CHOW FILTRATION ON REPRESENTATION RINGS
OF ALGEBRAIC GROUPS

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Abstract. We introduce and study a filtration on the representation ring $R(G)$ of an
affine algebraic group $G$ over a field. This filtration, which we call Chow filtration, is an
analogue of the coniveau filtration on the Grothendieck ring of a smooth variety. We
compare it with the other known filtrations on $R(G)$. For any $n \geq 1$, we compute the
Chow filtration on $R(G)$ for the special orthogonal group $G := O^+(2n+1)$. In particular,
we show that the graded group associated with the filtration is torsion-free. On the other
hand, the Chow ring of the classifying space of $G$ over any field of characteristic $\neq 2$
is known to contain non-zero torsion elements. As a consequence, any sufficiently good
approximation of the classifying space delivers an example of a smooth quasi-projective
variety $X$ such that its Chow ring is generated by Chern classes and at the same time
contains non-zero elements vanishing under the canonical epimorphism onto the graded
ring associated with the coniveau filtration on the Grothendieck ring of $X$.

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1. INTRODUCTION

Let $G$ be an affine group scheme of finite type over a field and let $R(G)$ be its rep-
resentation ring (the Grothendieck ring of the category of finite-dimensional linear $G$-
representations). In this paper we introduce and study a ring filtration on $R(G)$ which is
an analogue of the coniveau filtration on the Grothendieck ring of a smooth variety. We

Date: 7 September 2018.

Key words and phrases. Algebraic groups; representation rings; Chow groups. Mathematical Subject

The work of the first author has been supported by a Discovery Grant from the National Science and
Engineering Board of Canada. The work of the second author has been supported by the NSF grant
DMS #1801530.
call it *Chow filtration* because of a close relation with the Chow ring of the classifying space of $G$. In topology, for finite groups, a similar filtration has been considered by Atiyah in [1].

The ring $R(G)$ is an augmented $\lambda$-ring and therefore has two other important filtrations: the filtration by powers of the augmentation ideal and Grothendieck’s $\gamma$-filtration. The latter can be defined via Chern classes and for this reason we call it Chern filtration in the paper. It has been shown by Totaro (see [19, proof of Theorem 3.1]) that the Chern filtration is equivalent to the augmentation filtration in the sense that they deliver isomorphic completions of $R(G)$. We show that the Chow filtration is also equivalent to them (see Corollary 4.8).

Our initial motivation came from following conjecture raised in [9]:

**Conjecture 1.1.** Let $G$ be a split semisimple algebraic group over a field, let $P$ be a special parabolic subgroup of $G$, let $E$ be a standard generic $G$-torsor, and let $X$ be the quotient variety $E/P$. Then the canonical epimorphism of the Chow ring $\text{CH}(X)$ onto the graded ring associated with the coniveau filtration on the Grothendieck ring of $X$ is an isomorphism.

Using computations of Chow rings of classifying spaces of special parabolic subgroups, it has been shown in [10] that for $X$ as above the ring $\text{CH}(X)$ is generated by Chern classes (of vector bundles). It was not immediately clear to the authors of the present paper that this condition alone is insufficient for the conclusion of the conjecture to hold. Examples showing that it indeed is are produced here (see Theorem 5.5). Unexpectedly, they are also related to computations of Chow rings for classifying spaces of algebraic groups. In fact, analogous examples are first obtained with a classifying space in place of $X$. And then one takes for $X$ a variety which is (in a certain specific sense) its sufficiently good approximation.

Before we can construct the example with a classifying space of an algebraic group $G$, we need to introduce the Chow filtration on the ring $R(G)$ (which can be viewed as the Grothendieck ring of the classifying space of $G$). This is done in §4, where we also study some basic properties of the filtration introduced and relate it to the other two filtrations on $R(G)$ in various ways.

Proving Theorem 5.5, we are using a computation of the Chow ring for the classifying space of the orthogonal group $O^+(2n + 1)$ made first by B. Totaro over $\mathbb{C}$ and then by L. A. Molina Rojas and A. Vistoli over an arbitrary field of characteristic $\neq 2$. For the sake of completeness, we do the corresponding computation over a field of characteristic 2 in Appendix O. It turns out that the answer in characteristic 2 differs from the answer in other characteristics. In particular, examples of Theorem 5.5 do not extend to characteristic 2.

## 2. Chern classes

Let $R$ be $\lambda$-ring (see [6]). We say that $R$ is an enhanced $\lambda$-ring if there are given an (augmentation) homomorphism of $\lambda$-rings $\text{rk} : R \to \mathbb{Z}$, that is $\text{rk}(\lambda^i(a)) = (\text{rk}(a))^i$, and an involution (duality automorphism) $R \to R$, $a \mapsto a^\vee$ such that $\text{rk}(a^\vee) = \text{rk}(a)$ for all $a \in R$. 
Example 2.1. Let $X$ be a smooth variety over a field $F$ and $R = K(X)$, the Grothendieck ring $K_0(X)$ of classes of locally-free sheaves on $X$. Exterior powers of locally-free sheaves yield structure of a $\lambda$-ring on $R$. The rank map $R \to \mathbb{Z}$ is an augmentation and duality on $R$ is given by dual sheaves. Thus, $R$ is an enhanced $\lambda$-ring.

Example 2.2. Let $G$ be an algebraic group over $F$ and $R = R(G)$, the representation ring of $G$. Exterior powers of representations yield structure of a $\lambda$-ring on $R$. The dimension map $R \to \mathbb{Z}$ is an augmentation and duality on $R$ is given by dual representations. Thus, $R$ is an enhanced $\lambda$-ring.

Let $R$ be a $\lambda$-ring. Recall that the total $\lambda$-operation
\[
\lambda_t : R \to R[[t]],
\]
where $t$ is a variable, is defined by $\lambda_t(a) = \sum_{i \geq 0} \lambda^i(a)t^i$ and the total $\gamma$-operation
\[
\gamma_t : R \to R[[t]]
\]
satisfies $\gamma_t := \lambda_{/(1-t)}$. We have $\gamma^0 = 1$, $\gamma^1 = \text{id}$ and $\gamma^n = \sum_{i=1}^{n} \binom{n-1}{i-1} \lambda^i$ for all $n \geq 1$.

If $R$ is an enhanced $\lambda$-ring, we define Chern classes $c^R_i : R \to R$ for all $i \geq 0$ by
\[
c^R_i(a) := \gamma^i(\text{rk}(a) - a^\vee).
\]
The total Chern class $c^R_i$ satisfies $c^R_i(a + b) = c^R_i(a)c^R_i(b)$. We have $c^R_0 = 1$ and $c^R_1(a) = \text{rk}(a) - a^\vee$.

We define Chern filtration
\[
R = R^{[0]} \supseteq R^{[1]} \supseteq \ldots
\]
where $R^{[i]}$ is the subgroup generated by all products
\[
c^R_i(a_1) \cdots c^R_i(a_n)
\]
with $n \geq 0$, $a_1, \ldots, a_n \in R$, and $i_1 + \cdots + i_n \geq i$. (Note that since $c^R_i(a) = c^R_i(a - \text{rk}(a))$ for every $a \in R(G)$, it suffices to take $a_i$ with $\text{rk}(a_i) = 0$.) In other words, Chern filtration is the smallest ring filtration with the property that $c^R_i(a) \in R^{[i]}$ for any $a \in R$ and any $i \geq 0$ (cf., [8, Definition 2.6]).

We write Chern$R = \bigoplus_{i \geq 0} \text{Chern}^iR$ for the graded ring associated with the Chern filtration.

Let $I = \text{Ker}(\text{rk}) \subset R$ be the augmentation ideal. For any $a \in I$, we have
\[
a = \gamma^1(a) = -\gamma^1(\text{rk}(a) - a) = -c^R_1(a^\vee) \in R^{[1]}.
\]

It follows that $I \subset R^{[1]}$ and hence $I^i \subset R^{[i]}$ for all $i$.

Example 2.3. If $X$ is a smooth variety over a field $F$, the Chern classes on $R = K(X)$ coincide with the $K$-theoretic Chern classes as defined in [16, Example 3.6.1]. We write $c^K_i$ for $c^R_i$. In particular, $c^K_1(x) = 1 - x^{-1}$, where $x$ is the class of a line bundle on $X$. 
3. Grothendieck ring of smooth varieties

By a variety we mean an integral separated scheme of finite type over a field. Let $X$ be a smooth variety over a field $F$. We write $K(X)$ for the Grothendieck ring $K_0(X)$ of classes of locally-free sheaves on $X$. We introduce three filtrations on $K(X)$.

The augmentation ideal $I(X) \subset K(X)$ is the kernel of the (augmentation) ring homomorphism $K(X) \to \mathbb{Z}$ given by the rank of locally-free sheaves. The augmentation filtration on $K(X)$ is given by powers $I(X)^i$, $i \geq 0$ of the augmentation ideal.

By Example 2.1, the ring $K(X)$ is an enhanced $\lambda$-ring. In particular, it has a has Chern filtration (the same as Grothendieck’s $\gamma$-filtration, see [6])

$$K(X) = K(X)^{[0]} \supset K(X)^{[1]} \supset \ldots.$$ 

The class in $K(X)$ of any coherent sheaf on $X$ is obtained by taking the alternating sum of the terms of any its finite locally free resolution. For any $i \geq 0$, let $K(X)^{(i)} \subset K(X)$ be the subgroup generated by the classes of coherent sheaves whose support has codimension at least $i$. The finite filtration

$$K(X) = K(X)^{(0)} \supset K(X)^{(1)} \supset \cdots \supset K(X)^{(\dim X + 1)} = 0$$

thus obtained is a ring filtration known under various names in the literature: the filtration by codimension of support, topological filtration, geometrical filtration, coniveau filtration. We call it Chow filtration because of its close relation to the Chow ring $\text{CH}(X)$ (see below).

By [6, Theorem 3.9 of Chapter V], $c^K_i(a) \in K(X)^{(i)}$ for all $a \in K(X)$ and all $i$. It follows that the three filtrations are related by the inclusions:

$$I(X)^i \subset K(X)^{[i]} \subset K(X)^{(i)}$$

for any $i$.

Remark 3.1. Some of these inclusions are equalities: $I(X) = K(X)^{[1]} = K(X)^{(1)}$ and $K(X)^{(2)} = K(X)^{(2)}$ for all $X$, [8, Proposition 2.14(2)]. The inclusion of the Chern filtration into Chow filtration yields a graded ring homomorphism $\text{Chern}(X) \to \text{Chow}(X)$ of the associated graded rings that is neither injective nor surjective in general, but becomes an isomorphism after tensoring with $\mathbb{Q}$, [6, Proposition 5.5 of Chapter VI]. In particular, the kernel and cokernel of $\text{Chern}(X) \to \text{Chow}(X)$ are torsion groups.

There is a well defined surjective graded ring homomorphism

$$\varphi : \text{CH}(X) \to \text{Chow}(X)$$

taking the class of a closed subvariety $Z \subset X$ of codimension $i$ to the class of its structure sheaf $O_Z$ in $\text{Chow}^i(X)$. The kernel of $\varphi^i : \text{CH}^i(X) \to \text{Chow}^i(X)$ is killed by multiplication by $(i-1)!$ (see [5, Example 15.3.6]). In particular, $\varphi^i$ is an isomorphism for $i \leq 2$.

Proposition 3.2. The homomorphism $\varphi$ commutes with Chern classes, that is $\varphi(c_i(a)) = c^K_i(a)$ modulo $K(X)^{(i+1)}$ for every $a \in K(X)$.

Proof. In view of the splitting principle (see [6, Lemma 3.8 of Chapter V]) it suffices to consider the case $i = 1$ and $a = [\mathcal{L}(Z)]$ for an irreducible divisor $j : Z \subset X$, where $\mathcal{L}(Z)$
is the locally-free sheaf on $X$ associated with $Z$. We have an exact sequence of sheaves on $X$:

$$0 \to \mathcal{L}(-Z) \to O_X \to j_*O_Z \to 0.$$  

It follows that

$$\varphi(c_i(a)) = \varphi([Z]) = [j_*O_Z] = [O_X] - [\mathcal{L}(-Z)] = 1 - a^{-1} = c_i^K(a). \quad \square$$

**Proposition 3.3.** The following holds for any integer $i \geq 0$.

1. If $K(X)$ is generated by classes of line bundles, then $I(X)^i = K(X)^{[i]}$.

2. If $CH(X)$ is generated by Chern classes, then $K(X)^{[i]} = K(X)^{(i)}$.

3. If the group Chern$K(X)$ is torsion-free, then $K(X)^{[i]} = K(X)^{(i)}$.

**Proof.** (1) If $l \in K(X)$ is the class of a line bundle, then $c_i^K(l) = 1 - l^{-1} \in I(X)$ and $c_i^K(-l) = (l^{-1} - 1)i \in I(X)^i$ for any $i$. If $a, b \in K(X)$ are such that $c_i^K(a), c_i^K(b) \in I(X)^i$ for all $i$, then $c_i^K(a + b) = \sum_j c_j^K(a)c_{i-j}^K(b) \in I(X)^i$.

(2) By Proposition 3.2, for any $i$, the ring epimorphism $\varphi : CH(X) \to ChowK(X)$ takes the Chow-theoretical Chern class $c_i(a) \in CH^i(X)$ with $a \in K(X)$ to the class modulo $K(X)^{(i+1)}$ of the $K$-theoretical Chern class $c_i^K(a)$. It follows that the ring Chow$K(X)$ is generated by Chern classes. By descending induction on $i$, we see that $K(X)^{[i]} = K(X)^{(i)}$.

(3) If Chern$K(X)$ is torsion-free, the homomorphism Chern$K(X) \to ChowK(X)$ is injective. The equality $K(X)^{[i]} = K(X)^{(i)}$ follows by ascending induction on $i$. \quad \square

**Corollary 3.4.** If $CH(X)$ is generated by $CH^1(X)$, then $I(X)^i = K(X)^{[i]} = K(X)^{(i)}$.

**Proof.** By descending induction on $i$, we see that the subring of $K(X)$ generated by line bundles contains $K(X)^{(i)}$. Therefore $K(X)$ is generated by line bundles. The statement under proof follows then from Proposition 3.3 (1) and (2). \quad \square

Note that the graded ring Chern$K(X)$, associated with the Chern filtration, is always generated by Chern classes. So, if the Chern and Chow filtrations on $K(X)$ coincide, the ring Chow$K(X)$ is generated by Chern classes. This however does not imply that the Chow ring CH$K(X)$ is generated by Chern classes:

**Example 3.5.** Let $L/F$ be a biquadratic field extension with char $F \neq 2$ and let $T$ be the corresponding (3-dimensional) torus of elements of norm 1. Denote by $E_i$, $i = 1, 2, 3$ all quadratic subextensions, by $T_i$ the subtorus in $T$ of norm 1 elements in the extension $L/E_i$ and by $\xi_i$ the class of the sheaf $O_{T_i}$ in $K_0(T)$. Then by [13, Example 9.15(2)],

$$K(T)^{(i)} = \begin{cases} 
\prod_{k=1}^3 \mathbb{Z}/2\mathbb{Z} \cdot \xi_k, & \text{if } i = 1; \\
\mathbb{Z}/2\mathbb{Z} \cdot (\xi_1 + \xi_2 + \xi_3), & \text{if } i = 2; \\
0, & \text{if } i = 3.
\end{cases}$$

In particular, $K(T)$ is generated by the classes of line bundles, $K(T)^{[i]} = K(T)^{(i)}$ for all $i$, $CH^1(T)$ is a group of order 4 generated by the three elements $a_k := c_1(\xi_k)$ of order 2 (with the relation $a_1 + a_2 + a_3 = 0$). $CH^2(T)$ is a cyclic group of order 2 generated by $a_1a_2 = a_1a_3 = a_2a_3$ and $a_k^2 = 0$ for all $k$. Therefore, all polynomials in Chern classes in $CH^3(T)$ are trivial. On the other hand, by [11, Proposition 5.3], the order of the class of the identity in $CH^3(T)$ is equal to 2, hence $CH^3(T) \neq 0$ and Chow ring CH$K(T)$ is not generated by Chern classes.
The following statement will be used in the next section:

**Lemma 3.6.** Let \( f : X \to Y \) be a flat morphism of smooth varieties such that the pull-back homomorphisms \( K(Y) \to K(X) \) and \( CH(Y) \to CH(X) \) are isomorphisms. Then the induced monomorphisms \( K(Y)^{(i)} \to K(X)^{(i)} \) and \( K(Y)^{(i)} \to K(X)^{(i)} \) are isomorphisms.

**Proof.** To prove the statement on the Chow filtration, we proceed by descending induction on \( i \). Looking at the commutative diagram

\[
\begin{array}{ccc}
CH^i(Y) & \xrightarrow{\sim} & CH^i(X) \\
\downarrow & & \downarrow \\
\text{Chow}^i K(Y) & \to & \text{Chow}^i K(X)
\end{array}
\]

we see that the bottom map is surjective. By the induction hypothesis, the map

\[
K(Y)^{(i+1)} \to K(X)^{(i+1)}
\]

is an isomorphism, hence \( K(Y)^{(i)} \to K(X)^{(i)} \) is surjective. This proves the statement on the Chow filtration.

The statement on the Chern filtration does not require the assumption on \( CH(Y) \to CH(X) \). It follows from the fact that the isomorphism \( K(Y) \to K(X) \) respects the enhanced \( \lambda \)-structures.

\[\square\]

4. **Representation rings of algebraic groups**

By an *algebraic group* we mean an affine group scheme of finite type over a field. Let \( G \) be an algebraic group over a field \( F \) and let \( R(G) \) be its *representation ring* – the Grothendieck ring of the category of finite-dimensional linear \( G \)-representations. The augmentation ideal \( I(G) \subset R(G) \) is the kernel of the (augmentation) ring homomorphism \( R(G) \to \mathbb{Z} \) given by dimension of \( G \)-representations. The augmentation filtration on \( R(G) \) is given by powers \( I(G)^i, i \geq 0 \) of the augmentation ideal.

The ring \( R(G) \) is an enhanced \( \lambda \)-ring by Example 2.2. We simply write \( c_i^R \) for the Chern classes \( c_i^{R(G)} \). Recall that there is Chern filtration

\[
R(G) = R(G)^{[0]} \supset R(G)^{[1]} \supset \ldots
\]

with the property that \( c_i^R(x) \in R(G)^{[i]} \) for all \( x \in R(G) \) and any \( i \geq 0 \). As usual, we write \( \text{Chern} R(G) \) for the associated graded ring.

Our next goal is to define the Chow filtration on \( R(G) \). Let \( V \) be a generically free \( G \)-representation over \( F \). By [2, Exposée V, Théorème 8.1], there is a nonempty \( G \)-invariant open subset \( U \subset V \) and a \( G \)-torsor \( U \to U/G \) for some variety \( U/G \) over \( F \). We say that \( U/G \) is an *\( n \)-approximation* of \( BG \) if \( \text{codim}_V (V \setminus U) \geq n \).

**Example 4.1** (cf., [19, Remark 1.4]). Embed \( G \hookrightarrow \text{GL}(m) \) with \( m > 0 \) and choose an integer \( N \geq 0 \). Let \( U \) be the open subset of all injective linear maps \( F^m \to F^{m+N} \) in the vector space \( V \) of all linear maps \( F^m \to F^{m+N} \). We have \( \text{codim}_V (V \setminus U) = N + 1 \). The group \( \text{GL}(m+N) \) acts linearly on \( V \) and acts transitively on \( U \) with the stabilizer
of the canonical inclusion $F^m \hookrightarrow F^m \oplus F^N = F^{m+N}$. The group $G$ acts on $U$ via $GL(m)$ and

$$U/G = GL(m+N)/\left(\begin{array}{cc} G & * \\
0 & GL(N) \end{array}\right).$$

Thus, $U/G$ is an $(N+1)$-approximation of $B\Gamma$. Note that $U/GL(m)$ is naturally isomorphic to the Grassmannian variety $Gr(m, m+N)$.

Let $U/G$ be an $n$-approximation of $B\Gamma$. In [19], Totaro defined graded Chow ring $CH(B\Gamma)$ by

$$CH^i(B\Gamma) = CH^i(U/G)$$

for $i < n$. This is independent of the choice of approximation.

Note that $CH(B\Gamma)$ coincides with the $G$-equivariant Chow ring of $Spec F$, [3].

Let $E \rightarrow X$ be a $G$-torsor, where $X$ is a smooth variety. We have a canonical ring homomorphism

$$\alpha_E : R(G) \rightarrow K(X),$$

taking the class of a $G$-representation $W$ to the class of the vector bundle

$$(W \times E)/G \rightarrow X.$$

Since $\alpha_E$ is a homomorphism of enhanced $\lambda$-rings, $\alpha_E$ commutes with Chern classes $c_i^R$ and $c_i^K$ respectively.

If $U/G$ is an approximation of $B\Gamma$, we have a ring homomorphism

$$\alpha_U : R(G) \rightarrow K(U/G)$$

given by the $G$-torsor $U \rightarrow U/G$. The map $\alpha_U$ is the composition of the homotopy invariance isomorphism

$$R(G) = K^G(Spec F) \sim K^G(V)$$

in equivariant $K$-theory and the surjective restriction homomorphism

$$K^G(V) \rightarrow K^G(U) = K(U/G)$$

(see [18, Theorems 2.7 and 4.1]). Thus, $\alpha_U$ is surjective.

