FIELDS OF u-INVARIANT 11

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Abstract. The u-invariant of a field is the highest dimension of a non-degenerate anisotropic quadratic form over this field. As known since the 50es, the set of possible finite values of the u-invariant starts with 1, excludes 3, 5, 7, and includes all 2-powers. It was shown by Alexander Merkurjev in the end of the 80es that this set contains 6 and – a couple of years later – all positive even integers. Oleg Izhboldin proved by the end of the 90es that 9 is also there. In the second half of the 00s, this result has been extended to all larger numbers of the form a 2-power plus 1 by Alexander Vishik. Here we show that the value 11 is taken. The result still holds if we restrict to fields of any fixed characteristic. We also provide somewhat simpler arguments concerning the u-invariant 9.

Contents

1. Introduction

The u-invariant of a field is the highest dimension of a non-degenerate anisotropic quadratic form over this field. As known since the 50es (see [6], raising the question), the set of possible finite values of the u-invariant starts with 1, excludes 3, 5, 7, and includes all 2-powers (see [12, §6 of Chapter XI] for a modern exposition). It was shown by Alexander Merkurjev in the end of the 80es (see [14]) that this set contains 6 and – a couple of years later – all positive even integers (see [15] or [4, Theorem 38.4]). Oleg Izhboldin proved by the end of the 90es that 9 is also there (see [5]). In the second half

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of the 00s, this result has been extended to all larger numbers of the form a 2-power plus 1 by Alexander Vishik (see [21]).

Here we show that the value 11 is taken. The result also holds if we restrict to fields of any fixed characteristic or even to overfields of a fixed field:

Theorem 1.1. *For any field F_0, there is an extension field $F \supset F_0$ satisfying $u(F) = 11$.*

An overview of the proof and the proof itself are given in §5. Its most advanced ingredient is given by the Steenrod operations on the modulo 2 Chow groups invented by Vladimir Voevodsky ([22]), which became available over arbitrary fields due to Patrick Brosnan ([1], see also [4, Chapter XI]) and Eric Primozic ([18]). Although the symmetric operations [20] of Alexander Vishik in algebraic cobordism [13] of Marc Levine and Fabien Morel do not show up directly in the proof, they provide the ideology behind it. Finally, the most important classical ingredient of the proof is Index Reduction Formula for quadrics ([15], see also [4, Theorem 30.5]), discovered by Alexander Merkurjev.

2. Chow groups and quadrics

Let φ be a non-degenerate quadratic form over a field F. We consider the projective quadric $X = X_1$ defined by φ and write $\bar{X}$ for X over an algebraic closure $\bar{F}$ of F. Recall (see, e.g., [4, §68]) that the Chow group $\mathrm{CH}(\bar{X})$ is free with the basis

$$h^0, \dots, h^n, l_n, \dots, l_0,$$

where n is such that $\dim X = \dim \varphi - 2$ equals $2n$ or $2n+1$. The element $h \in \mathrm{CH}^1(\bar{X})$ is the hyperplane section class, $h^i \in \mathrm{CH}(\bar{X})$ is its ith power, and $l_i \in \mathrm{CH}_i(\bar{X})$ is the class of an i-dimensional projective space lying on $\bar{X}$. One has

$$hl_i = l_{i-1} \ \text{ and } \ h^{n+1} = 2l_{\dim X - n - 1},$$

where $l_{-1} := 0$.

We are going to work with the modulo 2 Chow group

$$\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$$

and write $\overline{\mathrm{Ch}}(X) \subset \mathrm{Ch}(\bar{X})$ for the image of the change of field homomorphism

$$\mathrm{Ch}(X) \to \mathrm{Ch}(\bar{X}).$$

(This notation will be applied not for quadrics only.)

Note that $l_i \in \overline{\mathrm{Ch}}(X)$ form some i if and only if the Witt index $\mathrm{ind}(\varphi)$ is at least $i+1$ (see [4, Corollary 72.6]).

3. Orthogonal grassmannians

Let F be a field and let φ be a non-degenerate quadratic F-form of dimension $2n+1$ with $n \geqslant 1$. For $m \in \{1, \dots, n\}$, let X_m be the mth grassmannian of φ, i.e., the variety of its m-dimensional totally isotropic subspaces. In particular, X_1 is the projective quadric $\varphi = 0$.

We write $c_1, \dots, c_{2n-m+1} \in \mathrm{CH}(X_m)$ for the Chern classes of the quotient of the rank $2n+1$ trivial vector bundle by the tautological (rank m) vector bundle on X_m. Since $c_{2n-m+1} = 0$, this last Chern class does not show up below.

We write $\bar{X}_m$ for the variety X_m over an algebraic closure $\bar{F}$ of F.

As shown in [21, §2] (see also [3]), the images of some of the above Chern classes in $\mathrm{CH}(\bar{X}_m)$ (for which we use the same notation)[1] are divisible by 2:

$$c_i = 2e_i \ \text{ for } i \geqslant n-m+1,$$

where the elements $e_{n-m+1}, \dots, e_{2n-m}$ are images (respectively) of the elements $l_{n-1}, \dots, l_0 \in \mathrm{CH}(\bar{X}_1)$ (introduced in §2) under the composition

$$\mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}_{1\subset m}) \to \mathrm{CH}(\bar{X}_m)$$

of the pullback followed by pushforward with respect to the projections

$$\bar{X}_1 \leftarrow \bar{X}_{1\subset m} \to \bar{X}_m,$$

where $X_{1\subset m} \subset X_1 \times X_m$ is the variety of 2-flags.

Note that the elements

$$e_{n-m+1} \in \mathrm{CH}^{n-m+1}(\bar{X}_m), \dots, e_{2n-m} \in \mathrm{CH}^{2n-m}(\bar{X}_m)$$

are indexed by their codimensions whereas the elements

$$l_{n-1} \in \mathrm{CH}_{n-1}(\bar{X}_1), \dots, l_0 \in \mathrm{CH}_0(\bar{X}_1)$$

– by their dimensions. For $m = 1$ we have

$$e_n = l_{n-1}, \dots, e_{2n-1} = l_0.$$

Recall that we are writing $\overline{\mathrm{CH}}(X_m)$ for the image of the change of field homomorphism

$$\mathrm{CH}(X_m) \to \mathrm{CH}(\bar{X}_m).$$

Note that the index of the even Clifford algebra $C_0(\varphi)$ is 2^{n-r} for some $r \in \{0, \dots, n\}$.

Proposition 3.1. *If* $\operatorname{ind} C_0(\varphi) = 2^{n-r}$, *then* $\overline{\mathrm{CH}}(X_m) \subset \mathrm{CH}(\bar{X}_m)$ *is contained in the subring* $A_r \subset \mathrm{CH}(\bar{X}_m)$ *generated by*

$$c_1, \dots, c_{2n-m-r}, e_{2n-m-r+1}, \dots, e_{2n-m}.$$

(For $r = 0$*, this list consists of the Chern classes only; for* $r = 1$*, the list consists of the single* e_{2n-m} *and the Chern classes.)*

Proof. By Index Reduction Formula for quadrics [4, