ISOTROPY OF 6-DIMENSIONAL QUADRATIC FORMS OVER FUNCTION FIELDS OF QUADRICS

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Abstract. Let F be a field of characteristic different from 2 and ϕ be an anisotropic 6-dimensional quadratic form over F. We study the last open cases in the problem of describing the quadratic forms ψ such that ϕ becomes isotropic over the function field $F(\psi)$.

Contents

0. Introduction

Let F be a field of characteristic different from 2 and let ϕ and ψ be two anisotropic quadratic forms over F. An important problem in the algebraic theory of quadratic forms is to find conditions on ϕ and ψ so that $\phi_{F(\psi)}$ is isotropic.

More precisely, one studies the question whether the isotropy of ϕ over $F(\psi)$ is standard in a sense. In this paper we will use the following definition of "*standard isotropy*":

Definition. Let ϕ and ψ be anisotropic quadratic forms such that $\phi_{F(\psi)}$ is isotropic. We say that the isotropy of $\phi_{F(\psi)}$ is *standard*, if at least one of the following conditions holds:

- ψ is similar to a subform in ϕ;
- there exists a subform $\phi_0 \subset \phi$ with the following two properties:
  - the form ϕ_0 is a Pfister neighbor,
  - the form $(\phi_0)_{F(\psi)}$ is isotropic.

Otherwise, we say that the isotropy is *non-standard*.

In the case when $\dim\phi \le 5$, the isotropy of $\phi_{F(\psi)}$ is always standard ([24], [3]). For 6-dimensional quadratic forms, the problem was studied by A. S. Merkurjev ([15]), D. Leep ([13]), D. W. Hoffmann ([4]), A. Laghribi ([10], [11]), and the authors ([5]). It was proved that the isotropy of a 6-dimensional quadratic form ϕ over the function field of a quadratic form ψ is always standard except (possibly) for the following case (see [10], [5]):

- $\dim\psi = 4$, $d_\pm\,\psi \neq 1$, $d_\pm\,\phi \neq 1$, and $\operatorname{ind} C_0(\phi) = 2$.

In the present paper we study the isotropy of $\phi_{F(\psi)}$ for quadratic forms ϕ and ψ satisfying these conditions (with $\dim\phi = 6$).

Note that the condition $\operatorname{ind} C_0(\phi) = 2$ implies that there exist $a, b, c, d \in F^*$ such that ϕ is similar to the form $\langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$. Since ϕ can be replaced by a similar form, we can assume that $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$. Note that in this case $[C_0(\phi)] = [(a, b)_{F(\sqrt{d})}] = [C_0(\rho)]$, where ρ is defined as follows: $\rho = \langle -a, -b, ab, d\rangle$.

Since $\dim\psi = 4$, there exist $u, v, \delta \in F^*$ such that ψ is similar to the quadratic form $\langle -u, -v, uv, \delta\rangle$. Since $d_\pm\,\psi \neq 1$, we have $\delta \notin F^{*2}$. Thus our main problem is reduced to the following

Question. *Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ be anisotropic quadratic forms over F with $d, \delta \notin F^{*2}$. Suppose that $\phi_{F(\psi)}$ is isotropic. Is the isotropy standard?*

This question naturally splits into the following four cases:

(1) $d = \delta$ as elements of F^*/F^{*2},
(2) $d \neq \delta$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$,
(3) $d \neq \delta$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 2$,
(4) $d \neq \delta$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 4$.

We prove that in the cases (1), (2), and (4) the isotropy of $\phi_{F(\psi)}$ is always standard (see Theorem 8.5, Propositions 8.6 and 8.7). This statement gives rise to the following one (which is Theorem 8.8):

Theorem. *Let ϕ be an anisotropic quadratic form of dimension ≤ 6 and ψ be such that $\phi_{F(\psi)}$ is isotropic. Then isotropy is standard except (possibly) the following case:* $\dim\phi = 6$, $\dim\psi = 4$, $1 \neq d_\pm\,\phi \neq d_\pm\,\psi \neq 1$, *and* $\operatorname{ind} C_0(\phi) = 2 = \operatorname{ind} C_0(\phi) \otimes_F C_0(\psi)$.

The proof of this theorem is based on a computation of the second Chow group for certain homogeneous varieties. Namely, we show that the question on the standard isotropy can be reduced to a question on the group $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$, where $\rho = \langle -a, -b, ab, d\rangle$ and X_ψ and X_ρ are the projective quadrics corresponding to ψ and ρ. In the cases (1), (2) and (4), we compute the group $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$ completely (see Theorems 5.7, 5.1, 5.8, and Lemma 7.7):

Theorem. *Let ψ and ρ be 4-dimensional quadratic forms. Then the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$ *is zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover,*

- *if* $\det\psi = \det\rho$ *or if* $\operatorname{ind} C_0(\psi)\otimes_F C_0(\rho) = 4$*, then the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi\times X_\rho)$ *is trivial;*
- *in the case* $\operatorname{ind} C_0(\psi)\otimes_F C_0(\rho) = 1$*, the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi\times X_\rho)$ *is trivial if and only if* ρ *and* ψ *contain similar 3-dimensional subforms.*

In the case (3) where $d \neq \delta$ and $\operatorname{ind} C_0(\phi)\otimes_F C_0(\psi) = 2$, we show that our main question is equivalent to the following one (see §9): *is the group* $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho)$ *trivial for any 4-dimensional quadratic forms* ψ *and* ρ *such that* $1 \neq \det\psi \neq \det\delta \neq 1$ *and* $\operatorname{ind} C_0(\psi)\otimes C_0(\rho) = 2$*?* As shown in [6], the answer to this qiestion is negative, i.e. a counterexample exists.

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1. Terminology, notation, and backgrounds

Quadratic forms. By $\phi \perp \psi$, $\phi \simeq \psi$, and $[\phi]$ we denote respectively orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F. To simplify notation, we write $\phi_1+\phi_2$ instead of $[\phi_1]+[\phi_2]$. For a quadratic form ϕ of dimension n, we set $d_\pm\,\phi = (-1)^{n(n-1)/2}\det\phi \in F^*/F^{*2}$. The maximal ideal of $W(F)$ generated by the classes of the even-dimensional forms is denoted by $I(F)$. The anisotropic part of ϕ is denoted by ϕ_{an}. We denote by $\langle\langle a_1,\dots,a_n\rangle\rangle$ the n-fold Pfister form

$$\langle 1,-a_1\rangle\otimes\cdots\otimes\langle 1,-a_n\rangle$$

and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to an n-fold Pfister form we denote by $GP_n(F)$. For any field extension L/F, we put $\phi_L = \phi\otimes_F L$, $W(L/F) = \ker(W(F)\to W(L))$, and $I^n(L/F) = \ker(I^n(F)\to I^n(L))$.

For a quadratic form ϕ and a field extension L/F, we denote by $D_L(\phi)$ the set of the non-zero values of the quadratic form ϕ_L.

For a quadratic form ϕ of dimension ≥ 3, we denote by X_ϕ the projective variety given by the equation $\phi = 0$. We set $F(\phi) = F(X_\phi)$ and $F(\phi,\psi) = F(X_\phi\times X_\psi)$ for quadratic forms ϕ and ψ of dimensions ≥ 3.

Algebras. We consider only finite-dimensional F-algebras.

For a simple F-algebra A, by $\operatorname{ind}(A)$ we denote the Schur index of A. For an algebra B of the form $B = A\times\cdots\times A$ with simple A, we set $\operatorname{ind} B = \operatorname{ind} A$.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ. By $C_0(\phi)$ we denote the even part of $C(\phi)$. For any collection $\rho_1,\dots,\rho_m$ of quadratic forms, the algebra $C_0(\rho_1)\otimes_F\cdots\otimes_F C_0(\rho_m)$ is of the form $A\times\cdots\times A$ with simple A. Therefore, we get a well-defined positive integer $\operatorname{ind} C_0(\rho_1)\otimes_F\cdots\otimes_F C_0(\rho_m)$.

If $\phi \in I^2(F)$ then $C(\phi)$ is a central simple algebra. Hence we get a well-defined element $[C(\phi)]$ in the 2-part $\operatorname{Br}_2(F)$ of the Brauer group $\operatorname{Br}(F)$ which we denote by $c(\phi)$.

Cohomology groups. By $H^*(F)$ we denote the graded ring of Galois cohomology $H^*(F, \mathbb{Z}/2\mathbb{Z}) \stackrel{\text{def}}{=} H^*(\text{Gal}(F_{\text{sep}}/F), \mathbb{Z}/2\mathbb{Z})$. For any field extension L/F, we set $H^*(L/F) = \ker\big(H^*(F) \to H^*(L)\big)$.

We use the standard canonical isomorphisms $H^0(F) = \mathbb{Z}/2\mathbb{Z}$, $H^1(F) = F^*/F^{*2}$, and $H^2(F) = \text{Br}_2(F)$. So any element $a \in F^*$ gives rise to an element of $H^1(F)$ which we denote by (a). The cup product $(a_1) \cup \dots \cup (a_n)$ we denote by $(a_1, \dots, a_n)$.

For $n = 0, 1, 2$, there is a homomorphism $e^n : I^n(F) \to H^n(F)$ defined as follows: $e^0(\phi) = \dim \phi \pmod 2$, $e^1(\phi) = d_\pm\, \phi$, and $e^2(\phi) = c(\phi)$. Moreover there exists a homomorphism $e^3 : I^3(F) \to H^3(F)$ which is uniquely determined by the condition $e^3(\langle\!\langle a_1, a_2, a_3\rangle\!\rangle) = (a_1, a_2, a_3)$ (see [1]). The homomorphism e^n is surjective and $\ker e^n = I^{n+1}(F)$ for $n = 0, 1, 2, 3$ (see [14], [17], and [22]).

We also work with the cohomology groups $H^n(F, \mathbb{Q}/\mathbb{Z}(i))$, $(i = 0, 1, 2)$, defined by B. Kahn (see [7]). For any field extension L/F, we set

$$H^*(L/F, \mathbb{Q}/\mathbb{Z}(i)) = \ker\big(H^*(F, \mathbb{Q}/\mathbb{Z}(i)) \to H^*(L, \mathbb{Q}/\mathbb{Z}(i))\big)\ .$$

For $n = 1, 2, 3$, the group $H^n(F)$ is naturally identified with the 2-part of $H^n(F, \mathbb{Q}/\mathbb{Z}(n-1))$.

***K*-theory and Chow groups.** For a smooth algebraic F-variety X, its Grothendieck ring is denoted by $K(X)$. This ring is supplied with the filtration by codimension of support (which respects the multiplication). For a ring (or a group) with filtration A, we denote by G^*A the adjoint graded ring (resp., the adjoint graded group). There is a canonical surjective homomorphism of the graded Chow ring $\text{CH}^*(X)$ onto $\text{G}^*K(X)$, its kernel consists only of torsion elements and is trivial in the 0-th, 1-st, and 2-nd graded components ([25, §9]).

2. The group $H^3(F(\rho_1, \rho_2)/F)$

The main result of this section (in view of our further purposes) is Corollary 2.13.

By a *homogeneous variety* we always mean a *projective* homogeneous variety.

Proposition 2.1 ([20])**.** *For any homogeneous F-variety X, there is a natural (in X and in F) epimorphism*

$$\tau_X\colon H^3\big(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)\big) \twoheadrightarrow \text{Tors}\,\text{CH}^2(X)\ .$$

Proposition 2.2. *For any homogeneous varieties $X_1, \dots, X_m$ over F, the quotient*

$$\frac{H^3\big(F(X_1 \times \dots \times X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)}{H^3\big(F(X_1)/F, \mathbb{Q}/\mathbb{Z}(2)\big) + \dots + H^3\big(F(X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)}$$

is canonically isomorphic to

$$\frac{\text{Tors}\,\text{CH}^2(X_1 \times \dots \times X_m)}{pr_1^*\,\text{Tors}\,\text{CH}^2(X_1) + \dots + pr_m^*\,\text{Tors}\,\text{CH}^2(X_m)}$$

where $pr_1^*, \dots, pr_m^*$ are the pull-backs with respect to the projections $pr_1, \dots, pr_m$ of the product $X_1 \times \dots \times X_m$ on $X_1, \dots, X_m$.