Composing $\alpha_U$ with (classical) Chern classes on $U/G$ yields Chern classes

$$c_i : R(G) \rightarrow CH^i(B\Gamma).$$

Lemma 4.2. Let $E \rightarrow X$ be a $G$-torsor and $U/G$ an $n$-approximation of $B\Gamma$. Then

$$\alpha_U^{-1}(K(U/G)^{(n)}) \subset \alpha_E^{-1}(K(X)^{(n)})$$

Proof. In the commutative diagram

$$\begin{array}{ccc}
R(G) & \xrightarrow{\alpha_U} & K(U/G) \\
\downarrow & \downarrow & \downarrow \beta \\
K((U \times E)/G) & \xrightarrow{\varepsilon} & K((V \times E)/G) \\
& \xrightarrow{\alpha_E} & K(X)
\end{array}$$

with $\delta$ the equivalence.
the map $\delta$ is the pull-back with respect to the vector bundle $(V \times E)/G \to X$ so that $\delta$ is an isomorphism of groups with Chow filtrations by Lemma 3.6. In particular,

$$\alpha_E^{-1}(K(X)^{(n)}) = \alpha_{V \times E}^{-1}(K((V \times E)/G)^{(n)}).$$

The homomorphism $\varepsilon$ is the restriction to the open subset $(U \times E)/G \subset (V \times E)/G$ with complement of codimension at least $n$. Therefore $\varepsilon$ is surjective on the terms of Chow filtrations and the kernel of $\varepsilon$ is contained in $K((V \times E)/G)^{(n)}$ by localization property in $K$-theory. It follows that

$$\alpha_{V \times E}^{-1}(K((V \times E)/G)^{(n)}) = \alpha_{U \times E}^{-1}(K((U \times E)/G)^{(n)}).$$

As $\beta$ respects Chow filtrations, we have

$$\alpha_U^{-1}(K(U/G)^{(n)}) \subset \alpha_{U \times E}^{-1}(K((U \times E)/G)^{(n)}).$$

The result follows. \hfill $\square$

It follows from Lemma 4.2 that the subgroup $(\alpha_U)^{-1}(K(U/G)^{(n)})$ of $R(G)$ does not depend on the choice of an $n$-approximation $U/G$. We set

$$R(G)^{(n)} := (\alpha_U)^{-1}(K(U/G)^{(n)})$$

for any $n$-approximation $U/G$ of $BG$. This way we get the Chow filtration

$$R(G) = R(G)^{(0)} \supset R(G)^{(1)} \supset \ldots$$

on $R(G)$.

It also follows from Lemma 4.2 that for a $G$-torsor $E \to X$, the map $\alpha_E : R(G) \to K(X)$ takes $R(G)^{(n)}$ into $K(X)^{(n)}$, i.e., $\alpha_E$ respects Chow filtrations.

As in Section 3, we have

$$I(G)^n \subset R(G)^{[n]} \subset R(G)^{(n)}$$

for all $n$. (However none of the filtrations is finite in general.) The second inclusion induces a ring homomorphism $\text{Chern} R(G) \to \text{Chow} R(G)$ which is neither injective nor surjective in general.

Let $U/G$ be an $(n+1)$-approximation of $BG$. The composition

$$\text{CH}^n(BG) = \text{CH}^n(U/G) \xrightarrow{\varphi} \text{Chow}^n K(U/G) = \text{Chow}^n R(G)$$

yields a surjective graded ring homomorphism

$$\varphi : \text{CH}(BG) \to \text{Chow} R(G).$$

The kernel of $\varphi^i : \text{CH}^i(BG) \to \text{Chow}^i R(G)$ is killed by multiplication by $(i-1)!$. In particular, the maps $\varphi^i$ are isomorphisms for $i \leq 2$. By Proposition 3.2, $\varphi(c_i(a)) = c_i^R(a)$ modulo $R(G)^{(i+1)}$ for every $a \in R(G)$.

**Remark 4.4.** For any approximation $X$ of $BG$, the epimorphism $R(G) \to K(X)$ maps $I(G)^n$ onto $I(X)^n$, $R(G)^{[n]}$ onto $K(X)^{[n]}$, and $R(G)^{(n)}$ onto $K(X)^{(n)}$ for any $n$. The statement on the Chern filtration holds because the epimorphism $R(G) \to K(X)$ commutes with Chern classes. For the statement on $R(G)^{(n)}$ consider the diagram (4.3) in the proof of Lemma 4.2 with $X = E/G$ being an approximation and $U/G$ a $k$-approximation of $BG$ for some $k > \dim(X)$ and $\geq n$. Then the epimorphism $K(X) \to K((U \times E)/G)$
is an isomorphism of rings with Chow filtrations. The map \( R(G) \to K(U/G) \) maps \( R(G)^{(n)} \) onto \( K(U/G)^{(n)} \) by the definition of \( R(G)^{(n)} \) and \( K(U/G)^{(n)} \) is mapped onto \( K((U \times E)/G)^{(n)} = K(X)^{(n)} \).

**Remark 4.5.** By the very definition of Chow filtration on \( R(G) \), for any \( n \)-approximation \( X \) of \( BG \), the kernel of \( R(G) \to K(X) \) is contained in \( R(G)^{(n)} \). This statement can be partially inverted: if \( X \) is any approximation of \( BG \) such that \( \ker(R(G) \to K(X)) \subset R(G)^{(n)} \) for some \( n \), then \( R(G)^{(k)} \) is the inverse image of \( K(X)^{(k)} \) for all \( k \leq n \). (This does not mean that \( X \) is an \( n \)-approximation but does mean that \( X \) – like a \( n \)-approximation – can be used for computation of the Chow filtration on \( R(G) \) in codimensions up to \( n \).)

Indeed, by Remark 4.5, the inverse image of \( K(X)^{(k)} \) is

\[
R(G)^{(k)} + \ker(R(G) \to K(X)).
\]

Similarly, the inverse image of \( K(X)^{(k)} \) is \( R(G)^{(k)} \) if the kernel is contained in \( R(G)^{(n)} \) and the inverse image of \( I(X)^{(k)} \) is \( I(G)^{(k)} \) if the kernel is contained in \( I(G)^{(n)} \).

**Lemma 4.6.** For any \( G \)-torsor \( E \to X \) with \( X \) a smooth variety one can find an integer \( n \) such that \( \ker(\alpha_E) \) contains \( R(G)^{(n)} \).

**Proof.** Let \( n = \dim(X) + 1 \). Since \( \alpha_E \) respects Chow filtration, we have

\[
\alpha_E(R(G)^{(n)}) \subset K(X)^{(n)} = 0.
\]

**Lemma 4.7** (cf. the beginning of Remark 4.5). For any \( n, k \) and any \( G \), there exists a \( k \)-approximation \( X \) of \( BG \) such that the kernel of the epimorphism \( R(G) \to K(X) \) is contained in \( I(G)^{(n)} \).

**Proof.** Let us fix an embedding \( G \hookrightarrow \text{GL}(m) \) for some \( m \). For any \( N \), consider an \((N+1)\)-approximation \( U/G = \text{GL}(m+N)/H \) of \( BG \) as in Example 4.1, where \( H = \begin{pmatrix} G & * \\ 0 & \text{GL}(N) \end{pmatrix} \).

Note that \( R(H) = R(G \times \text{GL}(N)) = R(G) \otimes R(\text{GL}(N)) \) since the unipotent radical of \( H \) acts trivially on all irreducible representations of \( H \).

By [12, Theorem 41],

\[
K(U/G) = K(\text{GL}(m+N)/H) = \mathbb{Z} \otimes_{\text{R(GL(m+N))}} R(H) = \mathbb{Z} \otimes_{\text{R(GL(m+N))}} [R(G) \otimes R(\text{GL}(N))].
\]

Under this identification, the homomorphism \( \alpha_U : R(G) \to K(U/G) \) (which we denote below by \( \alpha^G \)) coincides with the natural (surjective) homomorphism

\[
R(G) \to \mathbb{Z} \otimes_{\text{R(GL(m+N))}} [R(G) \otimes R(\text{GL}(N))].
\]

It follows that

\[
\alpha^G = \alpha^{\text{GL}(m)} \otimes_{\text{R(GL(m))}} R(G)
\]

and therefore, the natural homomorphism

\[
\ker(\alpha^{\text{GL}(m)} \otimes_{\text{R(GL(m))}} R(G)) \to \ker(\alpha^G)
\]

is surjective.

