$.*

PROOF. The sheaf $\mathscr{U}$ admits a resolution composed of the sheaves $\mathscr{O}_X(-d+1), \dots, \mathscr{O}_X(-1), \mathscr{O}_X, \mathscr{O}_X(1)$, and this resolution gives us the following equality in $K(X)$:

$$\begin{aligned}[\mathscr{U}] = {} & (C^0_{d+1} + C^1_{d+1} + \cdots + C^d_{d+1})\xi^{d-1} - (C^0_{d+1} + C^1_{d+1} + \cdots + C^{d-1}_{d+1})\xi^{d-2} \\ & + \cdots + (-1)^{d-2}(C^0_{d+1} + C^1_{d+1} + C^2_{d+1})\xi \\ & + (-1)^{d-1}(C^0_{d+1} + C^1_{d+1}) + (-1)^d C^0_{d+1}\xi^{-1}.\end{aligned}$$

Hence

$$(1+\xi)[\mathscr{U}] = (2^{d+1}-1)\xi^d + (-1)^d\xi^{-1} - \sum_{k=1}^{d} C_{d+1}^k(-1)^{d+1-k}\xi^{k-1} = 2^{d+1}\xi^d,$$

since $(1-\xi)^{d+1} = h^{d+1} = 0$. Therefore $(1+\xi)[\mathscr{U}(d)] = (1+\xi)[\mathscr{U}]\cdot\xi^{-d} = 2^{d+1}$. On the other hand, $1+\xi = 2-h$ and $(2-h)(h^d + 2h^{d-1} + \cdots + 2^d) = 2^{d+1}$. Consequently, $[\mathscr{U}(d)] = h^d + 2h^{d-1} + \cdots + 2^d$. ●

PROOF OF THE PROPOSITION. Let us examine the case when $\varphi \notin I^2(F)$. Thus $C_0(\varphi)$ is a simple algebra and hence there exists a unique (up to isomorphism) simple $C_0(\varphi)$-module; we will denote this module by J. We have $K_0(C_0(\varphi)) = \mathbb{Z}\cdot[J]$ and $[C_0(\varphi)] = 2^s[J]$. Further, if $u = [\mathscr{U}\otimes_{C_0} J] \in K(X)$, then $2^s u = [\mathscr{U}]$ and hence, by [1], $K(X) = H + \mathbb{Z}\cdot u$. Let us replace u by $u\xi^{-d}$. Then $2^s u = [\mathscr{U}(d)]$ as before, and $K(X) = H + \mathbb{Z}\cdot u$, since $\xi^{-d}H = H$. Thus the assertion of the proposition results from the equality $2^{p+1}\mathfrak{l}_p = h^d + 2h^{d-1} + \cdots + 2^p h^{d-p}$ (for any $p < m$). In the case $\varphi \in I^2(F)$ the algebra $C_0(\varphi)$ is isomorphic to $A \times A$ and hence there are two nonisomorphic simple $C_0(\varphi)$-modules, say, J and J'; the group $K_0(C_0(\varphi))$ equals $\mathbb{Z}\cdot[J]\oplus\mathbb{Z}\cdot[J']$. Let us set

$$u = [\mathscr{U}\oplus_{C_0} J], \qquad u' = [\mathscr{U}\oplus_{C_0} J'].$$

Then

$$K(X) = H + \mathbb{Z}\cdot u + \mathbb{Z}\cdot u'.$$

Let $\overline{u}$ and $\overline{u}'$ be the Swan generators of the group $K(\overline{X})$. Then

$$K(\overline{X}) = H + \mathbb{Z}\cdot\overline{u} + \mathbb{Z}\cdot\overline{u}' = H + \mathbb{Z}\cdot\mathfrak{l}_m + \mathbb{Z}\cdot\mathfrak{l}'_m;$$
$$2^{m-s}\overline{u} = u, \qquad 2^{m-s}\overline{u}' = u'.$$

Consequently,

$$K(X) = H + \mathbb{Z}\cdot u + \mathbb{Z}\cdot u' = H + \mathbb{Z}\cdot 2^{m-s}\overline{u} + \mathbb{Z}\cdot 2^{m-s}\overline{u}'$$
$$= H + \mathbb{Z}\cdot 2^{m-s}\mathfrak{l}_m + \mathbb{Z}\cdot 2^{m-s}\mathfrak{l}'_m. \quad ●$$

(3.7). By the proposition $\mathfrak{l}_{s-1} \in K(X)$, whence $\mathfrak{l}_{s-2} = h\mathfrak{l}_{s-1}$, $\mathfrak{l}_{s-3} = h\mathfrak{l}_{s-2}, \ldots, \mathfrak{l}_0 = h\mathfrak{l}_1$ belong to $K(X)$. Let $p_k = \min\{p \mid \mathfrak{l}_k \in K_{(p)}(X)\}$. It is clear that $p_0 < p_1 < \cdots < p_{s-1}$ and $p_k \geq k$. Moreover, if the form φ is anisotropic, then $p_k > k$, since the equality $p_k = k$ for some k would imply that $\mathfrak{l}_0 = h^k\mathfrak{l}_k$ belonged to $K_{(0)}(X)$, which is obviously not true for a quadric without rational points.