Proof. Set $X = X_1 \times \cdots \times X_m$. The homomorphism τ_X of Proposition 2.1 induces an epimorphism

$$f\colon \frac{H^3\big(F(X_1 \times \cdots \times X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)}{H^3\big(F(X_1)/F, \mathbb{Q}/\mathbb{Z}(2)\big) + \cdots + H^3\big(F(X_m)/F, \mathbb{Q}/\mathbb{Z}(2)\big)} \twoheadrightarrow$$
$$\twoheadrightarrow \frac{\operatorname{Tors}\operatorname{CH}^2(X_1 \times \cdots \times X_m)}{pr_1^* \operatorname{Tors}\operatorname{CH}^2(X_1) + \cdots + pr_m^* \operatorname{Tors}\operatorname{CH}^2(X_m)}$$

with the kernel $\ker f = \ker \tau_X / (\ker \tau_{X_1} + \cdots + \ker \tau_{X_m})$.

The kernel of τ_X is computed (for any homogeneous X) in [16]: let A be the separable F-algebra associated with X ([16, §2]) and denote by E the center of A; then $\ker \tau_X = \{N_{E/F}(\bar{x} \cup [A]) \mid \text{with } x \in E^*\}$ where $[A]$ is the class of A in the Brauer group $\operatorname{Br}(E) = H^2(E, \mathbb{Q}/\mathbb{Z}(1))$, $\bar{x}$ is the class of $x \in E^*$ in $H^1(E, \mathbb{Q}/\mathbb{Z}(1))$, $\bar{x} \cup [A] \in H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is the cup-product and $N_{E/F}$ is the norm map.

Denote by $A_1, \dots, A_m$ the separable algebras associated with $X_1, \dots, X_m$ respectively. Then $A = A_1 \times \cdots \times A_m$ and $E = E_1 \times \cdots \times E_m$. Thus for any $x \in E^*$

$$N_{E/F}(\bar{x} \cup [A]) = N_{E_1/F}(\bar{x}_1 \cup [A_1]) + \cdots + N_{E_m/F}(\bar{x}_m \cup [A_m]) \ ,$$

where x_i is the E_i-component of x, which proves that $\ker f = 0$. □

Corollary 2.3. *Let $X_1, \dots, X_m$ and $X'_1, \dots, X'_m$ be homogeneous varieties such that X_i is stably birationally equivalent to X'_i for $i = 1, \dots, m$. The quotient*

$$\frac{\operatorname{Tors}\operatorname{CH}^2(X_1 \times \cdots \times X_m)}{pr_1^* \operatorname{Tors}\operatorname{CH}^2(X_1) + \cdots + pr_m^* \operatorname{Tors}\operatorname{CH}^2(X_m)}$$

is isomorphic to the quotient

$$\frac{\operatorname{Tors}\operatorname{CH}^2(X'_1 \times \cdots \times X'_m)}{pr_1^* \operatorname{Tors}\operatorname{CH}^2(X'_1) + \cdots + pr_m^* \operatorname{Tors}\operatorname{CH}^2(X'_m)} \ .$$

□

Lemma 2.4. *For any homogeneous variety X of dimension ≤ 2, the group $\operatorname{CH}^2(X)$ is torsion-free.*

Proof. Since X is a homogeneous variety, $K(X)$ is a torsion-free group ([18]). Since $\dim X \leq 2$, the term $K(X)^{(3)}$ of the topological filtration is trivial. Hence $K(X)^{(2/3)}$ is a torsion-free group. By [25, §9], $\operatorname{CH}^2(X) \simeq K(X)^{(2/3)}$. Hence $\operatorname{Tors}\operatorname{CH}^2(X) = 0$. □

Corollary 2.5. *Under the conditions of Corollary 2.3 suppose additionally that the varieties $X_1, \dots, X_m; X'_1, \dots, X'_m$ have the dimensions ≤ 2. Then there is an isomorphism*

$$\operatorname{Tors}\operatorname{CH}^2(X_1 \times \cdots \times X_m) \simeq \operatorname{Tors}\operatorname{CH}^2(X'_1 \times \cdots \times X'_m).$$

Proof. Obvious in view of Corollary 2.3 and Lemma 2.4. □

Lemma 2.6. *Let X_1 and X_2 be homogeneous varieties. If the variety $(X_2)_{F(X_1)}$ has a rational point, then $H^3(F(X_1{\times}X_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X_1)/F, \mathbb{Q}/\mathbb{Z}(2))$.*

Proof. Since the homogeneous variety $(X_2)_{F(X_1)}$ has a rational point, it is rational, i.e. the field extension $F(X_1 \times X_2)/F(X_1)$ is purely transcendental. □

Corollary 2.7. *Let X_1 and X_2 be projective quadrics of the dimensions ≤ 2. If the quadric $(X_2)_{F(X_1)}$ is isotropic (e.g., if X_2 is isotropic or if $X_1 \simeq X_2$) then* $\operatorname{Tors}\operatorname{CH}^2(X_1 \times X_2) = 0$.

Proof. Follows from Lemma 2.6, Proposition 2.2 and Lemma 2.4. □

Lemma 2.8. *For any quadratic form ρ of dimension ≥ 3, we have*

$$2H^3(F(\rho)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0.$$

In other words, $H^3(F(\rho)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\rho)/F)$.

Proof. Let $u \in H^3(F(\rho)/F, \mathbb{Q}/\mathbb{Z}(2))$. There exists a field extension L/F such that ρ_L is isotropic and $[L : F] \leq 2$. Since ρ_L is isotropic, $u_L = 0$. Using the transfer homomorphism, we have $[L : F] \cdot u = 0$. Hence $2u = 0$. □

Corollary 2.9. *For any quadratic form ρ of dimension ≥ 3 the homomorphism $H^3(F(\rho)/F) \to \operatorname{Tors}\operatorname{CH}^2(X_\rho)$, induced by the epimorphism of Proposition 2.1, is surjective. In particular,* $2\operatorname{Tors}\operatorname{CH}^2(X_\rho) = 0$. □

Lemma 2.10. *Let ρ_1 and ρ_2 be quadratic form of dimension ≥ 3. Then*

$$2H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0.$$

In other words, $H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(\rho_1, \rho_2)/F)$.

Proof. Let ρ_1' and ρ_2' be 3-dimensional subforms in ρ_1 and ρ_2 respectively. Clearly $H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) \subset H^3(F(\rho_1', \rho_2')/F, \mathbb{Q}/\mathbb{Z}(2))$. Thus, replacing ρ_1 by ρ_1' and ρ_2 by ρ_2', one can reduce the proof to the case $\dim \rho_1 = \dim \rho_2 = 3$. In this case, $\dim X_{\rho_1} \times X_{\rho_2} = 2$; therefore $\operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} \times X_{\rho_2}) = 0$ (Lemma 2.4). For $i = 1, 2$, the conic X_{ρ_i} is isomorphic to the Severi-Brauer variety of the algebra $C_i \overset{\text{def}}{=} C_0(\rho_i)$. Applying [19, Thm. 4.1], we obtain an epimorphism

$$F^* \otimes U \twoheadrightarrow H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2))$$

where U is the subgroup of $\operatorname{Br}(F)$ generated by $[C_1]$ and $[C_2]$. Since $2[C_1] = 2[C_2] = 0$, it follows that $2H^3(F(\rho_1, \rho_2)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. □

Corollary 2.11. *Let ρ_1 and ρ_2 be quadratic forms of dimension ≥ 3. Then the homomorphism*

$$H^3(F(\rho_1, \rho_2)/F) \to \operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} {\times} X_{\rho_2})$$

induced by the epimorphism of Proposition 2.1, is surjective. In particular, $2\operatorname{Tors}\operatorname{CH}^2(X_{\rho_1} {\times} X_{\rho_2}) = 0$. □

Corollary 2.12. *For any quadratic forms ρ_1 and ρ_2 of dimension ≥ 3, there is a natural isomorphism*

$$\frac{H^3(F(\rho_1,\rho_2)/F)}{H^3(F(\rho_1)/F)+H^3(F(\rho_2)/F)} \simeq \frac{\operatorname{Tors}\operatorname{CH}^2(X_{\rho_1}\times X_{\rho_2})}{pr_1^*\operatorname{Tors}\operatorname{CH}^2(X_{\rho_1})+pr_2^*\operatorname{Tors}\operatorname{CH}^2(X_{\rho_2})}\,.$$

Proof. Follows from Proposition 2.2 and Lemmas 2.8 and 2.10. □

Corollary 2.13. *For any quadratic forms ρ_1 and ρ_2 with $3 \leq \dim\rho_i \leq 4$ $(i = 1, 2)$, there is a natural isomorphism*

$$\frac{H^3(F(\rho_1,\rho_2)/F)}{H^3(F(\rho_1)/F)+H^3(F(\rho_2)/F)} \simeq \operatorname{Tors}\operatorname{CH}^2(X_{\rho_1}\times X_{\rho_2}).$$

Proof. Follows from Corollary 2.12 and Lemma 2.4. □

3. The Grothendieck group of a quadric

In this section, ρ is an $(n + 2)$-dimensional quadratic form over F (where $n \geq 1$), V is the vector space of definition of ρ, $\mathbb{P}$ is the projective space of the vector space dual to V, and $X = X_\rho \subset \mathbb{P}$ is the n-dimensional projective quadric determined by ρ.

We are mainly interested in the case when $n = 2$, i.e. when X is a surface.

The even Clifford algebra $C_0(\rho)$ of the form ρ is denoted in this section by C. Let $\mathcal{U}$ be the Swan's sheaf on X [26, §6]. It is an $(C\otimes_F\mathcal{O}_X)$-module locally free as $\mathcal{O}_X$-module (note that the algebra C is canonically self-opposite; thus it is not necessary to distinguish between left and right action of C).

We denote by h the class of a general hyperplane section of X, i.e. the pull-back of the class of a hyperplane with respect to the imbedding $X \hookrightarrow \mathbb{P}$. The subring of $K(X)$ generated by h is denoted by H; it coincides with the image of the pull-back homomorphism $K(\mathbb{P}) \to K(X)$. Some further evident assertions on H are collected in

Lemma 3.1. *The abelian group H is freely generated by $1, h, \ldots, h^n$. The topological filtration on $K(X)$ induces on H the filtration by powers of h, i.e. for every $0 \leq r \leq n$, the term $H^{(r)}$ is generated by all h^j with $r \leq j \leq n$. In particular, the adjoint graded group G^*H is torsion-free.* □

In the case when X splits (i.e. when ρ is hyperbolic) and $n = 2$, a *line class* (resp., *point class*) refers to the class in $K(X)$ of a line (resp., of a closed rational point) lying on X.

Lemma 3.2 ([8]). *Suppose that X splits and* $\dim X = 2$.

