SOME ADDITIONAL MATERIAL

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1. On the formulas evaluating Steenrod operations for grassmannians of split quadratic forms

Let F be a field and let φ be a non-degenerate split quadratic form over F of odd dimension $2n+1$. For $m \in \{1, \dots, n\}$, let X_m be the mth grassmannian of φ. In particular, X_1 is the projective quadric of φ.

We work with the modulo 2 Chow group $\mathrm{Ch}(X_m)$ and consider its basis given by the products

$$c_1^{\alpha_1} \dots c_{n-m}^{\alpha_{n-m}} e_{n-m+1}^{\alpha_{n-m+1}} \dots e_{2n-m}^{\alpha_{2n-m}},$$

of the elements $c_1, \dots, c_{n-m}, e_{n-m+1}, \dots, e_{2n-m}$ (indexed by their codimensions), defined as in [2, §3], where the exponents $\alpha_1, \dots, \alpha_{2n-m}$ are such that $\alpha_1 + \dots + \alpha_{2n-m} \leqslant m$ and $\alpha_i \leqslant 1$ for $i \geqslant n-m+1$.

A formula evaluating the Steenrod operation St^j for $j \geqslant 0$ on the element e_i for $i \geqslant n-m+1$ is given in [3, Proposition 2.9]:

$$\mathrm{St}^j(e_i) = \sum_{k=0}^{\min\{j,n-m\}} \binom{i-k}{j-k} e_{i+j-k} c_k, \tag{1.1}$$

where $e_i := 0$ for $i > 2n-m$.

Formula (1.1) is valid for any m. However, there is a difference between the case of $m=1$ and the case of $m>1$. For $m>1$, the nonzero products $e_{i+j-k}c_k$ on the right of (1.1) constitute a part of the basis of $\mathrm{Ch}(X_m)$ and so are “final” in a sense.

For $m=1$ each of these products is equal to e_{i+j} so that (1.1) simplifies to

$$\mathrm{St}^j(e_i) = b e_{i+j}$$

with b being the sum of all binomial coefficients appearing in (1.1). By induction on j, it is easy to show that $b = \binom{i+1}{j}$ for $j \leqslant n-1$ (for higher j the value of b does not matter because $e_{i+j} = 0$) and so, (1.1) becomes the formula of [1, Corollary 78.5].

References

[1] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[2] Karpenko, N. A. Fields of u-invariant 11. Preprint (21 Apr 2026, 15 pages). Available on author's web page.

[3] Vishik, A. Fields of u-invariant 2^r+1. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.

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