(3.8). THEOREM. *Let X be a quadric defined by an anisotropic form $\varphi \notin I^2(F)$; let $p_0, p_1, \ldots, p_{s-1}$ be the numbers defined above. Then*

$$TG_pK(X) = \begin{cases} \mathbb{Z}/2\cdot\mathfrak{l}_k & \text{if } p = p_k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $|TGK(X)| = 2^{s(\varphi)}$; if $s(\varphi) = 0$, then the group $GK(X)$ is torsion-free.

(3.9). Lemma. *Under the assumptions of Theorem* (3.8) *the equality* $K_p(X) = \mathbb{Z}\cdot \mathfrak{l}_{k_0} + H_{(p)}$ *holds for* $k_0 = \max\{k | \mathfrak{l}_k \in K_{(p)}(X)\}$.

Proof. The inclusion $\supset$ is evident. We prove the reverse inclusion. Let $x = a\mathfrak{l}_{s-1} + \sum a_i h^i \in K_{(p)}$. Using the equality $2\mathfrak{l}_j = h^{d-j} + \mathfrak{l}_{j-1}$, we represent x in the form $x = b\mathfrak{l}_k + \sum b_i h^i$ with odd b. If $k \leq k_0$, then the first summand belongs to $K_{(p)}$; therefore the second belongs to $K_{(p)} \cap H = H_{(p)}$. Let $k > k_0$. One may assume that $k = k_0 + 1$ (replacing x by xh^{k-k_0+1}); hence $b = 1$ (since $2\mathfrak{l}_{k_0+1} = h^{d-k_0-1} + \mathfrak{l}_{k_0}$). So $\mathfrak{l}_{k_0+1} + \sum b_i h^i \in K_{(p)}$, whence $\sum b_i h^i \in K_{(p)}$, which contradicts the choice of the number k_0. ●

Proof of the theorem. For $p \neq p_k$ the group G_pK coincides, by the lemma, with the torsion-free group $H_{(p)}/H_{(p-1)}$. For $p = p_k$ the group G_pK is generated by the elements $\mathfrak{l}_k$ and $\hbar^{d-p}$; in addition $\mathfrak{l}_k \neq 0$, $2\mathfrak{l}_k = 0$, and $\hbar^{d-p} \notin TG_pK$. Therefore $TG_pK = \mathbb{Z}/2\cdot l_k$. ●

The picture is more complicated in the case when $\varphi \in I^2(F)$. Let $r = r(\varphi)$ be a number such that $(2^r, 0)$ is the second generator of $\overline{\mathrm{CH}}^m X$ (see (2.7)).

(3.10). Theorem. *Let X be a quadric defined by an anisotropic form $\varphi \in I^2(F)$. Then the graded group $TGK(X)$ can be decomposed into a direct sum $T^{\mathrm{I}} \oplus T^{\mathrm{II}}$ of graded subgroups (T^{I} will be called "a torsion subgroup of the first kind", and T^{II} "a torsion subgroup of the second kind"), so that*

1) *T^{I} has the same properties as the torsion subgroup in the case $\varphi \notin I^2(F)$, namely*

$$T_p^{\mathrm{I}} = \begin{cases} \mathbb{Z}/2\cdot \mathfrak{l}_k & \text{if } p = p_k, \\ 0 & \text{otherwise}; \end{cases}$$

2) *$T_p^{\mathrm{II}} = 0$ for $p \leq m$, $|T^{\mathrm{II}}| = 2^t$, where t is a nonnegative integer satisfying the inequality $r + s - m \leq t \leq s$, and T_p^{II} is a cyclic group for every p. In particular, $|TGK(X)| = 2^{s+t}$; if $s(\varphi) = 0$, then the group $GK(X)$ is torsion-free.*

Proof. According to the statement of the theorem it is clear that T^{I} is the subgroup of TGK generated by the elements $\mathfrak{l}_0, \mathfrak{l}_1, \dots, \mathfrak{l}_{s-1}$. We define the group T^{II}. Let $\mathfrak{l} = 2^{m-s}\mathfrak{l}_m \in K(X)$, $t = \min\{t' | 2^{t'}\cdot \mathfrak{l} \in K_{(m)}\}$, and $q_k = \min\{q | 2^k \mathfrak{l} \in K_{(q)}\}$ for $k = 0, 1, \dots, t-1$. It is clear that $q_0 \geq q_1 \geq \dots \geq q_{t-1} > m$. If q is not equal to any of the numbers q_k, then we put $T_q^{\mathrm{II}} = 0$. Otherwise let T_q^{II} be the subgroup of TG_qK generated by the element $2^{k_0}\mathfrak{l}$, where $k_0 = \min\{k | q = q_k\}$. Note that in both cases $|T_q^{\mathrm{II}}| = 2^{\mathrm{card}\{k|q=q_k\}}$, so that, in particular, $|T^{\mathrm{II}}| = 2^t$. Proceeding as in the proof of Theorem (3.8), it is easy to show that $TGK = T^{\mathrm{I}} \oplus T^{\mathrm{II}}$. ●

(3.11). Example. Let p be a natural number such that $p < m$ if m is odd. If the form φ is anisotropic and for some quadratic extension E/F the Witt index of the form φ_E is greater than p, then $TG_pK(X) = \mathbb{Z}/2$.