1. *For any two different lines in X, their classes in $K(X)$ coincide if and only if the lines have no intersection. There are exactly two different line classes in $K(X)$.*
2. *The classes in $K(X)$ of any two closed rational points of X coincide, i.e. there is only one point class in $K(X)$.*

3. *Denote by l and l' the different line classes and by p the point class in $K(X)$. The abelian group $K(X)$ is freely generated by the elements $1, l, l', p$.*
4. *The second term $K(X)^{(2)}$ of the topological filtration on $K(X)$ is generated by p; the term $K(X)^{(1)}$ is generated by l, l', p.*
5. *The multiplication in $K(X)$ is determined by the formulas $l^2 = 0 = (l')^2$ and $l \cdot l' = p$.*
6. *$h = l + l' - p$.*

□

In the case when the quadric X is arbitrary (not necessary of dimension 2, not necessary split), we dispose of the following information on $K(X)$:

Lemma 3.3. 1. *The group $K(X)$ is torsion-free and, for any field extension E/F, the restriction homomorphism $K(X) \to K(X_E)$ is injective.*

2. *The class $[\mathcal{U}(n)] \in K(X)$ of the n times twisted Swan's sheaf equals*

$$2^n + 2^{n-1}h + \cdots + 2h^{n-1} + h^n \ .$$

3. *The homomorphism $\mathfrak{u} : K(C) \to K(X)$ given by the functor of taking tensor product $\mathcal{U}(n) \otimes_C (-)$ induces an epimorphism $K(C) \twoheadrightarrow K(X)/H$.*
4. *If C is a skewfield, then $K(X) = H$.*
5. *For any autoisometry ξ of the quadratic form ρ, the diagram*

$$\begin{array}{ccc} K(C) & \xrightarrow{\mathfrak{u}} & K(X) \\ \uparrow & & \uparrow \\ K(C) & \xrightarrow{\mathfrak{u}} & K(X) \end{array}$$

commutes, where the vertical maps are induced by the automorphisms of C and of X given by ξ.

Proof. 1. Follows from [26, Theorem 9.1].
2. See [9, Lemma 3.6].
3. According to [26, Theorem 9.1], the functor $\mathcal{U} \otimes_C (-)$ induces an epimorphism $K(C) \to K(X)/H$. Since for any $r \in \mathbb{Z}$ (and in particular for $r = n$) the twisting by r gives an automorphism of $K(X)/H$, the functor $\mathcal{U}(n) \otimes_C (-)$ induces an epimorphism as well.
4. If C is a skewfield, then the image of this epimorphism is generated by $[\mathcal{U}(n)]$. Since $[\mathcal{U}(n)] \in H$ by Item 2, it follows that $K(X) = H$.
5. It is evident in view of the way the sheaf $\mathcal{U}$ is constructed (see [26, §6]). □

Lemma 3.4 ([12])**.** *The F-algebra $C = C_0(\rho)$ has the dimension $2^{n+1} = 2^{\dim \rho - 1}$ over F. Its isomorphism class depends only on the similarity class of ρ. Moreover,*

- *if n is odd, then C is a central simple F-algebra;*
- *if n is even, then $C \simeq C_0(\rho') \otimes_F F(\sqrt{d_\pm \rho})$ where ρ' is an arbitrary 1-codimensional subform of ρ.*

In particular, if ρ is an even-dimensional form of trivial discriminant, the algebra C is the direct product of two isomorphic central simple algebras; any automorphism of C should either interchange or stabilize the factors.

Lemma 3.5. *Suppose that* $\dim\rho$ *is even and* $d_\pm\,\rho$ *is trivial. Let* ξ *be an autoisometry of the quadratic space* (V,ρ) *having the determinant* -1. *Then the automorphism of* C *induced by* ξ *interchanges the simple components of* C.

Proof. Since $d_\pm\,\rho$ is trivial, there exists a basis $v_0,\dots,v_{n+1}$ of V such that

$$(v_0\cdots v_{n+1})^2 = 1 \in C\ .$$

Since $\xi(v_0)\cdots\xi(v_{n+1}) = (\det\xi)\cdot(v_0\cdots v_{n+1}) = -v_0\cdots v_{n+1}$, the automorphism of C induced by ξ interchanges the elements

$$e = (1+v_0\cdots v_{n+1})/2 \quad\text{and}\quad e' = (1-v_0\cdots v_{n+1})/2\ .$$

Since e and e' are orthogonal idempotents, they lie in different simple components of C. Therefore, the components of C are interchanged. □

Comparing Lemma 3.2 with Lemma 3.3 in the situation of a split quadric surface X, we get the following computation (note that here C is isomorphic to $M_2(F)\times M_2(F)$ and thus there exist exactly two, up to isomorphisms, simple C-modules; their classes are free generators of $K(C)$):

Lemma 3.6. *Suppose that* X *splits and* $\dim X = 2$. *There exist simple* C-*modules* M *and* M' *such that* $u = 1 + l$ *and* $u' = 1 + l'$ *where*

$$u \overset{\text{def}}{=} \mathfrak{u}([M]) = [\mathcal{U}(2)\otimes_C M],\quad u' \overset{\text{def}}{=} \mathfrak{u}([M']) = [\mathcal{U}(2)\otimes_C M'] \in K(X)\ .$$

Proof. Take as M an arbitrary simple C-module and denote by M' a (determined uniquely up to an isomorphism) simple C-module non-isomorphic to M. Since by Lemma 3.2 the elements $1, l, l', p$ generate $K(X)$, we have

$$u = a + bl + b'l' + cp$$

for certain $a, b, b', c \in \mathbb{Z}$. Now we are going to show that

$$u' = a + b'l + bl' + cp\ .$$

Let ξ be an autoisometry of the quadratic space (V,ρ) having determinant -1. By Lemma 3.5, the induced by ξ automorphism of $K(C)$ interchanges $[M]$ and $[M']$. Thus, by Item 5 of Lemma 3.3, the induced by ξ automorphism of $K(X)$ interchanges u and u'.

Since ρ splits, there exist 2-dimensional totally isotropic subspaces W and W' of V with 1-dimensional intersection and an autoisometry ξ of (V,ρ) having the determinant -1 interchanging W and W'. The line classes in $K(X)$ determined by W and W' are different (Item 1 of Lemma 3.2); therefore they coincide with l and l' (or vice versa: with l' and l).

Thus, we have found an automorphism of $K(X)$ interchanging u with u' and l with l' while leaving untouched 1 (of course) and p (since all the point classes coincide). Thereby, $u' = a + b'l + bl' + cp$.

Since $2([M]+[M'])=[C]\in K(C)$, we have: $2(u+u')=[\mathcal{U}(2)]$, and so, $2(u+u')=4+2h+h^2$ by Item 2 of Lemma 3.3. Since $K(X)$ is torsion-free, the last equality can be divided by 2. Replacing h by $l+l'-p$ and h^2 by $(l+l'-p)^2=2p$ (see Lemma 3.2), we obtain that $u+u'=2+l+l'$. From the other hand, $u+u'=2a+(b+b')l+(b'+b)l'+2c$; therefore $a=1$, $b+b'=1$ and $c=0$.

We have proved that

$$u=1+bl+(1-b)l' \quad \text{and} \quad u'=1+(1-b)l+bl'$$

for certain $b\in\mathbb{Z}$. It remains to show that $b=1$ or $b=0$.

It follows from Item 3 of Lemma 3.3 that the elements $u,u',1,h,h^2$ generate the group $K(X)$. Since $h^2=2p$ and $h=u+u'-2-p$, the elements $u,u',1,p$ also generate $K(X)$. So, the quotient $K(X)/(\mathbb{Z}\cdot 1+\mathbb{Z}\cdot p)$ which is according to Item 6 of Lemma 3.2 freely generated by l,l' is also generated by u,u'. Thus, the $\mathbb{Z}$-matrix

$$\begin{pmatrix} b & 1-b \\ 1-b & b \end{pmatrix}$$

is invertible, i.e. its determinant is ±1. Hence, $b=1$ or $b=0$. □

4. The Grothendieck group of a product of quadrics

In this and in the next sections, we work with two quadratic forms ρ_1 and ρ_2 of the dimensions ≥ 3. We use the notation of the previous section amplified by the index 1 or 2. So, for $i=1,2$, we have ρ_i, n_i (we are mainly interested in the case when $n_1=2=n_2$), V_i, $\mathbb{P}_i$, X_i, C_i, $\mathcal{U}_i$,h_i, H_i, l_i, l'_i and p_i. We set $n=(n_1,n_2)$, $\mathbb{P}=\mathbb{P}_1\times\mathbb{P}_2$, $X=X_1\times X_2$, and $C=C_1\otimes_F C_2$.

For any $x_1\in K(X_1)$ and $x_2\in K(X_2)$, we denote by $x_1\boxtimes x_2$ the product $pr_1^*(x_1)\cdot pr_2^*(x_2)\in K(X)$ where pr_1 and pr_2 are the projections of $X_1\times X_2$ on X_1 and X_2 respectively.

Denote by H the image of the pull-back homomorphism $K(\mathbb{P})\to K(X)$.

Lemma 4.1. *One has: $H=H_1\boxtimes H_2\subset K(X)$. The abelian group H is freely generated by all $h_1^{j_1}\boxtimes h_2^{j_2}$ with $0\leq j_1\leq n_1$ and $0\leq j_2\leq n_2$. Moreover, the filtration on H induced by the topological filtration on $K(X)$ looks as follows: for any $0\leq r\leq n_1+n_2$, the term $H^{(r)}$ is generated by all $h_1^{j_1}\boxtimes h_2^{j_2}$ with $j_1+j_2\geq r$. In particular, the adjoint graded group G^*H is torsion-free.* □

The following lemma is also evident; together with Lemma 3.2, it gives a complete description of the ring with filtration $K(X)$ in the split situation.

Lemma 4.2. *If X_1 and X_2 split then the map $K(X_1)\otimes K(X_2)\to K(X)$, $x_1\otimes x_2\mapsto x_1\boxtimes x_2$ is an isomorphism of rings with filtrations.* □

For an $\mathcal{O}_{X_1}$-module $\mathcal{F}_1$ and an $\mathcal{O}_{X_2}$-module $\mathcal{F}_2$, we denote by $\mathcal{F}_1\boxtimes\mathcal{F}_2$ the tensor product $pr_1^*(\mathcal{F}_1)\otimes_{\mathcal{O}_X}pr_2^*(\mathcal{F}_2)$. The sheaf $\mathcal{U}=\mathcal{U}_1\boxtimes\mathcal{U}_2$ has for $i=1,2$ the structures of a C_i-module commuting with each other. Thus, it is a C-module. Set $\mathcal{U}(n)=\mathcal{U}_1(n_1)\boxtimes\mathcal{U}_2(n_2)$. It is also a C-module. The functor of taking the tensor product $\mathcal{U}(n)\otimes_C(-)$ determines a homomorphism $\mathfrak{u}\colon K(C)\to K(X)$.

Lemma 4.3. 1. *The group* $K(X)$ *is torsion-free and, for any field extension* E/F*, the restriction homomorphism* $K(X) \to K(X_E)$ *is injective.*

2. *The homomorphism* $\mathfrak{u}\colon K(C) \to K(X)$*, defined right above, induces an epimorphism* $K(C) \twoheadrightarrow K(X)/\big(K(X_1) \boxtimes K(X_2)\big)$.
3. *If* C *is a skewfield, then* $K(X) = H$.