Since \( U/\text{GL}(m) = \text{Gr}(m, m+N) \), as computed in Example G.2, the kernel of \( \alpha^{\text{GL}(m)} \) is generated by some polynomials (namely, by the polynomials \( d_i^R \), \( i \geq N + 1 \)) of degree
at least $N+1$ in the Chern classes $c^R_1,\ldots,c^R_m \in R(GL(m))$ of the standard representation of $GL(m)$, where $c^R_i$ is of degree $i$. Therefore, $\text{Ker}(\alpha^G)$ is generated by polynomials in the images of $c^R_1,\ldots,c^R_m$ (these images are the Chern classes of the fixed embedding $G \hookrightarrow GL(m)$) of degree $> N$ and will indeed contain $I(G)^n$ for sufficiently large $N$ (say, for $N \geq mn$).

\textbf{Corollary 4.8.} For any group $G$ and any $n$, we have $I(G)^n \supset R(G)^{(N)}$ for some $N$. In particular, the completions of $R(G)$ with respect to the three filtrations are all the same.

\textit{Proof.} By Lemma 4.7, we find an approximation $X$ of $BG$ such that

$$I(G)^n \supset \text{Ker}(R(G) \rightarrow K(X)).$$

By Lemma 4.6, the kernel contains $R(G)^{(N)}$ for some $N$. \hfill $\square$

\textbf{Corollary 4.9.} We have $I(G) = R(G)^{(1)} = R(G)^{(2)}$ and $R(G)^{(2)} = R(G)^{(2)}$. The map $\text{Chern} R(G) \rightarrow \text{Chow} R(G)$ becomes an isomorphism after tensoring with $\mathbb{Q}$.

\textit{Proof.} Let $X$ be an approximation of $BG$ such that the kernel of $R(G) \rightarrow K(X)$ is contained in $I(G)^{n+1}$. By Remarks 4.4 and 4.5, $R(G)^{(n)}$ and $R(G)^{(n)}$ are inverse images of $K(X)^{(n)}$ and $K(X)^{(n)}$, respectively, moreover, $\text{Chern}^n R(G) = \text{Chern}^n K(X)$ and $\text{Chow}^n R(G) = \text{Chow}^n K(X)$. The statements follow from Remark 3.1. \hfill $\square$

Now we compute the groups $\text{Chow}^i(G)$ for $i \leq 1$. Clearly, $\text{Chow}^0(G) = R(G) / I(G) = \mathbb{Z}$.

\textbf{Lemma 4.10.} We have

$$\sum_{i=0}^{n} \gamma^i([V] - n) = \lambda^n([V]) \text{ in } R(G)$$

for every $G$-representation $V$ of dimension $n$.

\textit{Proof.} Since $\gamma_i(-1) = 1 - t$ and $\lambda_i([V]) = 0$ for $i > n$, we have

$$\gamma_i([V] - n) = \gamma_i([V])(1 - t)^n = \sum_{i=0}^{\infty} \lambda_i([V]) \frac{t^i}{(1-t)^i}(1-t)^n = \sum_{i=0}^{n} \lambda_i([V]) t^i(1-t)^{n-i}.$$

It follows that $\gamma^i([V] - n) = 0$ for $i > n$. Finally, plug in $t = 1$. \hfill $\square$

\textbf{Corollary 4.11.} For every $G$-representation $V$ of dimension $n$, we have

$$\lambda^n([V]) - [V] + n - 1 \in R(G)^{(2)}.$$

\textit{Proof.} Indeed, since $\gamma^0([V] - n) = 1$ and $\gamma^1([V] - n) = [V] - n$, we have

$$\lambda^n([V]) - [V] + n - 1 = \sum_{i=2}^{n} \gamma^i([V] - n) = \sum_{i=2}^{n} c^R_i([-V^\vee]) \in R(G)^{(2)}.$$

Let $G^* = \text{Hom}(G,G_m)$ be the character group of $G$. Consider the homomorphism $\det : R(G) \rightarrow G^*$ taking a representation $V$ of dimension $n$ to the character of the 1-dimensional representation $\wedge^n(V)$.

\textbf{Lemma 4.12.} $\det(R(G)^{(2)}) = 1$. 

Recall that $R(G)^{[2]}$ is generated by the products $c_i^R(a_1) \cdots c_i^R(a_n)$ with $i_1 + \cdots + i_n \geq 2$ and $a_t \in I(G)$. In view of the product formula $\det(ab) = \det(a)^{rk(b)} \det(b)^{rk(a)}$, it suffices to show that $\det c_i^R(V) = 1$ for any representation $\rho : G \to GL(V)$ and $i \geq 2$. By functoriality of the Chern classes with respect to $\rho$, it suffices to check the equality in the case $G = GL(V)$ and $V$ the standard representation of $GL(V)$.

Let $T$ be a split maximal torus of $G$. Since $G^*$ restricts injectively to $T^*$, it suffices to prove the formula for $T$. But $R(T)$ is generated by the classes of 1-dimensional representations, hence $R(T)^{[2]}$ is generated by products of at least two (first) Chern classes. The statement follows again from the product formula. \hfill $\square$

By Lemma 4.12, the map $\det$ yields a homomorphism $\text{Chow}^1(G) = \text{Chern}^1(G) \to G^*$.

**Proposition 4.13.** The homomorphism $\text{Chern}^1(G) \to G^*$ is an isomorphism.

*Proof.* We define a homomorphism $G^* \to \text{Chern}^1(G)$ by $f \mapsto [L_f] - 1$ modulo $R(G)^{[2]}$, where $L_f$ is a 1-dimensional representation of the character $f$. Both compositions of the maps between $\text{Chern}^1(G)$ and $G^*$ are the identities, one of them – in view of Corollary 4.11. \hfill $\square$

The next statement is an analogue of Proposition 3.3.

**Proposition 4.14.** The following holds for any integer $i \geq 0$.

1. If $R(G)$ is generated by classes of 1-dimensional representations, then $I(G)^i = R(G)^{[i]}$.
2. If $\text{CH}(BG)$ is generated by Chern classes (of $G$-representations), then $R(G)^{[i]} = R(G)^{(i)}$.
3. If the group $\text{Chern} R(G)$ is torsion-free, then $R(G)^{[i]} = R(G)^{(i)}$.

*Proof.* The proof of (1) is literally the same as in Proposition 3.3. If $l \in R(G)$ is the class of a 1-dimensional representation, then $c_i^R(l) = 1 - l^{-1} \in I(G)$ and $c_i^R(-l) = (l^{-1} - 1)^i \in I(G)^i$ for any $i$. If $a, b \in R(G)$ are such that $c_i^R(a), c_i^R(b) \in I(G)^i$ for all $i$, then $c_i^R(a + b) = \sum_j c_j^R(a)c_{i-j}^R(b) \in I(G)^i$.

(2) For any $i$, the surjective ring homomorphism $\varphi : \text{CH}(BG) \to \text{Chow} R(G)$ takes the Chow-theoretical Chern class $c_i(a) \in \text{CH}^i(BG)$ with $a \in R(G)$ to the class of $c_i^R(a) \in R(G)^{[i]} \subset R(G)^{(i)}$ modulo $R(G)^{(i+1)}$. It follows that the ring $\text{Chow} R(G)$ is generated by Chern classes. However, unlike the proof of Proposition 3.3(2), we are not able to show $R(G)^{[i]} = R(G)^{(i)}$ by descending induction on $i$ as the filtrations in question can be infinite. We use Lemma 4.7 instead.