Proof. For $p < m$, applying the norm function $N_{E/F}$ to the element $\mathfrak{l}_p \in K_{(p)}(X_E)$, we obtain the element $2\mathfrak{l}_p$, whence $\mathfrak{l}_{p-1} = 2\mathfrak{l}_p - h^{d-p} \in K_{(p)}(X)$ and consequently $TG_pK(X) = \mathbb{Z}/2 \cdot \iota_{p-1}$. For $p = m$ (if m is even) we have $N_{E/F}(\iota_m) = \iota_m + \iota'_m$, whence $\iota_{m-1} = \iota_m + \iota'_m - h^m \in K_{(m)}(X)$. •

Note that if, say,

$$\varphi = b_0(X_0^2 - aY_0^2) + b_1(X_1^2 - aY_1^2) + \cdots + b_p(X_p^2 - aY_p^2) + \psi(T_j),$$

then, under the assumptions of the example, the nonzero torsion element of the group $G_pK(X)$ is equal to $[\mathcal{O}_Z] - h^{d-p}$, where Z denotes the prime cycle defined by equations

$$X_0X_i - aY_0Y_i = 0 \quad (i = 0, 1, \ldots, p), \qquad T_j = 0$$

(Z is not a locally complete intersection).

We conclude this section with a description of the behavior of the group G^*K of a quadric under certain extensions of the base field.

(3.12). PROPOSITION. *Let E/F be an extension of one of the following types:*

1) *an algebraic (possibly infinite) extension of odd degree;*
2) *a purely transcendental extension.*

Then the homomorphism $\mathrm{res}_{E/F}: GK(X) \to GK(X_E)$ is an isomorphism.

PROOF. By virtue of [1] the filtration-preserving homomorphism $\mathrm{res}_{E/F}$: $K(X) \to K(X_E)$ is an isomorphism. Therefore, in order to prove that the induced homomorphism of the associated graded groups is also an isomorphism, it suffices to check that it is injective or surjective. If E/F is a finite extension of odd degree, then $\mathrm{res}_{E/F}: G^*K(X) \to G^*K(X_E)$ is injective, since the composition $N_{E/F} \circ \mathrm{res}_{E/F}$ is multiplication by the odd number $[E : F]$. For an infinite odd algebraic extension the injectivity of res follows from the permutability of G^*K with projective limits (1.1). Now let E/F be a purely transcendental extension. In this case we prove the surjectivity of $\mathrm{res}_{E/F}$. We may assume that E is the field of rational functions of an affine space $\mathbf{A}^n_F$. First we prove that the homomorphism $\mathrm{res}_{E/F}: \mathrm{CH}^*X \to \mathrm{CH}^*X_E$ is surjective on the Chow groups. In order to do this we decompose the homomorphism into the composition $\mathrm{CH}^*X \to \mathrm{CH}^*(X \times \mathbf{A}^n) \to \mathrm{CH}^*X_E$. The first arrow is an isomorphism because of the homotopy invariance of Chow groups (1.4). The second arrow coincides with the last homomorphism in the exact sequence (1.3.2) associated with the flat morphism $\mathrm{pr}: X \times \mathbf{A}^n \to \mathbf{A}^n$; therefore it is surjective. The surjectivity of res on G^*K now follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^* & \twoheadrightarrow & \mathrm{CH}^*X_E \\ \downarrow & & \downarrow \\ G^*K(X) & \to & G^*K(X_E) \end{array} \quad \bullet$$

4. The group $H^1(X, K_2)$

(4.1). THEOREM. *For $d \geq 3$ the natural homomorphism $F^* = H^0(X, K_1) \otimes \mathrm{CH}^1X \to H^1(X, K_2)$ is an isomorphism.*

(4.2). LEMMA. *Let X contain a rational point. Then*

$$F^* \xrightarrow{\sim} H^1(X, K_2), \qquad K_2(F) \xrightarrow{\sim} H^0(X, K_2).$$

PROOF. Let $\varphi = X_0X_1 + \cdots$, Y be the hyperplane section $X_0 = 0$ of the quadric X, and $U = X\backslash Y \cong \mathbb{A}^d$. We obtain a diagram with exact row

$$\begin{array}{ccccccccc} 0 \to H^0(X, K_2) & \xrightarrow{\beta} & H^0(U, K_2) & \to & H^0(Y, K_1) & \to & H^1(X, K_2) & \to & H^1(U, K_2) \\ \nwarrow^{\alpha} & & \| & & \| & & & & \| \\ & & K_2(F) & & F^* & & & & 0 \end{array}$$

where α is the inverse image of the structure morphism. The triangle is commutative; therefore the homomorphism β is surjective and hence the sequence decomposes into two isomorphisms. •

Now suppose X has no rational points and E/F is a quadratic extension such that φ_E is isotropic. It follows from the commutative diagram

$$\begin{array}{ccc} E^* & \xrightarrow{\sim} & H^1(X_E, K_2) \\ \uparrow & & \uparrow \text{res} \\ F^* & \xrightarrow{f} & H^1(X, K_2) \end{array}$$

that f is an injection. Moreover, $\operatorname{Im}\operatorname{res} \subset H^1(X_E, K_2)^G = E^{*G} = F^*$ ($G = \operatorname{Gal}(E/F)$), so that the homomorphism res splits the monomorphism f, and hence it suffices to prove

(4.3). PROPOSITION. *The homomorphism* $\operatorname{res}_{E/F}: H^1(X, K_2) \to H^1(X_E, K_2)$ *is injective for an arbitrary quadratic extension* $E = F(\sqrt{a})$.