Proof. 1. This statement is valid for any homogeneous variety X ([18]).
2. The isomorphism $K_*(X_1) \simeq K_*(F)^{\oplus n_1} \oplus K_*(C_1)$ of [26, Theorem 9.1] remains bijective after changing the base F to any field extension, i.e. for any field extension E/F, the homomorphism $K_*(\operatorname{Spec} E \times X_1) \to K_*(\operatorname{Spec} E)^{\oplus n_1} \oplus K_*(\operatorname{Spec} E, C_1)$ is bijective. Therefore, for any F-variety Y, the defined in the similar way homomorphism $K_*(Y \times X_1) \to K_*(Y)^{\oplus n_1} \oplus K_*(Y, C_1)$ is bijective (compare to the proof of Proposition 4.1 of [21, §7]). In particular, $K(X) \simeq K(X_2)^{\oplus n_1} \oplus K(X_2, C_1)$. Computing $K(X_2)$ and $K(X_2, C_1)$ using [26, Theorem 9.1] once again, one gets

$$K(X) \simeq K(F)^{\oplus n_1 n_2} \oplus K(C_1)^{\oplus n_2} \oplus K(C_2)^{\oplus n_1} \oplus K(C) \ .$$

The image of $K(F)^{\oplus n_1 n_2} \oplus K(C_1)^{\oplus n_2} \oplus K(C_2)^{\oplus n_1}$ in $K(X)$ is contained in $K(X_1) \boxtimes K(X_2)$ and the homomorphism $K(C) \to K(X)$ is induced by the functor of taking tensor product $\mathcal{U} \otimes_C (-)$. Thus $\mathfrak{u}\colon K(C) \to K(X)$ is modulo $K(X_1) \boxtimes K(X_2)$ an epimorphism.
3. If the algebra C is a skewfield then the image of $\mathfrak{u}$ is contained in H; moreover, the algebras C_1 and C_2 are skewfields as well and thus $K(X_i) = H_i$ for $i = 1, 2$. □

Corollary 4.4. *If* C *is a skewfield, then* $\mathrm{G}^* K(X)$ *is torsion-free. In particular,* $\operatorname{Tors} \mathrm{CH}^2(X) = 0$.

Proof. If C is a skewfield, then $K(X) = H$ by Item 3 of Lemma 4.3. Consequently, $\operatorname{Tors} \mathrm{G}^* K(X) = \operatorname{Tors} \mathrm{G}^* H = 0$ (see Lemma 4.1). □

5. CH^2 of a product of quadrics

The notation used in this section is introduced in the beginning of the previous one. However, each of the quadratic forms ρ_1 and ρ_2 is now supposed to have the dimension 3 or 4. So, each of X_i is either a quadric surface or a conic. We are mainly interested in the case when X_1 and X_2 are surfaces.

Theorem 5.1. *Suppose that* $\dim \rho_1 = 4 = \dim \rho_2$*, i.e. that* X_1 *and* X_2 *are surfaces. If* $\det \rho_1 = \det \rho_2$*, then* $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2) = 0$.

Proof. If one of the quadratic forms is isotropic, then $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2) = 0$ by Corollary 2.7. In the rest of the proof we assume that ρ_1 and ρ_2 are anisotropic.

As a next step, we are going to consider the case when $\det \rho_1 = \det \rho_2 = 1$.

Lemma 5.2. *Any projective quadric surface defined by a quadratic form of determinant 1 is stably birationally equivalent to a conic.*

Proof. Suppose that we are given a quadric determined by a 4-dimensional quadratic form ρ with $\det\rho = 1$. Take the conic determined by an arbitrary 3-dimensional subform $\rho' \subset \rho$. Since ρ' becomes isotropic over $F(\rho)$ and vice versa, ρ becomes isotropic over $F(\rho')$, the quadrics given by ρ' and ρ are stably birationally equivalent. $\square$

Suppose that $\det\rho_1 = \det\rho_2 = 1$ and choose some conics X'_1 and X'_2 stably birationally equivalent to X_1 and X_2 respectively. Applying Corollary 2.5, we get an isomorphism of $\operatorname{Tors}\operatorname{CH}^2(X_1 \times X_2)$ onto the group $\operatorname{Tors}\operatorname{CH}^2(X'_1 \times X'_2)$ which is trivial by Lemma 2.4.

Therefore, we may assume that $d \neq 1$ where $d = \det\rho_1 = \det\rho_2$.

As a next step of the proof of Theorem, we consider the case when the F-algebras $C_1 \overset{\text{def}}{=} C_0(\rho_1)$ and $C_2 \overset{\text{def}}{=} C_0(\rho_2)$ are isomorphic. In this case, the forms ρ_1 and ρ_2 becomes similar over the field $F(\sqrt{d})$. Thus by a theorem of Wadsworth ([27, Theorem 7]), they are already similar over F. Therefore the quadrics X_1 and X_2 are isomorphic and consequently $\operatorname{Tors}\operatorname{CH}^2(X) = 0$ by Corollary 2.7.

It remains only to consider the situation when the forms ρ_1 and ρ_2 are anisotropic, $d \neq 1$ and $C_1 \not\simeq C_2$. Set $c = \operatorname{ind} C$. We have: $c = 2$ or $c = 4$.

Fix a separable closure $\bar F$ of the field F. For the algebra $C_{\bar F}$, the variety $X_{\bar F}$, etc. we shall use the notation $\bar C$, $\bar X$, etc.

For $i = 1, 2$, denote by M_i and M'_i the (determined uniquely up to an isomorphism and up to the order) non-isomorphic simple $\bar C_i$-modules. There are exactly 4 different isomorphism classes of simple C-modules; they are represented by $M_1 \boxtimes M_2$ ($M_1 \boxtimes M_2$ is by definition the tensor product $M_1 \otimes M_2$ considered as $\bar C$-module in the natural way), $M_1 \boxtimes M'_2$, $M'_1 \boxtimes M_2$, and $M'_1 \boxtimes M'_2$. Denote by m_i the class of M_i and by m'_i the class of M'_i in $K(\bar C_i)$. The abelian group $K(\bar C)$ is freely generated by $m_1 \boxtimes m_2$ ($m_1 \boxtimes m_2$ is defined as follows: for $i = 1, 2$, one takes the image of $m_i \in K(C_i)$ with respect to the map $K(C_i) \to K(C)$ and than takes the product of the images in the ring $K(C)$), $m_1 \boxtimes m'_2$, $m'_1 \boxtimes m_2$, and $m'_1 \boxtimes m'_2$. We identify $K(C)$ with a subgroup in $K(\bar C)$ via the restriction map $K(C) \hookrightarrow K(\bar C)$.

Lemma 5.3. *The subgroup $K(C) \subset K(\bar C)$ is generated by*

$$c \cdot (m_1 \boxtimes m_2 + m'_1 \boxtimes m'_2) \quad \text{and} \quad c \cdot (m_1 \boxtimes m'_2 + m'_1 \boxtimes m_2)\ .$$

Proof. Denote by L the quadratic extension $F(\sqrt{d})$ of the field F, where $d = \det\rho_1 = \det\rho_2$. The algebra C_L is the direct product of 4 copies of a central simple L-algebra of index c. Evidently, the subgroup $K(C_L)$ of $K(\bar C)$ is freely generated by $c \cdot m_1 \boxtimes m_2$, $c \cdot m_1 \boxtimes m'_2$, $c \cdot m'_1 \boxtimes m_2$, and $c \cdot m'_1 \boxtimes m'_2$.

Now we are going to determine $K(C)$ as a subgroup in $K(C_L)$. Computing the norm $N_{L/F} \colon K(C_L) \to K(C)$, we get:

$$x \overset{\text{def}}{=} N_{L/F}(c \cdot m_1 \boxtimes m_2) = c \cdot (m_1 \boxtimes m_2 + m'_1 \boxtimes m'_2)\ ;$$

$$x' \overset{\text{def}}{=} N_{L/F}(c \cdot m_1 \boxtimes m'_2) = c \cdot (m_1 \boxtimes m'_2 + m'_1 \boxtimes m_2)\ .$$

Thus, the elements x and x' are in $K(C)$. Note that:

- x and x' can be included in a system of free generators of the free abelian group $K(C_L)$ (e.g. x, x', $c \cdot m_1 \boxtimes m_2$, and $c \cdot m_1 \boxtimes m_2'$);
- $K(C)$ is a free abelian group of rank 2 (because the algebra C is the direct product of two copies of a simple algebra, since for $i = 1, 2$ one has: $C_i = C_i' \otimes_F L$ for a central simple F-algebra C_i');
- $K(C)$ is a subgroup of $K(C_L)$ containing x and x'.

Consequently, $K(C)$ is generated by x and x'. □

We identify $K(X)$ with a subgroup in $K(\bar{X})$ via the restriction map $K(X) \hookrightarrow K(\bar{X})$ (which is injective by Item 1 of Lemma 4.3). For $i = 1, 2$, let l_i, l_i' be the different line classes and p_i the point class in $K(\bar{X}_i)$ (see Lemma 3.2).

Corollary 5.4. *The group $K(X)$ is generated modulo H by $c \cdot (l_1 \boxtimes l_2 + l_1' \boxtimes l_2')$ and $c \cdot p_1 \boxtimes p_2$.*

Proof. According to Item 2 of Lemma 4.3, the map $\mathfrak{u}\colon K(C) \to K(X)/H$ is surjective. By Lemma 5.3, the group $K(C)$ is generated by

$$c \cdot (m_1 \boxtimes m_2 + m_1' \boxtimes m_2') \quad \text{and} \quad c \cdot (m_1 \boxtimes m_2' + m_1' \boxtimes m_2)\ .$$

Applying Lemma 3.6, we can compute the images of these generators in $K(X)$: up to the order, they are

$$c \cdot \big((1 + l_1) \boxtimes (1 + l_2) + (1 + l_1') \boxtimes (1 + l_2')\big) \quad \text{and}$$
$$c \cdot \big((1 + l_1) \boxtimes (1 + l_2') + (1 + l_1') \boxtimes (1 + l_2)\big)\ .$$

One can modify the first expression as follows (the formulas of Lemma 3.2 are in use):

$$\begin{aligned} &c \cdot \big((1 + l_1) \boxtimes (1 + l_2) + (1 + l_1') \boxtimes (1 + l_2')\big) = \\ &\quad = c \cdot \big(2 + (l_1 + l_1') \boxtimes 1 + 1 \boxtimes (l_2 + l_2') + l_1 \boxtimes l_2 + l_1' \boxtimes l_2'\big) = \\ &\quad = c \cdot \big(2 + (h_1 + h_1^2/2) \boxtimes 1 + 1 \boxtimes (h_2 + h_2^2/2) + l_1 \boxtimes l_2 + l_1' \boxtimes l_2'\big) \equiv \\ &\quad \equiv c \cdot (l_1 \boxtimes l_2 + l_1' \boxtimes l_2') \pmod{H} \end{aligned}$$

(note that c is divisible by 2). The analogous modification can be made for the second expression as well. Thus, the group $K(X)$ is generated modulo H by $c \cdot (l_1 \boxtimes l_2 + l_1' \boxtimes l_2')$ and $c \cdot (l_1 \boxtimes l_2' + l_1' \boxtimes l_2)$. Taking the sum of these generators, we get:

$$\begin{aligned} &c \cdot (l_1 \boxtimes l_2 + l_1' \boxtimes l_2') + c \cdot (l_1 \boxtimes l_2' + l_1' \boxtimes l_2) = \\ &\quad = c \cdot (l_1 + l_1') \boxtimes (l_2 + l_2') = c \cdot (h_1 + h_1^2/2) \boxtimes (h_2 + h_2^2/2) \equiv \\ &\quad \equiv c \cdot (h_1^2/2) \boxtimes (h_2^2/2) = c \cdot p_1 \boxtimes p_2 \end{aligned}$$

(where the congruence is modulo H). □

Lemma 5.5. 1. *$c \cdot (l_1 \boxtimes l_2 + l_1' \boxtimes l_2') \in K(X)^{(2)}$;*
2. *$c \cdot p_1 \boxtimes p_2 \in K(X)^{(3)}$;*

3. *for any* $0 \neq r \in \mathbb{Z}$, *the set* $r \cdot c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) + H$ *has no intersection with* $K(X)^{(3)}$.