For $X$ as in Lemma 4.7 (with any given $n$ and arbitrary $k$), since $\text{CH}(BG)$ is generated by Chern classes and $\text{CH}(BG) \to \text{CH}(X)$ is an epimorphism mapping Chern classes to Chern classes, the ring $\text{CH}(X)$ is also generated by Chern classes. It follows by Proposition 3.3(2) that $K(X)^{[n]} = K(X)^{(n)}$. Therefore $R(G)^{[n]} = R(G)^{(n)}$ by Remark 4.6.

(3) If $\text{Chern} R(G)$ is torsion-free, the homomorphism $\text{Chern} R(G) \to \text{Chow} R(G)$ is injective. The equality $R(G)^{[i]} = R(G)^{(i)}$ follows by ascending induction on $i$. \hfill $\square$

**Corollary 4.15.** If $\text{CH}(BG)$ is generated by $\text{CH}^1(BG)$, then $I(G)^i = R(G)^{[i]} = R(G)^{(i)}$. 

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Proof. If $\text{CH}(BG)$ is generated by $\text{CH}^1(BG)$, then $I(G)^i + R(G)^{(i+1)} = R(G)^{(i)}$ for any $i$ so that $I(G)^i + R(G)^{(j)} = R(G)^{(i)}$ for any $j > i$. By Corollary 4.8, $R(G)^{(j)} \subset I(G)^i$ for some $j$, hence $R(G)^{(i)} \subset I(G)^i$.

We finish this section by an example of $G$ with the Chern filtration on $R(G)$ different from the Chow filtration:

**Example 4.16.** For $G := O^+(2n)$ with any $n \geq 3$ over the field of complex numbers, the Chern filtration on $R(G)$ differs from the Chow filtration. Indeed, according to [4, Corollary 2], the Chow ring $\text{CH}(BG)$ is not generated by Chern classes. By [4, Theorem 1] (see also [15]), the Chern subring of $\text{CH}(BG)$ contains every element of finite order of the group $\text{CH}(BG)$. Since the kernel of the epimorphism $\text{CH}(BG) \rightarrow \text{Chow} R(G)$ consists of elements of finite order, the two above statements together imply that the ring $\text{Chow} R(G)$ is not generated by Chern classes. Since the ring $\text{Chern} R(G)$ is generated by Chern classes (for any $G$), the two filtrations (for $G = O^+(2n)$) are not the same.

5. Orthogonal groups

Let $G$ be a split reductive group (over an arbitrary field), $T \subset G$ a split maximal torus, $W$ the Weyl group. We have a natural homomorphism of graded rings

$$\text{CH}(BG) \rightarrow \text{CH}(BT)^W = S(T^*)^W,$$

where $T^*$ is the character group of $T$ and $S(T^*)$ is its symmetric ring. Similarly, we have a ring homomorphism

$$R(G) \rightarrow R(T)^W = Z[T^*]^W.$$

**Proposition 5.1** ([17, Théorème 4], see also [14, Theorem 22.38]). The homomorphism $R(G) \rightarrow Z[T^*]^W$ is an isomorphism.

Recall that an algebraic group $G$ is special, if all $G$-torsors over field extensions of the base field are trivial.

**Proposition 5.2** ([3, Proposition 6]). The homomorphism $\text{CH}(BG) \rightarrow S(T^*)^W$ is an isomorphism provided that $G$ is special.

**Example 5.3** (cf. [19, §15], [15, §3]). For the special groups $G = \text{GL}(n), \text{SL}(n), \text{Sp}(2n)$ (over an arbitrary field), let $c_i \in \text{CH}^i(BG)$ be the $i$th Chern class of the standard $G$-representation. Then the following Chow rings are polynomial rings in the listed algebraically independent elements:

$$\text{CH}(B\text{GL}(n)) = Z[c_1, c_2, \ldots, c_n],$$

$$\text{CH}(B\text{SL}(n)) = Z[c_2, \ldots, c_n], \quad c_1 = 0,$$

$$\text{CH}(B\text{Sp}(2n)) = Z[c_2, c_4, \ldots, c_{2n}], \quad c_{odd} = 0.$$

Let us fix an integer $n \geq 1$ and consider the symplectic group $H := \text{Sp}(2n)$ (over an arbitrary field). By Example 5.3, the group $\text{CH}(BH)$ is torsion-free. Since the kernel of the ring epimorphism $\text{CH}(BH) \rightarrow \text{Chow} R(H)$ consists of torsion elements only, it follows that this map is an isomorphism. In particular, the group $\text{Chow} R(H)$ is torsion-free.
Since the ring CH(BH) is generated by Chern classes, we conclude by Proposition 4.14(2) that the Chow filtration on R(H) coincides with the Chern filtration. It follows that the group ChernR(H) is torsion-free.

The Weil groups and character groups of maximal tori (as modules over the Weil groups) of Sp(2n) and O+(2n + 1) are isomorphic. Set G = O+(2n + 1). By Proposition 5.1, there is an isomorphism of enhanced lambda rings R(H) \simeq R(G). It induces an isomorphism ChernR(H) \simeq ChernR(G). In particular, the group ChernR(G) turns out to be torsion-free. By Proposition 4.14(3), this implies that the Chern filtration on R(G) coincides with the Chow filtration. We conclude that the group ChowR(G) is torsion-free.

The ring CH(BG) has been computed for G = O+(2n + 1) over the complex numbers in [19, §16]; it has been then computed for an arbitrary base field of characteristic ≠ 2:

Proposition 5.4 ([15, Theorem 5.1]). For G = O+(2n + 1) (with any n ≥ 1) over a field of characteristic ≠ 2, the ring homomorphism \( \mathbb{Z}[c_1, \ldots, c_{2n+1}] \to CH(BG) \) of the polynomial ring, mapping \( c_i \) to the \( i \)th Chern class of the standard \( G \)-representation, is surjective; its kernel is generated by \( c_1 \) and all \( 2c_i \) with odd \( i \).

It follows that for \( G \) as in Proposition 5.4, the kernel of the epimorphism \( CH^3(BG) \to Chow^3R(G) \) is non-zero.

Theorem 5.5. For any field \( F \) of char \( F \neq 2 \), there exists a smooth quasi-projective variety \( X \) over \( F \) such that its Chow ring is generated by Chern classes (of vector bundles over \( X \)) and at the same time the kernel of the epimorphism \( CH(X) \to ChowK(X) \) is non-zero. Specifically, let \( X \) be any 4-approximation of \( O^+(2n + 1) \) (for any \( n \geq 1 \)). Then the ring \( CH(X) \) is generated by Chern classes and \( Ker(CH^3(X) \to Chow^3K(X)) \neq 0 \).

Proof. We have a ring epimorphism \( CH(BG) \to CH(X) \) with \( G = O^+(2n + 1) \), mapping Chern classes to Chern classes. Since \( CH(BG) \) is generated by Chern classes (Proposition 5.4), the ring \( CH(X) \) is also generated by Chern classes.

Since \( X \) is a 4-approximation, the epimorphism \( CH^3(BG) \to CH^3(X) \) is an isomorphism so that \( CH^3(X) \) contains a non-trivial torsion by Proposition 5.4.