PROOF. We exploit two results about quadratic extensions. The first is Hilbert's Theorem 90 for K_2 [2]. The second is the following description of $(K_2E)^G$ ([3], p. 16): any $\mu \in (K_2E)^G$ can be represented in the form $\xi_E + \{\sqrt{a}, c\}$, where $\xi \in K_2(F)$, $c \in F_0^*$ (F_0 denotes the algebraic closure in F of the prime subfield), and $\{-1, c\} = 0 \in K_2(F_0)$.

Consider the commutative diagram

$$\begin{array}{ccccc} & & K_2F(X) & \xrightarrow{d} & \coprod_{x\in X^1} F(x)^* \\ & & \downarrow r & & \downarrow r \\ 0 \to K_2E & \to & K_2E(X) & \xrightarrow{d} & \coprod_{x\in X_E^1} E(x)^* \end{array}$$

Let $\nu \in \coprod_{x\in X^1} F(x)^*$, $\eta = r\nu$, and $\eta = d\mu$ for some $\mu \in K_2E(X)$. It is required to find $\xi \in K_2F(X)$ such that $\nu = d\xi$. Let σ be a generator of the Galois group G. The element η is σ-invariant; consequently $\mu - \sigma\mu \in \operatorname{Ker} d = K_2E$ (Lemma (4.2)). Since $\mu - \sigma\mu \in \operatorname{Ker}(K_2E \xrightarrow{N} K_2F)$, it follows from Hilbert's Theorem 90 that there exists $t \in K_2E$ such that $\mu - \sigma\mu = t - \sigma t$. Let us replace μ by $\mu - t$. Then as before $d\mu = \eta$ and, moreover, now $\mu \in (K_2E(X))^G$. Using the description of $(K_2E(X))^G$ for the quadratic extension $E(X)/F(X)$ and considering that F is algebraically closed in $F(X)$, we obtain a representation of μ in the form $\mu = r\xi + \Theta$, where $\xi \in K_2F(X)$, $\Theta \in K_2E$. Now $rd\xi = dr\xi = \eta = r\nu$, whence $d\xi = \nu$. The proposition and the theorem are proved. •

(4.4). COROLLARY. *For any quadric* X *all the differentials in the spectral* BGQ*-sequence that come from the terms* $E_r^{1,-2}(X)$ *for* $r \geq 2$, *are equal to zero.*

PROOF. If $d < 3$, there is nothing to prove. Let $d \geq 3$. We have $K_1^{(1/2)}(X) = E_\infty^{1,-2}(X) \hookrightarrow E_2^{1,-2}(X) = H^1(X, K_2)$. It follows from the commutative diagram

$$\begin{array}{ccc} K_1^{(1/2)}(X) & \hookrightarrow & H^1(X, K_2) \\ \nwarrow & & \nearrow \\ & F^* & \end{array}$$

that $E_\infty^{1,-2} = E_2^{1,-2}$, which is equivalent to the assertion to be proved. •

Using in addition the fact that the differentials coming from the terms $E_r^{0,-1}$ for $r \geq 2$ are also zero, we obtain

(4.5). COROLLARY. *The mapping* $\mathrm{CH}^3 X \to G^3 K(X)$ *is an isomorphism.*

5. Quadrics of dimensions 3 and 4

The results of the preceding sections allow us to compute the Chow groups of quadrics of dimension no higher than 4. For such quadrics $\mathrm{CH}^p X \to G^p K(X)$ is an isomorphism for all p. We begin with auxiliary results.

(5.1). LEMMA [5]. *For any odd-dimensional quadratic form* φ *the equality*

$$[C_0(\varphi)] = [C(\varphi \perp \langle -d_\pm \varphi \rangle)]$$

holds in the Brauer group $\mathrm{Br}\, F$ *of the field* F.

(5.2). COROLLARY. *For an odd-dimensional form* φ *the following statements are equivalent*:

1) $[C_0(\varphi)]$ *is trivial in* $\mathrm{Br}\, F$;
2) $[\varphi \perp \langle -d_\pm \varphi \rangle] \in I^3(F)$.

PROOF. Observe that $[\varphi \perp \langle -d_\pm \varphi \rangle] \in I^2(F)$ and the homomorphism $I^2/I^3 \to {}_2\mathrm{Br}\, F$, $[\overline{\psi}] \mapsto [C(\psi)]$ is an isomorphism [6]. •

Since the Chow groups of conics and of 2-dimensional quadrics are described in §2, we begin with the case $d = 3$.

(5.3). THEOREM. *Let* X *be a three-dimensional quadric without rational ponts. If* $\varphi \sim \langle\langle a, b \rangle\rangle \perp \langle c \rangle$, *then* $TCH^2 X = \mathbb{Z}/2$; *otherwise* $\mathrm{TCH}^2 X = 0$.

PROOF. Considering that $\mathrm{TCH}^1 X = 0$, from Theorem (3.8) we obtain $\mathrm{TCH}^2 X = \mathbb{Z}/2$ if $s(\varphi) \geq 1$; $\mathrm{TCH}^2 X = 0$ if $s(\varphi) = 0$. The condition $s \geq 1$ means that $[C_0(\varphi)] = 0$ in $\mathrm{Br}\, E$ for some quadratic extension $E = F(\sqrt{a})$, that is (Corollary (5.2)), $[\varphi \perp \langle -d_\pm \varphi \rangle]_E \in I^3(E)$. Since the dimension of the form $\varphi \perp \langle -d_\pm \varphi \rangle$ is equal to $6 < 2^3$, the last relation means that the form $([\varphi \perp \langle -d_\pm \varphi \rangle])_E$ is hyperbolic [5], i.e., the Witt index of the form φ_E is equal to 2, whence $\varphi \sim \langle\langle a, b \rangle\rangle \perp \langle c \rangle$ for some $b, c \in F^*$. •

We pass to the study of 4-dimensional quadrics.