Proof. 1. It is evident that $c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \in K(\bar{X})^{(2)}$. Since $K(X)^{(2)} = K(\bar{X})^{(2)} \cap K(X)$ (see e.g. [23, Lemme 6.3, (i)]), we are done.
2. If we multiply the element $c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \in K(X)^{(2)}$ by the element $h_1 \boxtimes 1 \in K(X)^{(1)}$, we get:

$$K(X)^{(3)} \ni c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \cdot (h_1 \boxtimes 1) = \\ = c(p_1 \boxtimes l_2 + p_1 \boxtimes l'_2) = c \cdot p_1 \boxtimes (h_2 + p_2) = \\ = c \cdot p_1 \boxtimes h_2 + c \cdot p_1 \boxtimes p_2 \ .$$

Since $c \cdot p_1 \boxtimes h_2 \in H^{(3)} \in K(X)^{(3)}$, it follows that $c \cdot p_1 \boxtimes p_2 \in K(X)^{(3)}$.
3. By Lemmas 3.2 and 4.2, the abelian group $K(\bar{X})$ is freely generated by the products $x_1 \boxtimes x_2$ where x_i is one of the elements $1, l_i, l'_i, p_i$; moreover, the term $K(\bar{X})^{(3)}$ of the filtration is generated by $l_1 \boxtimes p_2$, $l'_1 \boxtimes p_2$, $p_1 \boxtimes l_2$, $p_1 \boxtimes l'_2$ and $p_1 \boxtimes p_2$. In particular, $4K(\bar{X})^{(3)} \subset H$.

Suppose that, for certain $0 \neq r \in \mathbb{Z}$, the intersection of $r \cdot c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) + H$ with $K(X)^{(3)}$ is non-empty. Then $4r \cdot c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2) \in H$, a contradiction. □

Corollary 5.6. *Let us supply the quotient* $K(X)/H$ *with the filtration induced from* $K(X)$. *Then* $\operatorname{Tors} \mathrm{G}^2(K(X)/H) = 0$.

Proof. By Corollary 5.4 and Lemma 5.5, $\mathrm{G}^2(K(X)/H)$ is an infinite cyclic group (generated by the residue of $c(l_1 \boxtimes l_2 + l'_1 \boxtimes l'_2)$). □

To finish the proof of Theorem 5.1, consider the exact sequence

$$0 \to \mathrm{G}^2 H \to \mathrm{G}^2 K(X) \to \mathrm{G}^2(K(X)/H) \to 0 \ .$$

The left-hand side term is torsion-free by Lemma 4.1 while the right-hand side term is torsion-free by Corollary 5.6. Consequently, the middle term is a torsion-free group as well. □

Theorem 5.7. *The order of the group* $\operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2)$ *is at most* 2.

Proof. Since $2 \operatorname{Tors} \mathrm{CH}^2(X_1 \times X_2) = 0$ by Corollary 2.11, it suffices to show that the torsion in $\mathrm{CH}^2(X_1 \times X_2)$ is a cyclic group.

By Corollary 2.7, it suffices to consider only the case when the both quadratic forms ρ_1 and ρ_2 are anisotropic.

Set as usual $X = X_1 \times X_2$, $C_i = C_0(\rho_i)$ and $C = C_1 \otimes_F C_2$. Suppose that the algebra C is simple. Then $K(C)$ is a cyclic group and therefore, by Item 2 of Lemma 4.3, the quotient $K(X)/H$ is cyclic as well. Moreover, C_1 and C_2 are division algebras (since they are simple and the quadratic forms are anisotropic) and therefore $K(X_i) = H_i$ for $i = 1, 2$ by Item 4 of Lemma 3.3. Supplying $K(X)/H$ with the filtration induced from $K(X)$, we get an exact sequence of the adjoint graded groups

$$0 \to \mathrm{G}^* H \to \mathrm{G}^* K(X) \to \mathrm{G}^*(K(X)/H) \to 0 \ .$$

Take any $r \geq 0$. Since $\mathrm{G}^r H$ is torsion-free (Lemma 4.1), $\mathrm{Tors}\,\mathrm{G}^r K(X)$ is mapped injectively into $G^r(K(X)/H)$. Since $K(X)/H$ is cyclic, $G^r(K(X)/H)$ is cyclic as well and thus so is also $\mathrm{Tors}\,\mathrm{G}^r K(X)$. In particular, the group $\mathrm{Tors}\,\mathrm{CH}^2(X) \simeq \mathrm{Tors}\,\mathrm{G}^2 K(X)$ is cyclic.

Now suppose that C is *not* simple. Then

either: $\dim X_1 = 2 = \dim X_2$ and $\det X_1 = \det X_2$,

or: for $i = 1$ or for $i = 2$, one has: $\dim X_i = 2$ and $\det X_i = 1$.

In the first case, the torsion in $\mathrm{CH}^2(X)$ is 0 by Theorem 5.1. In the second case, we replace the surface X_i by a stably birationally equivalent conic (see Lemma 5.2 and Corollary 2.5). □

Theorem 5.8. *If* $\mathrm{ind}\, C_0(\rho_1) \otimes_F C_0(\rho_2) = 4$*, then* $\mathrm{Tors}\,\mathrm{CH}^2(X_1 \times X_2) = 0$.

Proof. We set $C = C_0(\rho_1) \otimes_F C_0(\rho_2)$ and suppose that $\mathrm{ind}\, C = 4$.

If C is a simple algebra, then it is a skewfield and we are done by Corollary 4.4.

If C is *not* simple, then

either: $\dim X_1 = 2 = \dim X_2$ and $\det \rho_1 = \det \rho_2$,

or: for $i = 1$ or for $i = 2$, one has: $\dim X_i = 2$ and $\det X_i = 1$.

In the first case, the torsion in $\mathrm{CH}^2(X_1 \times X_2)$ is 0 by Theorem 5.1. In the second case, we replace the surface X_i by a stably birationally equivalent conic (see Lemma 5.2 and Corollary 2.5). □

Theorem 5.9. *Suppose that* $\dim \rho_1 = 4$*,* $\det \rho_1 \neq 1$ *and that for a certain 3-dimensional subform* ρ_1' *of* ρ_1 *one has:*

$$\mathrm{ind}\, C_0(\rho_1) \otimes_F C_0(\rho_2) = \mathrm{ind}\, C_0(\rho_1') \otimes_F C_0(\rho_2)\ .$$

Then $\mathrm{Tors}\,\mathrm{CH}^2(X_1 \times X_2) = 0$.

Proof. Applying the same arguments as above, we may assume that

- the forms ρ_1 and ρ_2 are anisotropic and
- one of the following alternative conditions holds:
  - the dimension of ρ_2 equals 3 or
  - the dimension of ρ_2 is 4 and $\det \rho_1 \neq \det \rho_2 \neq 1$.

We are going to show that, under the assumptions made, $\mathrm{Tors}\,\mathrm{G}^2 K(X_1 \times X_2) = 0$.

The algebra C is now simple; it has the index 1, 2, or 4. Set $c = \mathrm{ind}\, C$. The group $K(C)$ is generated by $(c/4) \cdot [C]$ where $[C] \in K(C)$ is the class of C.

Consider the case when $\dim \rho_2 = 4$.

It follows from Item 2 of Lemma 4.3 that $K(X)$ is generated modulo H by the element $(c/4)[\mathcal{U}(2,2)]$. Applying Item 2 of Lemma 3.3, one computes that $[\mathcal{U}(2,2)] = (4 + 2h_1 + h_1^2) \boxtimes (4 + 2h_2 + h_2^2) \in K(X)$. Thus, $K(X)$ is generated modulo H also by $x \stackrel{\mathrm{def}}{=} (c/4)(2 \cdot h_1 \boxtimes h_2^2 + 2 \cdot h_1^2 \boxtimes h_2 + h_1^2 \boxtimes h_2^2)$. Since we have the exact sequence

$$0 \to \mathrm{G}^* H \to \mathrm{G}^* K(X) \to \mathrm{G}^*(K(X)/H) \to 0$$

with torsion-free G^*H, it would suffice to show that $x \in K(X)^{(3)}$.

Consider the conic X_1' determined by ρ_1' and denote by $\mathcal{U}_1'$ the Swan's sheaf on X_1'. The product $\mathcal{U}_1'(1)\boxtimes\mathcal{U}_2(2)$ of the twisted Swan's sheaves has a structure of module over $C' \stackrel{\text{def}}{=} C_1' \otimes C_2$; its class in $K(X')$, where $X' \stackrel{\text{def}}{=} X_1' \times X_2$ is equal to $(2 + h_1') \boxtimes (4 + 2h_2 + h_2^2)$ where h_1' is the class in $K(X_1')$ of a hyperplane section of X_1'. Since $\operatorname{ind} C' = \operatorname{ind} C = c$, the latter product can be divided by $(4/c)$ in $K(X')$, i.e.

$$K(X') \ni x' \stackrel{\text{def}}{=} (c/4)(2 \cdot 1 \boxtimes h_2^2 + 2 \cdot h_1' \boxtimes h_2 + h_1' \boxtimes h_2^2) .$$

Since $4x' \in K(X')^{(2)}$ and the group $G^1K(X') = CH^1(X')$ is torsion-free (see e.g. [23, Lemme 6.3, (i)]), it follows that $x' \in K(X')^{(2)}$. Since the image of x' with respect to the push-forward given by the closed imbedding $X' \hookrightarrow X$ coincides with x and $\operatorname{codim}_X X' = 1$, the element x is in $K(X)^{(3)}$.

Now suppose that $\dim \rho_2 = 3$.

If $c = 1$, then the quadric $(X_2)_{F(X_1)}$ is isotropic and therefore $\operatorname{Tors} CH^2(X) = 0$ by Corollary 2.7. Thus we may assume that c is divisible by 2.

The group $K(X)$ is now generated modulo H by $(c/4)[\mathcal{U}(2,1)]$ and $[\mathcal{U}(2,1)] = (4 + 2h_1 + h_1^2) \boxtimes (2 + h_2) \in K(X)$. Thus, $K(X)$ is generated modulo H also by $x \stackrel{\text{def}}{=} (c/4)(h_1^2 \boxtimes h_2)$ and it suffices to show that $x \in K(X)^{(3)}$.

The class in $K(X')$ of the product $\mathcal{U}_1'(1)\boxtimes\mathcal{U}_2(1)$ of the twisted Swan's sheaves is equal this time to $(2+h_1')\boxtimes(2+h_2)$ and can be divided by $(4/c)$ in $K(X')$, i.e.

$$K(X') \ni x' \stackrel{\text{def}}{=} (c/4)(h_1' \boxtimes h_2) .$$

Since $x' \in K(X')^{(2)}$ and the image of x' with respect to the push-forward given by the closed imbedding $X' \hookrightarrow X$ coincides with x, the element x is in $K(X)^{(3)}$. □

Corollary 5.10. *If ρ_1 and ρ_2 contain similar 3-dimensional subforms, then* $\operatorname{Tors} CH^2(X_1 \times X_2) = 0$.

Proof. If $\dim \rho_1 = 3$ or if $\det \rho_1 = 1$, then the quadric $(X_2)_{F(X_1)}$ is isotropic and so we are done by Corollary 2.7.

Therefore, we may assume that $\dim \rho_1 = 4$ and $\det \rho_1 \neq 1$. These are the first two conditions of Theorem 5.9. We state that also the last condition of Theorem 5.9 is satisfied. Indeed, denote by $\rho_1' \subset \rho_1$ and $\rho_2' \subset \rho_2$ the similar 3-dimensional subforms. According to Lemma 3.4, the F-algebras $C_0(\rho_1')$ and $C_0(\rho_2')$ are isomorphic and $C_0(\rho_i) = C_0(\rho_i')_{F(\sqrt{\det \rho_i})}$ for $i = 1, 2$. Therefore, $\operatorname{ind} C_0(\rho_1) \otimes_F C_0(\rho_2) = 1 = \operatorname{ind} C_0(\rho_1') \otimes_F C_0(\rho_2)$. □

6. The group $I^3(F(\rho,\psi)/F)$

The following assertion is obvious:

Lemma 6.1. *Let $\rho = \langle -a, -b, ab, d\rangle$ be a quadratic form over F. For any $k \in F^*$ the following conditions are equivalent.*

(1) *$k \in D_F(\langle\langle d\rangle\rangle)$;*

(2) $\langle\langle a,b,k\rangle\rangle = \rho\,\langle\langle k\rangle\rangle$;
(3) $\rho\,\langle\langle k\rangle\rangle \in P_3(F)$. □

Lemma 6.2. *Let $\rho = \langle -a,-b,ab,d\rangle$ be a quadratic form over F. Then*
1. $P_3(F(\rho)/F) = \{\langle\langle a,b,k\rangle\rangle \mid k \in D_F(\langle\langle d\rangle\rangle)\}$,
2. $H^3(F(\rho)/F) = \{(a,b,k) \mid k \in D_F(\langle\langle d\rangle\rangle)\}$.