At the same time, we have an epimorphism \( ChowR(G) \to ChowK(X) \) which is an isomorphism in codimensions < 4. Since \( ChowR(G) \) is torsion-free, the group \( Chow^3K(X) \) is also torsion-free. \( \square \)

Remark 5.6. If \( n \) is large enough and \( X \) is an \( r \)-approximation of \( BO^+(2n + 1) \), then \( CH(X) \) is generated by Chern classes and \( Ker(CH^i(X) \to Chow^iK(X)) \neq 0 \) for all \( i \neq 4 \) with \( 3 \leq i < r \).

Appendix G. General linear group

In this appendix, we provide some computations in the representation ring of a general linear group needed in the main part.

Example G.1. Let \( G := GL(n) \), \( T \subset G \) the maximal torus of diagonal matrices. We have \( R(T) = \mathbb{Z}[x_1^{±1}, x_2^{±1}, \ldots, x_n^{±1}] \) a Laurent polynomial ring, where \( x_i = \exp(t_i) \) and \( t_i \) are canonical generators of \( T^* \). The Weyl group \( W \) is the \( n \)th symmetric group permuting the \( t_i \)'s. It follows that

\[
R(G) = R(T)^W = \mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_n^{-1}],
\]
Consider the homomorphism $\sigma_k := \sigma_k(x)$ are standard symmetric functions in the $x_i$’s. Note that $\sigma_k = \lambda^k(x)$, the $k$th exterior power of $x$, where $x = x_1 + x_2 + \cdots + x_n$ is the class of the standard representation of $G$.

Since taking the dual $a \mapsto a^\vee$ of a representation yields an automorphism of the rings $R(T)$ and $R(G)$ taking $x_i$ to $x_i^{-1}$, we have $\lambda^i(x^\vee) = \lambda^{n-i}(x)\lambda^n(x)^{-1}$ and $\lambda^n(x^\vee) = \lambda^n(x)^{-1}$. Therefore,

$$R(G) = \mathbb{Z}[\lambda^1(x^\vee), \lambda^2(x^\vee), \ldots, \lambda^n(x^\vee), \lambda^n(x^\vee)^{-1}].$$

The Chern class $c_i^R$ is the $i$th standard elementary symmetric function in the first Chern classes $c_i^R(x_j) = 1 - x_j^{-1}$, $j = 1, \ldots, n$. In particular, $c_i^R$ is the sum of $(-1)^j \lambda^i(x^\vee)$ and an integer linear combination of $\lambda^j(x^\vee)$ with $j < i$.

We also have

$$\sigma_n^{-1} = \prod [1 - (1 - x_i^{-1})],$$

hence

$$\sigma_n^{-1} = c_0^R - c_1^R + c_2^R - \cdots + (\pm 1)^n c_n^R.$$ 

It follows that

$$R(G) = \mathbb{Z}[c_1^R, c_2^R, \ldots, c_n^R, s_n^{-1}],$$

where $c_i^R$ are algebraically independent and

$$s_n = \sum_{i=0}^n (-1)^i c_i^R.$$

**Example G.2.** Consider the homomorphism

$$\alpha^{\text{GL}(m)} : R(\text{GL}(m)) \to \mathbb{Z} \otimes_{R(\text{GL}(m+N))} [R(\text{GL}(m)) \otimes R(\text{GL}(N))] = K(\text{Gr}(m, m+N))$$

from the proof of Lemma 4.7. The target group is canonically isomorphic to

$$\left( R(\text{GL}(m)) \otimes R(\text{GL}(N)) \right) / J,$$

where $J$ is the ideal generated by the image of the augmentation ideal of $R(\text{GL}(m+N))$ under $R(\text{GL}(m+N)) \to R(\text{GL}(m)) \otimes R(\text{GL}(N))$. Recall (see Example G.1) that

$$R(\text{GL}(m)) = \mathbb{Z}[c_1^R(x), c_2^R(x), \ldots, c_n^R(x), s_n^{-1}(x)],$$

$$R(\text{GL}(N)) = \mathbb{Z}[c_1^R(y), c_2^R(y), \ldots, c_m^R(y), s_m^{-1}(y)],$$

where $x$ and $y$ are the classes of the standard representations of $\text{GL}(m)$ and $\text{GL}(N)$ respectively. Note that the class of the standard representation of $\text{GL}(m+N)$ restricted to $\text{GL}(m) \times \text{GL}(N)$ is the sum of $x$ and $y$. Moreover, the augmentation ideal of $R(\text{GL}(m+N))$ is generated by $c_i^R(x+y)$ for $i = 1, 2, \ldots, m+N$. It follows that modding out $J$ amounts to imposing the relation

$$c_i^R(x+y) = c_i^R(x) \cdot c_i^R(y) = 1,$$

where $c_i^R$ is the total Chern class.

Invert the polynomial $c_i^R := c_i^R(x)$ formally:

$$(c_i^R)^{-1} = \sum_{i \geq 0} d_i^R t^i,$$
where $d^R_i$ is a homogeneous polynomial in $c_j^R(x)$ of degree $i$. Then

$$K(\text{Gr}(m, m + N)) = \mathbb{Z}[c_1^R, c_2^R, \ldots, c_m^R]/(d^R_i, i \geq N + 1).$$

(Not that the elements $s_m(x)$ and $s_N(y)$ are 1 plus nilpotents in $K(\text{Gr}(m, m + N))$, hence, $s_m(x)$ and $s_N(y)$ are automatically invertible in $K(\text{Gr}(m, m + N)).$)

**Remark G.3.** The ideal $(d^R_i, i \geq N + 1)$ is generated by $d^R_i$ for $i = N + 1, \ldots, N + m$ only. Indeed, if $j > 0$, $d^R_{N+m+j}$ is the negative of the linear combination $c_1^R d^R_{N+m+j-1} + \cdots + c_m^R d^R_{N+j}$ and hence the statement follows by induction on $j$.

It follows that the homomorphism

$$\alpha^{\text{GL}(m)} : R(\text{GL}(m)) \to K(\text{Gr}(m, m + N))$$

is surjective, and its kernel is generated by polynomials in $c_1^R, c_2^R, \ldots, c_m^R$ of degree at least $N + 1$.

**APPENDIX O. ORTHOGONAL GROUPS IN CHARACTERISTIC 2**

In this appendix, we determine $\text{CH}(BG)$ for $G = O(2n + 1)$ and $G = O^+(2n + 1)$ over a field $F$ of characteristic 2. We start with some observations valid over a field of any characteristic.

For any $m \geq 1$, the factor variety $\text{GL}(m)/O(m)$ is isomorphic to the variety of non-degenerate quadratic forms of dimension $m$ that is an open subset in the affine space of all quadratic forms of dimension $m$ (see, e.g., [7, §3]). Therefore, by [19, Proposition 14.2] (see also [7, Proposition 5.1]), $\text{CH}(BO(m))$ is generated by Chern classes $c_1, c_2, \ldots, c_m$ of the standard representation of $O(m)$.

For any $n \geq 1$, we have $O(2n + 1) = \mu_2 \times O^+(2n + 1)$. Hence the restriction

$$\text{CH}(BO(2n + 1)) \to \text{CH}(BO^+(2n + 1))$$

is surjective. Therefore, the Chow ring $\text{CH}(BO^+(2n + 1))$ of the classifying space of the group $O^+(2n + 1)$ is also generated by the Chern classes of the standard representation.