(5.4). PROPOSITION. *Let* φ *be a 6-dimensional anisotropic form.*
I. *If* $d_\pm \varphi = -\det \varphi = 1$, *then* $s(\varphi) = 0$.
II. *If* $-\det \varphi \neq 1$, *then*

1) $s(\varphi) = 2$ *if and only if* $\varphi \sim \langle\langle a \rangle\rangle \otimes \langle 1, b, c \rangle$;
2) $s(\varphi) = 1$ *if and only if* $\varphi \sim \langle\langle a, b \rangle\rangle \perp \langle c, d \rangle$, *where* $cd \notin D\langle -a, -b, ab \rangle$;

3) $s(\varphi) = 0$ *if and only if* φ *contains no* 4*-dimensional subform of discriminant* 1.

PROOF. Let $\varphi = \langle -1 \rangle \perp \psi$. Then $C_0(\varphi) \cong C(\psi)$ [5].

I. Let $-\det\varphi = 1$. In this case $C(\psi) \cong C_0(\psi) \times C_0(\psi)$, whence $s(\varphi) = s(\psi)$. If $s > 0$, then, as shown in the proof of Theorem (5.3), the form ψ contains a 4-dimensional subform of determinant 1 and consequently ψ represents -1 (since $\det\psi = -1$), which contradicts the anisotropy of the form φ.

II. Let $-\det\varphi \neq 1$. Then $C(\psi) \cong C_0(\psi) \otimes E$, where $E = F(\sqrt{-\det\varphi})$; moreover, $\psi_E \perp \langle -\det\psi_E \rangle = \varphi_E$.

1) The condition $s = 2$ means that the form φ_E is hyperbolic, that is, $\varphi \sim \langle\langle a \rangle\rangle \otimes \langle 1, b, c \rangle$, where $a = -\det\varphi$.

2) If $s = 1$, then φ_E is isotropic and consequently φ contains a 4-dimensional subform of determinant 1. Statement 3) is an easy consequence of statements 1) and 2). ●

Using Proposition (5.4), Theorems (3.8) and (3.10), and Example (3.11), we obtain the following theorem:

(5.5). THEOREM. *Let* X *be a* 4*-dimensional quadric without rational points.*

I. *If* $-\det\varphi = 1$, *then* CH^*X *is torsion-free and* CH^2X *is the subgroup of* $\mathbb{Z} \oplus \mathbb{Z}$ *generated by the elements* $(1, 1) = h^2$ *and* $(4, 0)$.

II. *If* $-\det\varphi \neq 1$, *then, in case* 1) *of Proposition* (5.4), $\mathrm{TCH}^2X = \mathbb{Z}/2 = \mathrm{TCH}^3X$; *in case* 2), $\mathrm{TCH}^2X = 0$ *and* $\mathrm{TCH}^3X = \mathbb{Z}/2$; *in case* 3), *the group* CH^*X *is torsion-free.*

(5.6). SUPPLEMENT. *If* $\varphi = a_0X_0^2 + \cdots + a_5X_5^2$ *and* $a_0a_1\cdots a_5 = -1$ (*case* I), *then* CH^2X *contains* $(4, 0) = [Z]$, *where* Z *is defined by*

$$a_0X_0^2 + a_1X_1^2 = 0, \quad a_2X_2^2 + a_3X_3^2 = 0, \quad X_0X_2X_4 + a_1a_3a_5X_1X_3X_5 = 0.$$

6. The group CH^2X

(6.1). THEOREM. *Let* X *be a projective quadric defined by a nondegenerate quadratic form* φ. *If* φ *is anisotropic, proportional to a subform of a* 3*-fold Pfister form, and* $\dim\varphi > 4$, *then* $\mathrm{TCH}^2X = \mathbb{Z}/2$. *In other cases* $\mathrm{TCH}^2X = 0$.

(6.2). LEMMA. *Let* U *be an affine quadric over a field* F *defined by the equation* $a_0 + a_1Y_1^2 + \cdots + a_nY_n^2 + b_1X_1^2 + \cdots + b_5X_5^2 = 0$ $(n \geq 1)$; *let* V *be the* 4*-dimensional quadric over the field* $F(y_1, \ldots, y_n)$ *defined by the equation* $(a_0 + a_1Y_1^2 + \cdots + a_ny_n^2) + b_1X_1^2 + \cdots + b_5X_5^2 = 0$. *Then the natural homomorphism* $\mathrm{CH}^2U \to \mathrm{CH}^2V$ *is an isomorphism.*