Proof. 1. See [3, Lemma 3.1].
2. Let $\rho_0 = \langle -a,-b,ab\rangle$. Clearly $H^3(F(\rho)/F) \subset H^3(F(\rho_0)/F)$. It follows from [1, Beweis vom Satz 5.6] that $H^3(F(\rho_0)/F) = (a,b)\cup H^1(F)$. Hence any element $u \in H^3(F(\rho)/F)$ has the form (a,b,x) where $x \in F^*$. Since $(a,b,x) \in H^3(F(\rho)/F)$, the Pfister form $\langle\langle a,b,x\rangle\rangle_{F(\rho)}$ is hyperbolic. It follows from the first assertion that there exists $k \in D_F(\langle\langle d\rangle\rangle)$ such that $\langle\langle a,b,x\rangle\rangle = \langle\langle a,b,k\rangle\rangle$. Hence $u = (a,b,x) = (a,b,k)$. □

Corollary 6.3. *Let $\rho_1,\dots,\rho_m$ be 4-dimensional quadratic forms over F. Then for a quadratic form ϕ the following conditions are equivalent:*
(1) $\phi \in I^3(F(\rho_1)/F) + \dots + I^3(F(\rho_m)/F) + I^4(F)$;
(2) $\phi \in P_3(F(\rho_1)/F) + \dots + P_3(F(\rho_m)/F) + I^4(F)$;
(3) $\phi \in I^3(F)$ *and* $e^3(\phi) \in H^3(F(\rho_1)/F) + \dots + H^3(F(\rho_m)/F)$.

Proof. (2)⇒(1)⇒(3). Obvious.
(3)⇒(2). Follows from Lemma 6.2. □

Corollary 6.4. *Let $\rho_1,\dots,\rho_m$ be 4-dimensional quadratic forms such that $H^3(F(\rho_1,\dots,\rho_m)/F) = H^3(F(\rho_1)/F) + \dots + H^3(F(\rho_m)/F)$. Then*

$$I^3(F(\rho_1,\dots,\rho_m)/F) \subset I^3(F(\rho_1)/F) + \dots + I^3(F(\rho_m)/F) + I^4(F).$$

□

Corollary 6.5. *Let $\rho = \langle -a,-b,ab,d\rangle$ and $\psi = \langle -u,-v,uv,\delta\rangle$ be quadratic forms over F. Then for any $\pi \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$ there exist $k_1,k_2 \in F^*$ with the following properties:*
1) $\langle\langle a,b,k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$ *and* $\langle\langle u,v,k_2\rangle\rangle = \psi\,\langle\langle k_2\rangle\rangle$;
2) $\pi \equiv \langle\langle a,b,k_1\rangle\rangle + \langle\langle u,v,k_2\rangle\rangle \pmod{I^4(F)}$.

Proof. By Corollary 6.3, we have $\pi \in P_3(F(\rho)/F) + P_3(F(\psi)/F) + I^4(F)$. Hence there exist $\pi_1 \in P_3(F(\rho)/F)$ and $\pi_2 \in P_3(F(\psi)/F)$ such that

$$\pi \equiv \pi_1 + \pi_2 \pmod{I^4(F)}\ .$$

By Lemma 6.2, there exist $k_1,k_2 \in F^*$ such that $\pi_1 = \langle\langle a,b,k_1\rangle\rangle$ and $\pi_2 = \langle\langle u,v,k_2\rangle\rangle$. Finally, Lemma 6.1 shows that $\langle\langle a,b,k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$, $\langle\langle u,v,k_2\rangle\rangle = \psi\,\langle\langle k_2\rangle\rangle$. □

7. The case of index 1

In this section, we study the group $H^3(F(\rho,\psi)/F)$ in the case where ρ, ψ are 4-dimensional quadratic forms with non-trivial discriminants and $\operatorname{ind} C_0(\rho)\otimes_F C_0(\psi) = 1$. In the case $d_\pm\,\rho = d_\pm\,\psi$ we obviously have $C_0(\rho) \simeq C_0(\psi)$. Hence

ρ is similar to ψ (see [27, Theorem 7]) and hence the group $H^3(F(\rho,\psi)/F)$ coincides with $H^3(F(\rho)/F)$. So it is sufficient to study only the case where $d_\pm\,\rho \neq d_\pm\,\psi$.

Replacing ρ and ψ by similar forms, we can rewrite our conditions as follows:

1) $\rho = \langle -a, -b, ab, d\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ with $a, b, d, u, v, \delta \in F^*$;
2) d, δ, and $d\delta$ are not squares in F^*;
3) $\operatorname{ind}((a,b)\otimes_F (u,v))_{F(\sqrt{d},\sqrt{\delta})} = 1$.

During this section we will suppose that the conditions 1)–3) hold.

We define the set $\Gamma(\rho,\psi)$ as

$$\{\gamma \in I^3(F) \mid \text{there exist } l_1, l_2 \in F^* \text{ such that } \gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle\}.$$

Lemma 7.1. *The set $\Gamma(\rho,\psi)$ is not empty.*

Proof. Since $\operatorname{ind}((a,b)\otimes_F (u,v))_{F(\sqrt{d},\sqrt{\delta})} = 1$, there exist $s, r \in F^*$ such that $(a,b)\otimes(u,v) = (d,s)\otimes(\delta,r)$. Set $l_1 = \delta s$, $l_2 = -\delta r$. It is sufficient to verify that $\gamma \overset{\text{def}}{=} l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \in I^3(F)$. We have

$$\begin{aligned}\gamma &= \delta s\rho - \delta r\psi + \langle 1, -d\delta\rangle = \delta(s\rho - r\psi + \langle \delta, -d\rangle) = \\ &= \delta(s(\langle\langle a,b\rangle\rangle - \langle\langle d\rangle\rangle) - r(\langle\langle u,v\rangle\rangle - \langle\langle \delta\rangle\rangle) + (\langle\langle d\rangle\rangle - \langle\langle \delta\rangle\rangle)) = \\ &= \delta(s\,\langle\langle a,b\rangle\rangle - r\,\langle\langle u,v\rangle\rangle + \langle\langle d,s\rangle\rangle - \langle\langle \delta,r\rangle\rangle)\ .\end{aligned}$$

Therefore $\gamma \in I^2(F)$ and $c(\gamma) = (a,b) + (u,v) + (d,s) + (\delta,r) = 0$. Hence $\gamma \in I^3(F)$. ☐

Lemma 7.2. $\Gamma(\rho,\psi) \subset I^3(F(\rho,\psi)/F)$.

Proof. Let $\gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \in \Gamma(\rho,\psi)$. We have $\dim(\gamma_{F(\psi,\rho)})_{an} \le \dim(\rho_{F(\rho)})_{an} + \dim(\psi_{F(\psi)})_{an} + \dim\langle\langle \delta d\rangle\rangle \le 2+2+2 = 6 < 8$. Since $\gamma \in I^3(F)$, the Arason-Pfister Hauptsatz shows that $\gamma_{F(\psi,\rho)}$ is hyperbolic. ☐

Corollary 7.3. *For any $\gamma \in \Gamma(\rho,\psi)$, we have $e^3(\gamma) \in H^3(F(\rho,\psi)/F)$.* ☐

Lemma 7.4. *Let $l, k \in F^*$ and let τ be a quadratic form such that $\tau\,\langle\langle k\rangle\rangle \in I^3(F)$. Then $l\tau - \langle\langle k\rangle\rangle\,\tau \equiv lk\tau \pmod{I^4(F)}$.*

Proof. $l\tau - \langle\langle k\rangle\rangle\,\tau - lk\tau = -\langle\langle l\rangle\rangle\,\langle\langle k\rangle\rangle\,\tau \in \langle\langle l\rangle\rangle\, I^3(F) \subset I^4(F)$. ☐

Lemma 7.5. *Let $\gamma \in \Gamma(\rho,\psi)$, $\pi_1 \in P_3(F(\rho)/F)$ and $\pi_2 \in P_3(F(\psi)/F)$. Then there exists $\gamma' \in \Gamma(\rho,\psi)$ such that $\gamma - \pi_1 - \pi_2 \equiv \gamma' \pmod{I^4(F)}$. Moreover, $\gamma + \pi_1 + \pi_2 \equiv \gamma' \pmod{I^4(F)}$.*

Proof. Let $l_1, l_2 \in F^*$ be such that $\gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle$. By Lemmas 6.1 and 6.2, there exist $k_1, k_2 \in F^*$ such that $\pi_1 = \rho\,\langle\langle k_1\rangle\rangle$, $\pi_2 = \psi\,\langle\langle k_2\rangle\rangle$. By Lemma 7.4, we have

$$l_1\rho - \pi_1 = l_1\rho - \langle\langle k_1\rangle\rangle\,\rho \equiv l_1k_1\rho \pmod{I^4(F)},$$
$$l_2\psi - \pi_2 = l_2\psi - \langle\langle k_2\rangle\rangle\,\psi \equiv l_2k_2\psi \pmod{I^4(F)}.$$

Hence $\gamma - \pi_1 - \pi_2 \equiv l_1k_1\rho + l_2k_2\psi + \langle\langle d\delta\rangle\rangle \pmod{I^4(F)}$. Setting $\gamma' = l_1k_1\rho + l_2k_2\psi + \langle\langle d\delta\rangle\rangle$, we get the required equation $\gamma - \pi_1 - \pi_2 \equiv \gamma' \pmod{I^4(F)}$.

The second equation $\gamma + \pi_1 + \pi_2 \equiv \gamma'$ (mod $I^4(F)$) is obvious in view of the congruence $\pi_i \equiv -\pi_i$ (mod $I^4(F)$) (for $i = 1, 2$). □

Corollary 7.6. $\Gamma(\rho,\psi)+I^3(F(\rho)/F)+I^3(F(\psi)/F)+I^4(F) = \Gamma(\rho,\psi)+I^4(F)$.

Proof. It is an obvious consequence of Corollary 6.3 and Lemma 7.5 □

Lemma 7.7. *The following conditions are equivalent:*

1. $I^3(F(\rho,\psi)/F) \subset I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$;
2. $\Gamma(\rho,\psi) \subset I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$;
3. *there exists* $\gamma \in \Gamma(\rho,\psi)$ *such that* $\gamma \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$;
4. $\Gamma(\rho,\psi)$ *contains a hyperbolic form, i.e.* $0 \in \Gamma(\rho,\psi)$;
5. *the quadratic forms* ψ *and* ρ *contain similar* 3*-dimensional subforms;*
6. $\operatorname{Tors}\operatorname{CH}^2(X_\rho \times X_\psi) = 0$;
7. $H^3(F(\rho,\psi)/F) = H^3(F(\rho)/F) + H^3(F(\psi)/F)$.