From now on assume that $\text{char}(F) = 2$. We have the following group homomorphisms:

$$O(2n) \to O^+(2n + 1) \to \text{Sp}(2n) \hookrightarrow \text{GL}(2n).$$

The first map takes an automorphism $\alpha$ of the standard (non-degenerate) $2n$-dimensional hyperbolic quadratic form $(V, q)$ to the automorphism $1 \oplus \alpha$ of $(V', q')$, where $V' = F \oplus V$ and $q'(a + v) = a^2 + q(v)$. Note that the subspace $F \subset V'$ coincides with the radical of the bilinear form of $q'$. The second map takes an automorphism $\beta$ of $(V', q')$ to the induced automorphism of $V = V'/F$ that preserves the associated nondegenerate alternating form on $V$. The first map and the composition $O(2n) \to \text{Sp}(2n)$ are the embeddings. The second map is a (non-central) isogeny.

Let $T$ be a split maximal torus in $O(2n)$. Its isomorphic images in $O^+(2n + 1)$ and $\text{Sp}(2n)$ are also maximal tori. Consider the composition

$$\text{CH}(BGL(2n)) \to \text{CH}(BSp(2n)) \to \text{CH}(BO^+(2n + 1)) \to \text{CH}(BO(2n)) \to \text{CH}(BT).$$

For any $i$, the Chern class $c_i$ in $\text{CH}(BGL(2n))$ clearly stays $c_i$ in $\text{CH}(BSp(2n))$ as well as in $\text{CH}(BO(2n))$. We claim that the class $c_i$ in $\text{CH}(BO^+(2n + 1))$ (which is defined via the embedding of $O^+(2n + 1)$ into $\text{GL}(2n + 1)$) stays $c_i$ in $\text{CH}(BO(2n))$ (which is defined
via the embedding of $O(2n)$ into $GL(2n)$). In other words, the classes $c_i$ in all groups correspond to each other.

To prove the claim, we use [19, Theorem 1.3], identifying the elements of $CH(BG)$ for reductive $G$ with assignments to every $G$-torsor over a smooth quasi-projective variety $X$ of an element in $CH(X)$. So, let $X$ be a smooth quasi-projective variety over $F$ and let us consider a vector bundle $E \to X$ of rank $2n + 1$ with a quadratic form representing an $O^+(2n + 1)$-torsor. As $O^+(2n + 1) \subset SL(2n + 1)$, we have $c_1(E) = 0$ (see Example 5.3). The radical $R \subset E$ of the associated bilinear form is a line sub-bundle of $E$. The factor bundle $E/R$ carries a non-degenerate alternating form, thus representing a $Sp(2n)$-torsor, hence $c_1(E/R) = 0$. It follows that $c_i(E) = c_i(E/R) + c_1(R)c_{i-1}(E/R) = c_i(E/R)$. The claim is proved.

The composition $CH(BSp(2n)) \to CH(BT)$ in the sequence above is a monomorphism. All groups are generated by the Chern classes $c_i$, therefore, all maps are surjective. The odd Chern classes are trivial in $CH(BSp(2n))$ and hence in $CH(BO^+(2n + 1))$ and in $CH(BO(2n))$. It follows that

$$CH(BSp(2n)) = CH(BO^+(2n + 1)) = CH(BO(2n)) = \mathbb{Z}[c_2, c_4, \ldots, c_{2n}].$$

Recall that $O(2n + 1) = \mu_2 \times O^+(2n + 1)$ and $CH(B\mu_2) = \mathbb{Z}[c_1]/(2c_1)$. It follows (see [19, §6]) that

$$CH(BO(2n + 1)) = \mathbb{Z}[c_1, c_2, c_4, \ldots, c_{2n}]/(2c_1).$$

### Appendix S. Symplectic group

The statement of Lemma S.1 below has already been proved in §5 (and used in the proof of Theorem 5.5). In this appendix we provide a more direct proof.

Let us fix some $n \geq 1$ and consider the symplectic group $G := Sp(2n)$ over an arbitrary field. The representation ring $R(T)$ of the standard split maximal torus $T = G_m^n$ is identified with the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ in variables $x_1, \ldots, x_n$. The Weyl group $W$ of $G$ with respect to $T$ is a semidirect product of the symmetric group $S_n$ with a direct product $\Pi$ of $n$ copies of $\mathbb{Z}/2\mathbb{Z}$. The action of $W$ on $R(T)$ is described as follows: the subgroup $S_n \subset W$ acts by permutations of the variables and the $i$th copy of $\mathbb{Z}/2\mathbb{Z}$ acts by exchanging $x_i$ and $x_i^{-1}$. It follows that the ring $R(T)^W$ of the elements invariant under the action of $W$ (this ring can be viewed as the representation ring of the product of $n$ copies of $SL(2)$, containing $T$ and contained in $G$) is the polynomial ring $\mathbb{Z}[y_1, \ldots, y_n]$ with $y_i := x_i + x_i^{-1}$. We prefer to view it as the polynomial ring $\mathbb{Z}[z_1, \ldots, z_n]$ with $z_i := 2 - y_i$. The ring $R(T)^W$ of $W$-invariant elements is the subring of symmetric polynomials in $z_1, \ldots, z_n$. So, $R(G) = R(T)^W = \mathbb{Z}[\sigma_1, \ldots, \sigma_n]$, where $\sigma_i$ is the $i$th elementary symmetric polynomial in $z_1, \ldots, z_n$.

Let us consider the Chern classes $c_1^R, \ldots, c_{2n}^R \in R(G)$ of the standard representation. We claim that the ring $R(G)$ is generated by the even Chern classes $c_2^R, c_4^R, \ldots, c_{2n}^R$. Indeed, the class in $R(G)$ of the standard representation of $G$ equals $y_1 + \cdots + y_n$. The total Chern class of $y_i \in R(T)^W$ equals

$$(1 + (1 - x_i^{-1})t)(1 + (1 - x_i)t) = 1 + z_i t + z_i t^2 = 1 + z_i (t + t^2) \in R(T)^W[t]$$
so that the total Chern class of $y_1 + \cdots + y_n$ equals
$$1 + \sigma_1(t + t^2) + \sigma_2(t + t^2)^2 + \cdots + \sigma_n(t + t^2)^n \in R(G)[t].$$

It follows that $c_{2n}^R = \sigma_n$. More generally, $c_i^R$ for any $i$ equals $\sigma_i$ plus a polynomial in higher sigmas so that a descending induction on $i$ gives the claim.

Let us consider the filtration on $R(G)$ given by the even Chern classes $c_2^R, \ldots, c_{2n}^R$: its $i$th term is generated by monomials in these Chern classes of degree $\geq i$, where the degree of $c_i^R$ is $i$. This filtration is contained in the Chern filtration on $R(G)$ and therefore also in the Chow filtration on $R(G)$. Let $C$ be the associated graded ring of this filtration. We recall that $CH(BG)$ is a polynomial ring in the even Chow Chern classes $c_2, c_4, \ldots, c_{2n} \in CH(BG)$. The homomorphism $CH(BG) \to C$, mapping $c_{2i}$ to the class of $c_{2i}^R$ in $C^{2i}$, is surjective. Composing it with the homomorphism $C \to Chow(R(G))$, induced by inclusion of filtrations, we get the standard epimorphism $CH(BG) \to ChowR(G)$ which is an isomorphism because $CH(BG)$ is torsion-free. It follows that $C \to Chow(R(G))$ (as well as $CH(BG) \to C$) is an isomorphism so that the filtration on $R(G)$, given by $c_2^R, \ldots, c_{2n}^R$, coincides with the Chow filtration as well as with the Chern filtration. In particular, we proved

**Lemma S.1.** For $G = Sp(2n)$, the group $ChernR(G)$ is torsion-free.

**References**


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