PROOF. By virtue of (1.3.2), the morphism $\pi\colon U \to \mathbf{A}^1$, $(y_1, \ldots, y_n, x_1, \ldots, x_5) \mapsto y_1$, gives the exact sequence $\coprod_{t\in(\mathbf{A}^1)^1} \mathrm{CH}^1U_t \to \mathrm{CH}^2U \to \mathrm{CH}^2\tilde{U} \to 0$, where U_t denotes the fiber of π over a closed point t and $\tilde{U}$—the fiber over the generic point—is the quadric over the field $F(y_1)$ defined by the equation $(a_0 + a_1y_1^2) + a_2Y_2^2 + \cdots = 0$. Since $\mathrm{CH}^1U_t = 0$ for each t, we obtain an isomorphism $\mathrm{CH}^2U \xrightarrow{\sim} \mathrm{CH}^2\tilde{U}$. Continuing in this manner, we obtain the result after n steps. ●

PROOF OF THE THEOREM. In the isotropic case there is nothing to prove; in the sequel φ is assumed to be anisotropic. For quadratic forms of dimension not higher than 6 the statement follows from results of the preceding section. Let $\varphi = a_0Y_0^2 + a_1Y_1^2 + \cdots + a_nY_n^2 + b_1X_1^2 + \cdots + b_5X_5^2$ $(n \geq 1)$ and Y be the hyperplane section $Y_0 = 0$ of the projective quadric X. The exact sequence $\mathrm{CH}^1 Y \to \mathrm{CH}^2 X \to \mathrm{CH}^2 U \to 0$ allows us to obtain $\mathrm{TCH}^2 X = \mathrm{CH}^2 U$; Lemma (6.2) gives us $\mathrm{CH}^2 U = \mathrm{CH}^2 V$ and then $\mathrm{CH}^2 V = \mathrm{TCH}^2 X'$, where X' denotes the 4-dimensional projective quadric over the field $F(y_1, \ldots, y_n)$ corresponding to the anisotropic form $\psi = (a_0 + a_1y_1^2 + \cdots + a_ny_n^2)X_0^2 + b_1X_1^2 + \cdots + b_5X_5^2$. The group $\mathrm{TCH}^2 X'$ was computed in §5. Since $-\det\psi \neq 1$, it follows from Theorem (5.5) that $\mathrm{TCH}^2 X' = \mathbb{Z}/2$ if ψ splits completely in some quadratic extension, $\mathrm{TCH}^2 X' = 0$ otherwise. The last step in the proof of the theorem is

(6.3). LEMMA. *The following statements are equivalent*:
a) ψ *splits completely in some quadratic extension*;
b) φ *is proportional to a subform of a 3-fold Pfister form.*

PROOF. We prove that a) implies b). Let $L = F(y_1, \ldots, y_n)$ and $E = L(\sqrt{-\det\psi})$. If a 6-dimensional anisotropic form splits completely in some quadratic extension, then the form splits completely if $\sqrt{-\det}$ is adjoint. The determinant of the form $\chi = \langle -b_1, \ldots, b_5, b_1, \ldots, b_5\rangle$ is equal to -1 and $\chi_E \cong \psi_E$ is completely splitting. Consequently, χ is isotropic, whence $\langle b_1, \ldots, b_5\rangle \sim \langle\langle a, b\rangle\rangle \perp \langle -c\rangle$. We may assume that $\psi = \langle(a_0 + a_1y_1^2 + \cdots + a_ny_n^2), 1, -a, -b, ab, -c\rangle$. The form $\langle\langle a, b\rangle\rangle$ splits if we adjoin the roots of $-\det\psi = c(a_0 + a_1y_1^2 + \cdots + a_ny_n^2)$ to L; consequently, $\det\psi \in D_L\langle -a, -b, ab\rangle$, i.e., $a_0 + a_1y_1^2 + \cdots + a_ny_n^2 \in D_L\langle ac, bc, -abc\rangle$. The last relation means [5] that $\langle a_0, a_1, \ldots, a_n\rangle$ is a subform of $\langle ac, bc, -abc\rangle$, whence φ is a subform of $\langle\langle a, b, c\rangle\rangle$. The converse implication b) $\Rightarrow$ a) is trivial. ●

(6.4). APPENDIX. Let $\varphi = X_0^2 - aX_1^2 - bX_2^2 + abX_3^2 - c(Y_0^2 - aY_1^2 - bY_2^2 + abY_3^2)$ be an anisotropic 3-fold Pfister form. Passing through the isomorphisms used in the proof of the theorem, one can show that the nonzero torsion element in $\mathrm{CH}^2 X$ is equal to $[Z] - h^2$, where the cycle Z is defined by the equations: $X_0^2 - aX_1^2 - bX_2^2 + abX_3^2 = 0$, $X_0Y_0 - aX_1Y_1 - bX_2Y_2 + abX_3Y_3 = 0$, $X_0Y_3 - X_1Y_2 + X_2Y_1 - X_3Y_0 = 0$. Removing the terms containing Y_3, we obtain a formula for a 7-dimensional subform of a 3-fold Pfister form.

7. The group $G_1K(X)$

(7.1). We will say that a quadratic form φ has property R if $\dim\varphi \geq 5$ and there exists a finite odd-dimensional extension E/F such that φ_E contains a subform proportional to a 2-fold Pfister form (in other words, a 4-dimensional subform of determinant 1).

(7.2). REMARK. We say that a form φ of dimension not less than 5 satisfies property R' if φ contains a 4-dimensional subform of determinant 1. The properties R and R' are equivalent if $\dim\varphi$ equals 5 or 6. However, one can give an example of a 7-dimensional form having R, but not R'.