Proof. (1)⇒(2). Obvious in view of Lemma 7.2.
(2)⇒(3). Obvious in view of Lemma 7.1.
(3)⇒(4). Let γ be such as in (3). By Corollary 6.3, there exist $\pi_1 \in P_3(F(\rho)/F)$ and $\pi_2 \in P_3(F(\psi)/F)$ such that $\gamma \in \pi_1+\pi_2+I^4(F)$. Hence $\gamma-\pi_1-\pi_2 \in I^4(F)$. By Lemma 7.5, there exists $\gamma' \in \Gamma(\rho,\psi)$ such that $\gamma-\pi_1-\pi_2 \equiv \gamma'$ (mod $I^4(F)$). Since $\gamma - \pi_1 - \pi_2 \in I^4(F)$, we have $\gamma' \in I^4(F)$. By definition of $\Gamma(\rho,\psi)$, $\dim(\gamma')_{an} \le 4 + 4 + 2 = 10 < 16$. Since $\gamma' \in I^4(F)$, the Arason-Pfister Hauptsatz shows that $\gamma' = 0$.
(4)⇒(5). Since $0 \in \Gamma(\rho,\psi)$, there exist $l_1, l_2 \in F^*$ such that $0 = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle$. Thus $l_1\rho+l_2\psi = -\langle\langle d\delta\rangle\rangle$. Hence $l_1\rho$ and $l_2\psi$ contain a common subform of the dimension $(\dim(\rho) + \dim(\psi) - \dim\langle\langle d\delta\rangle\rangle)/2 = (4 + 4 - 2)/2 = 3$.
(5)⇒(6). See Corollary 5.10.
(6)⇒(7). See Corollary 2.13.
(7)⇒(1). It is a particular case of Corollary 6.4. □

Proposition 7.8. *For an arbitrary element* $\gamma \in \Gamma(\rho,\psi)$, *one has*

$$H^3(F(\rho,\psi)/F) = H^3(F(\rho)/F) + H^3(F(\psi)/F) + e^3(\gamma)H^0(F)\ .$$

Proof. By Corollary 7.3, the element $e^3(\gamma)$ belongs to $H^3(F(\rho,\psi)/F)$. If $\operatorname{Tors}\operatorname{CH}^2(X_\rho \times X_\psi) = 0$ then by Corollary 2.13, we have $H^3(F(\rho,\psi)/F) = H^3(F(\rho)/F)+H^3(F(\psi)/F)$ and the proof is complete. If $\operatorname{Tors}\operatorname{CH}^2(X_\rho\times X_\psi) \ne 0$, Lemma 7.7 shows that $\gamma \notin I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$. Hence, by Corollary 6.3, $e^3(\gamma) \notin H^3(F(\rho)/F) + H^3(F(\psi)/F)$. To complete the proof it is sufficient to apply Corollary 2.13 and Theorem 5.7. □

Corollary 7.9. $I^3(F(\rho,\psi)/F) \subset I^3(F(\rho)/F) + I^3(F(\psi)/F) + \{\Gamma(\rho,\psi), 0\} + I^4(F)$.

Proof. Let $\tau \in I^3(F(\rho,\psi)/F)$. Choose an element $\gamma \in \Gamma(\rho,\psi)$. By Proposition 7.8, either $e^3(\tau) \in H^3(F(\rho)/F) + H^3(F(\psi)/F)$ or $e^3(\tau-\gamma) \in H^3(F(\rho)/F) + H^3(F(\psi)/F)$. It remains to apply Corollary 6.3. □

Proposition 7.10. *Let $\pi \in I^3(F(\rho,\psi)/F)$. Then at least one of the following conditions holds*

1) $\pi \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$;
2) $\pi \in \Gamma(\rho,\psi) + I^4(F)$.

Proof. Obvious in view of Corollaries 7.9 and 7.6. □

8. Main theorem

Proposition 8.1. *Let $\phi = \langle\langle a,b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ be an anisotropic quadratic form. Let $\psi = \langle -u,-v,uv,\delta\rangle$ and $\rho = \langle -a,-b,ab,d\rangle$. Then:*

1. *The following two conditions are equivalent:*
   (i) $\langle\langle a,b,c\rangle\rangle \in I^3(F(\rho,\psi)/F)$,
   (ii) $\phi_{F(\psi)}$ *is isotropic.*
2. *The following two conditions are equivalent:*
   (i) $\langle\langle a,b,c\rangle\rangle \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$,
   (ii) *there exits a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. Note that $\langle\langle a,b,c\rangle\rangle = \phi - c\rho = \rho - c\phi$.
(1i)⇒(1ii). Let $E = F(\psi)$. If the Pfister form $\langle\langle a,b,c\rangle\rangle_E$ is isotropic, its neighbor $(\langle\langle a,b\rangle\rangle \perp \langle -c\rangle)_E$ is isotropic too. Since $\langle\langle a,b\rangle\rangle \perp \langle -c\rangle \subset \phi$, the form ϕ_E is isotropic. Thus we can suppose that $\langle\langle a,b,c\rangle\rangle_E$ is anisotropic. By the assumption, $\langle\langle a,b,c\rangle\rangle \in I^3(F(\rho,\psi)/F) = I^3(E(\rho)/F)$. Hence the anisotropic Pfister form $\langle\langle a,b,c\rangle\rangle_E$ becomes isotropic over the function field of ρ_E. By the Arason-Pfister subform theorem, we have $k\rho_E \subset \langle\langle a,b,c\rangle\rangle_E$ where k is an arbitrary element of $D_E(\rho)\cdot D_E(\langle\langle a,b,c\rangle\rangle)$. Since $(ab)^{-1} \in D_E(\rho)$ and $-abc \in D_E(\langle\langle a,b,c\rangle\rangle)$ we can take $k = (ab)^{-1}\cdot(-abc) = -c$. Thus $-c\rho_E \subset \langle\langle a,b,c\rangle\rangle_E$. Hence $\dim((\langle\langle a,b,c\rangle\rangle \perp c\rho)_E)_{an} \le 8-4 = 4$. Since $\langle\langle a,b,c\rangle\rangle + c\rho = \phi$, it follows that $\dim(\phi_E)_{an} \le 4$. Hence $\phi_{F(\psi)} = \phi_E$ is isotropic.
(1ii)⇒(1i). Since $\phi_{F(\psi)}$ and $\rho_{F(\rho)}$ are isotropic, we have $\dim(\phi_{F(\psi)})_{an} \le 4$ and $\dim(\rho_{F(\rho)})_{an} \le 2$. Therefore $\dim(\langle\langle a,b,c\rangle\rangle_{F(\rho,\psi)})_{an} = \dim((\phi - c\rho)_{F(\rho,\psi)})_{an} \le 4+2 = 6$. By the Arason-Pfister theorem, $\langle\langle a,b,c\rangle\rangle_{F(\rho,\psi)}$ is hyperbolic. Hence $\langle\langle a,b,c\rangle\rangle \in I^3(F(\rho,\psi)/F)$.
(2i)⇒(2ii). By Corollary 6.5, there exist $k_1,k_2 \in F^*$ such that $\langle\langle a,b,k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$, $\langle\langle u,v,k_2\rangle\rangle = \psi\,\langle\langle k_2\rangle\rangle$, and

$$\langle\langle a,b,c\rangle\rangle \equiv \langle\langle a,b,k_1\rangle\rangle + \langle\langle u,v,k_2\rangle\rangle \pmod{I^4(F)}\ .$$

It follows from [2, Theorem 4.8] that the Pfister forms $\langle\langle a,b,c\rangle\rangle$, $\langle\langle a,b,k_1\rangle\rangle$, and $\langle\langle u,v,k_2\rangle\rangle$ are linked. Hence there exists $s \in F^*$ such that $s\,\langle\langle u,v,k_2\rangle\rangle = \langle\langle a,b,k_1\rangle\rangle - \langle\langle a,b,c\rangle\rangle$. Since $\langle\langle a,b,k_1\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle$ and $\langle\langle a,b,c\rangle\rangle = \rho - c\phi$, we have $s\,\langle\langle u,v,k_2\rangle\rangle = \rho\,\langle\langle k_1\rangle\rangle - (\rho - c\phi) = c\phi - k_1\rho$. Therefore $\phi - cs\,\langle\langle u,v,k_2\rangle\rangle = ck_1\rho$. Hence ϕ and $cs\,\langle\langle u,v,k_2\rangle\rangle$ contain a common subform of the dimension

$$\frac{1}{2}(\dim\phi + \dim(sc\,\langle\langle u,v,k_2\rangle\rangle) - \dim(ck_1\rho)) = \frac{1}{2}(6+8-4) = 5.$$

Let us denote such a form by ϕ_0. By the definition, we have $\phi_0 \subset \phi$. Since $\phi_0 \subset sc\langle\langle u,v,k_2\rangle\rangle$, it follows that ϕ_0 is a Pfister neighbor. Since $\langle\langle u,v,k_2\rangle\rangle = \psi\langle\langle k_2\rangle\rangle$, it follows that $\langle\langle u,v,k_2\rangle\rangle_{F(\psi)}$ is isotropic. Hence the Pfister neighbor $(\phi_0)_{F(\psi)}$ of $\langle\langle u,v,k_2\rangle\rangle_{F(\psi)}$ is isotropic as well.
(2ii)⇒(2i). Let ϕ_0 be a 5-dimensional Pfister neighbor such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic. Let us write ϕ in the form $\phi = \phi_0 \perp \langle s_0\rangle$. Since ϕ_0 is a Pfister neighbor, there exists $\pi \in GP_3(F)$ such that $\phi_0 \subset \pi$. We can write π in the form $\pi = \phi_0 \perp -\langle s_1, s_2, s_3\rangle$. Set $\gamma = \langle s_0, s_1, s_2, s_3\rangle$. We have

$$\gamma = \phi - \pi \equiv \phi = \langle\langle a,b,c\rangle\rangle + c\rho \equiv c\rho \pmod{I^3(F)}.$$

Since $\dim\gamma = \dim c\rho = 4$ it follows from the Wadsworth's theorem ([27, Theorem 7]) that γ is similar to $c\rho$. Hence there exists $k \in F^*$ such that $\gamma = ck\rho$. We have

$$\langle\langle a,b,c\rangle\rangle = \rho - c\phi = \rho - c(\gamma+\pi) = \rho - c(ck\rho+\pi) = \langle\langle k\rangle\rangle\rho - c\pi.$$

Now it is sufficient to verify that $\langle\langle k\rangle\rangle\rho \in I^3(F(\rho)/F)$ and $\pi \in I^3(F(\psi)/F)$. We have $\langle\langle k\rangle\rangle\rho = \langle\langle a,b,c\rangle\rangle + c\pi \in I^3(F)$. Since $\dim(\langle\langle k\rangle\rangle\rho_{F(\rho)})_{an} < 8$, the Arason-Pfister Hauptsatz shows that $\langle\langle k\rangle\rangle\rho_{F(\rho)}$ is hyperbolic. Thus $\langle\langle k\rangle\rangle\rho \in I^3(F(\rho)/F)$. Since $\phi_0 \subset \pi$ and $(\phi_0)_{F(\psi)}$ is isotropic, $\pi_{F(\psi)}$ is isotropic as well. Since $\pi \in GP_3(F)$, it follows that $\pi_{F(\psi)}$ is hyperbolic. Hence $\pi \in I^3(F(\psi)/F)$. □

Corollary 8.2. *Let $\phi = \langle\langle a,b\rangle\rangle \perp -c\langle\langle d\rangle\rangle$ be an anisotropic quadratic form. Let $\psi = \langle -u,-v,uv,\delta\rangle$ and $\rho = \langle -a,-b,ab,d\rangle$. Suppose that the group $\mathrm{CH}^2(X_\psi \times X_\rho)$ is torsion-free. Then the following conditions are equivalent:*

1. *$\phi_{F(\psi)}$ is isotropic;*
2. *there exits a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic*

Proof. (1)⇒(2). By Item 1 of Proposition 8.1, we know that $\langle\langle a,b,c\rangle\rangle \in I^3(F(\rho,\psi)/F)$. Since $\operatorname{Tors}\mathrm{CH}^2(X_\psi \times X_\rho) = 0$, Corollary 2.13 implies that

$$H^3(F(\rho,\psi)/F) = H^3(F(\rho)/F) + H^3(F(\psi)/F)];.$$

By Corollary 6.4, $I^3(F(\rho,\psi)/F) \subset I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$. Applying Proposition 8.1 once again, we are done.
(2)⇒(1). Obvious. □

Lemma 8.3. *Let ϕ be a 6-dimensional form and ψ be a 4-dimensional form. Suppose that ψ is similar to a subform in ϕ. Then* $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$.