(7.3). THEOREM. *Let X be a projective quadric defined by a nondegenerate quadratic form φ. If φ is anisotropic and has R, then $TG_1K(X) = \mathbb{Z}/2$. In other cases $TG_1K(X) = 0$.*

PROOF. By virtue of (3.12.1), we may assume that the base field has no odd extension (replacing F by the compositum of all odd extensions). Then the property R becomes R'. If φ is anisotropic and has R', then $TG_1K(X) = \mathbb{Z}/2$ by (3.11). If φ is isotropic, then obviously $TG_1K(X) = 0$.

Let φ be anisotropic and $TG_1K(X) = \mathbb{Z}/2$. It remains to prove that in this case φ has R'. We carry out the proof by induction on $\dim\varphi$. The first step—the case of $\dim\varphi = 5$—is covered by Theorem (5.3). Let $\dim\varphi \geq 6$ and Y be a hyperplane section of the quadric X. The section Y is a quadric corresponding to a subform ψ of the form φ, where $\dim\psi = \dim\varphi - 1$. If $TG_1K(Y) \neq 0$, then by the induction assumption ψ has R', whence φ also has R'. Next we assume that $TG_1K(Y) = 0$. Let us set $U = X\backslash Y$. For a quadric without rational points the exact sequence $0 \to TG_1K \to G_1K \to \overline{G}_1K \to 0$ admits the canonical splitting $\overline{G}_1K \hookrightarrow G_1K$, $\bar{h}^{d-1} \mapsto h^{d-1}$, which gives the canonical decomposition $G_1K = TG_1K \oplus \overline{G}_1K$. In the sequel the arrow $G_1K \to TG_1K$ will always denote the corresponding projection. The commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{CH}_1 Y & \to & \mathrm{CH}_1 X & \to & \mathrm{CH}_1 U & \to & 0 \\
\downarrow & & \downarrow & & & & \\
G_1K(Y) & \to & G_1K(X) & \to & TG_1K(X) & \to & 0
\end{array}$$

gives an epimorphism $\alpha\colon \mathrm{CH}_1 U \twoheadrightarrow TG_1K(X)$ (the bottom row is exact, since $G_1K(Y)$ is torsion-free). Moreover, writing the exact sequence (1.3.2)

$$\coprod_{x\in(\mathbf{A}^1)^1} \mathrm{CH}_1 U_x \to \mathrm{CH}_1 U \to \mathrm{CH}_0 U_\Theta \to 0$$

for the fibering $U \to \mathbf{A}^1$ and considering that $\mathrm{CH}_0 U_\Theta = 0$, we obtain an epimorphism $\beta\colon \coprod_{x\in(\mathbf{A}^1)^1} \mathrm{CH}_1 U_x \twoheadrightarrow \mathrm{CH}_1 U$. The group $\coprod \mathrm{CH}_1 U_x$ is generated by all pairs $(x, [Z])$ where x is a closed point on $\mathbf{A}^1$ and Z is a prime cycle on U_x. Therefore, for some of these pairs we obtain $\alpha\circ\beta(x, [Z]) \neq 0$. The morphism $U_x \hookrightarrow U$ is a closed imbedding and $\beta(x, [Z]) = [Z]$. Let $\overline{Z}$ be the closure of Z in X. Then $f([\overline{Z}]) \neq 0$, where f denotes the composition $\mathrm{CH}_1 X \twoheadrightarrow G_1K(X) \twoheadrightarrow TG_1K(X)$, as follows from the commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{CH}_1 U_x & \ni & [Z] \\
 & & \downarrow \beta & & \\
[\overline{Z}] \in \mathrm{CH}_1 X & \to & \mathrm{CH}_1 U & \ni & [Z] \\
\downarrow & \searrow^{f} & \downarrow \alpha & & \\
G_1K(X) & \to & TG_1K(X) & &
\end{array}$$

If $\deg x = 1$, then U_x is a hyperplane section of the affine quadric. Let Y' be the projective closure of U_x in X. Then Y' is a hyperplane section of X and

$\overline{Z} \subset Y'$. From the diagram

$$\begin{array}{ccc} [\overline{Z}] \in \mathrm{CH}_1 Y' & \to & \mathrm{CH}_1 X \quad \ni [\overline{Z}] \\ \downarrow & & \downarrow \\ G_1K(X) & \to & TG_1K(X) \end{array}$$

we obtain that $TG_1K(Y') \neq 0$, whence (by the induction assumption) φ has R'. It remains to examine the case of $\deg x \geq 2$. We have $F \subset F(x) \subset F(\overline{Z})$. Let E/F be a subextension of degree 2 of $F(x)/F$. We prove that $i(\varphi_E) \geq 2$. It follows from Lemma (2.5) that $[\overline{Z}] \in \mathrm{Im}(N\colon \mathrm{CH}_1 X_E \to \mathrm{CH}_1 X)$. The image of $[\overline{Z}]$ in $TG_1K(X)$ is nonzero; therefore the image of $(\overline{Z})$ in $G_1K(X)$ is equal to $\ell_0 + nh^{d-1}$ for some integer n. From the commutative square

$$\begin{array}{ccc} \mathrm{CH}_1 X_E & \longrightarrow & G_1K(X_E) \\ \Big\downarrow N & & \Big\downarrow N \\ \mathrm{CH}_1 X & \longrightarrow & G_1K(X) \end{array}$$

we obtain that $\ell_0 + nh^{d-1} \in \mathrm{Im}(N\colon G_1K(X_E) \to G_1K(X))$, i.e., $(2j+1)\ell_0 + nh^{d-1} \in \mathrm{Im}(N\colon K_{(1)}(X_E) \to K_{(1)}(X))$ for some j. Suppose that $\ell_1 \in K_{(1)}(X_E)$. Then each element of $K_{(1)}(X_E)$ is of the form $a\ell_0 + bh^{d-1}$ and $N(a\ell_0 + bh^{d-1}) = 2a\ell_0 + 2bh^{d-1}$, whence $(2j+1)\ell_0 + nh^{d-1} \notin \mathrm{Im}\, N$. Consequently, $\ell_1 \in K_{(1)}(X_E)$, i.e., $i(\varphi_E) \geq 2$ and hence φ has R'. The theorem is proved. ●

8. Quadrics of dimension 5 and 6

Using Theorems (3.8), (6.1), and (7.1), we obtain the computation of $G^*K(X)$ for a 5-dimensional quadric.

(8.1). THEOREM. *Let X be a 5-dimensional projective quadric defined by an anisotropic form φ. Then the following distribution variants of the torsion in $G^*K(X)$ are possible (the symbol $*$ denotes $\mathbb{Z}/2$):*

Condition on φ

codimension	$s(\varphi)=0$	$s(\varphi)=1$, φ has R	$s(\varphi)=1$, φ does not have R	$s(\varphi)=2$	$s(\varphi)=3$
2	0	0	0	0	*
3	0	0	*	*	*
4	0	*	0	*	*

(8.2). EXAMPLE. Let $\varphi = a_0X_0^2 + \cdots + a_5X_5^2 + a_6X_6^2$ be an anisotropic form and let $a_0a_1\cdots a_5 = -1$. Then $s(\varphi) = 1$. Let Z be the prime cycle defined by equations $X_6 = 0$, $a_0X_0^2 + a_1X_1^2 = 0$, $a_2X_2^2 + a_3X_3^2 = 0$, $X_0X_2X_4 + a_1a_3a_5X_1X_3X_5 = 0$. Then $[\mathcal{O}_Z] = 4\ell_1 \in G_2K(X)$, whence $[\mathcal{O}_Z] - 2h^3$ is a nonzero torsion element if φ does not have R.

In conclusion we present the computation of $G^*K(X)$ for a class of 6-dimensional quadrics.

(8.3). Theorem. *Let X be a 6-dimensional projective quadric defined by an anisotropic form φ of determinant 1. The following distribution variants of the torsion in $G^{*}K(X)$ are possible:*

Condition on φ

codimension	$s(\varphi)=0$	$s(\varphi)=1$, φ has R	$s(\varphi)=1$, φ does not have R	$s(\varphi)=2$	$s(\varphi)=3$
2	0	0	0	0	*
3	0	0	0	0	*
4	0	0	*	*	*
5	0	*	0	*	*

Note that torsion of the second kind appears only in the case $s(\varphi)=3$.

Proof. If $s(\varphi)=3$, then φ is proportional to a 3-fold Pfister form and hence it suffices to use Theorems (3.11) and (6.1). If $s(\varphi)=0$, then there is no torsion by virtue of (3.10). We split the remaining part of the proof into several lemmas. We assume below that the base field admits no extension of odd degree. ●

(8.3.1). Lemma. *If $s(\varphi)=2$, then φ has R'.*

Proof. Let $\varphi=\langle a\rangle\perp\psi$ and Y be the projective quadric corresponding to ψ. Since $s(\psi)=s(\varphi)=2$, it follows from Theorem (8.1) that $TG_1K(Y)\neq 0$, i.e., ψ has R'. Consequently, φ also has R'. ●

(8.3.2). Lemma. *If $s(\varphi)=2$, then φ is divisible by $\langle\langle a\rangle\rangle$ for some $a\in F^{*}$.*

Proof. It follows from (8.3.1) that $\varphi=f\cdot\langle\langle b_1, b_2\rangle\rangle\perp g\cdot\langle\langle c_1, c_2\rangle\rangle$, whence

$$[C(\varphi)]=\left[\begin{pmatrix} b_1, b_2 \\ F\end{pmatrix}\otimes\begin{pmatrix} c_1, c_2 \\ F\end{pmatrix}\right].$$

Since $s(\varphi)=2$, this tensor product is not a skew field, i.e., the form $\langle b_1, b_2, -b_1b_2, -c_1, -c_2, c_1c_2\rangle$ is isotropic. This fact means that there exists an element $a\in F^{*}$ that is the common value of the pure subforms of the 2-fold Pfister forms, whence φ is divisible by $\langle\langle a\rangle\rangle$. ●

By virtue of (3.11), Lemma (8.3.2) proves the Theorem in the case $s(\varphi)=2$. The last remaining case $s(\varphi)=1$ is handled by Theorem (7.3) and the following

(8.3.3). Lemma. *If $s(\varphi)=1$, then $\mathfrak{l}_0\in K_{(2)}(X)$.*

Proof. If ψ is a 7-dimensional subform of φ and Y is the corresponding hyperplane section, then $s(\psi)=1$ and hence, by Theorem (8.1), $\mathfrak{l}_0\in K_{(2)}(Y)$. Using the mapping $K_{(2)}(Y)\to K_{(2)}(X)$, we obtain that $\mathfrak{l}_0\in K_{(2)}(X)$. The theorem is proved. ●

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