Proof. We can suppose that $\psi \subset \phi$. Hence there exists a 2-dimensional form μ such that $\psi \perp \mu = \phi$. Let E be a field extension of F generated by $\sqrt{d_\pm\phi}$ and $\sqrt{d_\pm\psi}$. Obviously $\phi_E, \psi_E \in I^2(F)$ and $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = \operatorname{ind} C_0(\phi_E) \otimes_E C_0(\psi_E)$. Thus we can reduce our problem to the case where $\phi, \psi \in I^2(F)$. Then $\mu \in I^2(F)$. Since $\dim\mu = 2$, the form μ is hyperbolic. Hence $\phi = \psi \perp \mathbb{H}$. Therefore $C_0(\phi) = C_0(\psi) \otimes_F M_2(F)$. Hence $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$. □

Corollary 8.4. *Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ be an anisotropic quadratic form. Let $\psi = \langle -u, -v, uv, \delta\rangle$ and $\rho = \langle -a, -b, ab, d\rangle$. Suppose that* $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) \neq 1$. *Then the following conditions are equivalent:*

(1) *$\phi_{F(\psi)}$ is isotropic and the isotropy is standard;*
(2) *there exits a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic;*
(3) *$\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$;*
(4) *$(a, b, c) \in H^3(F(\rho)/F) + H^3(F(\psi)/F)$.*

Proof. (1)⇒(2). Let ϕ and ψ be such as in (1). Let us suppose that the condition (2) is not satisfied. Then by the definition of standard isotropy, ψ is similar to a subform of ϕ. By Lemma 8.3, we have $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$. This contradicts to our assumption.
(2)⇒(1). Obvious.
(3)⟺(4)⟺(1). Follows from Proposition 8.1 and Corollary 6.3. □

Theorem 8.5. *Let ϕ be an anisotropic 6-dimensional quadratic form and ψ be a 4-dimensional quadratic form with $d_\pm\,\psi = d_\pm\,\phi \neq 1$. Suppose that $\phi_{F(\psi)}$ is isotropic. Then there exits a 5-dimensional Pfister neighbor ϕ_0 such that $\phi_0 \subset \phi$ and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. If $\operatorname{ind} C_0(\phi) = 1$ then ϕ is a Pfister neighbor. In this case we can take ϕ_0 to be equal to an arbitrary 5-dimensional subform in ϕ. In the case $\operatorname{ind} C_0(\phi) = 4$, it follows from [5] that $\phi_{F(\psi)}$ is anisotropic and we have a contradiction. Thus we can assume that $\operatorname{ind} C_0(\phi) = 2$. Then ϕ is similar to a form of the kind $\langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$. Since $d_\pm\,\psi = d_\pm\,\phi$, there exist $u, v \in F^*$ such that ψ is similar to the form $\langle -u, -v, uv, d\rangle$. Replacing ϕ and ψ by similar forms, we can suppose that

$$\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle \text{ and } \psi = \langle -u, -v, uv, d\rangle\,.$$

Let $\rho = \langle -a, -b, ab, d\rangle$. It follows from Theorem 5.1 that $\operatorname{Tors}\operatorname{CH}^2(X_\psi \times X_\rho) = 0$. Now the result required follows immediately from Corollary 8.2. □

Proposition 8.6. *Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ be anisotropic quadratic forms. Suppose that* $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 4$. *Then the following conditions are equivalent:*

(1) *$\phi_{F(\psi)}$ is isotropic;*
(2) *There is a 5-dimensional subform $\phi_0 \subset \phi$ which is a Pfister neighbor and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. Let $\rho = \langle -a, -b, ab, d\rangle$. Clearly $C_0(\phi) = M_2(F) \otimes_F C_0(\rho)$. Hence $\operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 4$. It follows from Theorem 5.8 that $\operatorname{Tors}\operatorname{CH}^2(X_\rho \times X_\psi) = 0$. By Corollary 8.2, we are done. □

Proposition 8.7. *Let $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$ be anisotropic quadratic forms with $\delta \notin F^{*2}$. Suppose that* $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 1$. *Then the following conditions are equivalent:*

(1) *$\phi_{F(\psi)}$ is isotropic;*

(2) *Either ψ is similar to a subform in ϕ or there exists a 5-dimensional subform $\phi_0 \subset \phi$ which is a Pfister neighbor and $(\phi_0)_{F(\psi)}$ is isotropic.*

Proof. (1)⇒(2). Since ϕ is anisotropic, we have $d \notin F^{*2}$. In view of Theorem 8.5 is is sufficient to consider the case $d\delta \notin F^{*2}$. Let $\rho = \langle -a, -b, ab, d\rangle$. Since $C_0(\phi) = M_2(F) \otimes_F C_0(\rho)$, we have $\operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 1$. Thus all the assumptions of §7 hold. Propositions 7.10 and 8.1 show that at least one of the following conditions holds:

1) $\langle\langle a, b, c\rangle\rangle \in I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)$,
2) $\langle\langle a, b, c\rangle\rangle \in \Gamma(\rho, \psi) + I^4(F)$.

In the first case, Proposition 8.1 asserts that there exists a 5-dimensional subform $\phi_0 \subset \phi$ which is a Pfister neighbor and $(\phi_0)_{F(\psi)}$ is isotropic.

Thus we can suppose that $\langle\langle a, b, c\rangle\rangle \in \Gamma(\rho, \psi) + I^4(F)$. Let $\gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \in \Gamma(\rho, \psi)$ be such that $\langle\langle a, b, c\rangle\rangle \in \gamma + I^4(F)$. Since $\langle\langle a, b, c\rangle\rangle = \rho - c\phi$, we have

$$l_1\rho - l_1c\phi = l_1 \langle\langle a, b, c\rangle\rangle \equiv \langle\langle a, b, c\rangle\rangle \equiv \gamma = l_1\rho + l_2\psi + \langle\langle d\delta\rangle\rangle \pmod{I^4(F)}.$$

Hence $l_1c\phi + l_2\psi + \langle\langle d\delta\rangle\rangle \in I^4(F)$. Since $\dim(l_1c\phi + l_2\psi + \langle\langle d\delta\rangle\rangle)_{an} \le 6 + 4 + 2 = 12 < 16$, the Arason-Pfister Hauptsatz shows that $l_1c\phi + l_2\psi + \langle\langle d\delta\rangle\rangle = 0$. Therefore $\phi = -cl_1l_2\psi - cl_1 \langle\langle d\delta\rangle\rangle$. Since $\dim \phi = 6 = \dim(-cl_1l_2\psi \perp -cl_1 \langle\langle d\delta\rangle\rangle)$, we have $\phi = -cl_1l_2\psi \perp -cl_1 \langle\langle d\delta\rangle\rangle$. Hence ψ is similar to a subform in ϕ.

(2)⇒(1). Obvious. □

Together with results described in Introduction, Theorem 8.5, Propositions 8.6 and 8.7 give rise to the following

Theorem 8.8. *Let ϕ be an anisotropic quadratic form of dimension ≤ 6 and ψ be such that $\phi_{F(\psi)}$ is isotropic. If the isotropy is non-standard then*

- $\dim \phi = 6$ *and* $\dim \psi = 4$;
- $1 \ne d_\pm \phi \ne d_\pm \psi \ne 1$;
- $\operatorname{ind} C_0(\phi) = 2$; *and*
- $\operatorname{ind} C_0(\phi) \otimes_F C_0(\psi) = 2$. □

9. The case of index 2

Theorem 8.8 implies that if there exists a quadratic form ϕ of dimension ≤ 6 having a non-standard isotropy over the function field of a quadratic form ψ, then there are $a, b, c, d, u, v, \delta \in F^*$ such that $\phi \sim \langle\langle a, b\rangle\rangle \perp -c \langle\langle d\rangle\rangle$, $\psi \sim \langle -u, -v, uv, \delta\rangle$, $d, \delta, d\delta \notin F^{*2}$, and $\operatorname{ind}((a, b) \otimes_F (u, v))_{F(\sqrt{d},\sqrt{\delta})} = 2$.

Set $\rho = \langle -a, -b, ab, d\rangle$. By Corollary 8.2, if $\operatorname{Tors} \operatorname{CH}^2(X_\psi \times X_\rho) = 0$, then the isotropy is standard.

In this section we prove the following

Theorem 9.1. *Let $a, b, u, v, d, \delta \in F^{*2}$ be such that $d, \delta, d\delta \notin F^{*2}$. Let $\rho = \langle -a, -b, ab, d\rangle$ and $\psi = \langle -u, -v, uv, \delta\rangle$. Suppose that* $\operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 2$. *The following conditions are equivalent:*

(1) $\operatorname{Tors}\operatorname{CH}^2(X_\rho \times X_\psi) \neq 0$;
(2) *there exists* $c \in F^*$ *such that the quadratic form* $\phi = \langle\langle a, b\rangle\rangle \perp -c\,\langle\langle d\rangle\rangle$ *is isotropic over* $F(\psi)$*, but the isotropy is not standard.*

Proof. (2)⇒(1). Obvious in view of Corollary 8.2.
(1)⇒(2). Since $\operatorname{Tors}\operatorname{CH}^2(X_\rho \times X_\psi) \neq 0$, it follows from Corollary 2.13 that there exists $w \in H^3(F(\rho,\psi)/F)$ such that $w \notin H^3(F(\rho)/F) + H^3(F(\psi)/F)$. Let $\rho_0 = \langle -a, -b, ab\rangle$. It follows from Theorem 5.9 that $\operatorname{ind} C_0(\rho_0) \otimes_F C_0(\psi) \neq \operatorname{ind} C_0(\rho) \otimes_F C_0(\psi) = 2$. Therefore $\operatorname{ind} C_0(\rho_0) \otimes_F C_0(\psi) = 4$. By Theorem 5.8, we have $\operatorname{Tors}\operatorname{CH}^2(X_{\rho_0} \times X_\psi) = 0$. By Corollary 2.13, we have $H^3(F(\rho_0,\psi)/F) = H^3(F(\rho_0)/F) + H^3(F(\psi)/F)$. Hence

$$w \in H^3(F(\rho,\psi)/F) \subset H^3(F(\rho_0,\psi)/F) = H^3(F(\rho_0)/F) + H^3(F(\psi)/F).$$

Since $H^3(F(\rho_0)/F) = (a,b) \cup H^1(F)$, there exists $c \in F^*$ such that $w - (a,b,c) \in H^3(F(\psi)/F)$, i.e. $w \equiv (a,b,c) \pmod{H^3(F(\psi)/F)}$. By the assumption on w, we see that $(a,b,c) \in H^3(F(\rho,\psi)/F)$ and $(a,b,c) \notin H^3(F(\rho)/F) + H^3(F(\psi)/F)$. Therefore, $\langle\langle a,b,c\rangle\rangle \in I^3(F(\rho,\psi)/F$ and

$$\langle\langle a,b,c\rangle\rangle \notin I^3(F(\rho)/F) + I^3(F(\psi)/F) + I^4(F)\ .$$

By Proposition 8.1, the quadratic form $\phi_{F(\psi)}$ is isotropic. By Corollary 8.4, the isotropy is not standard. □

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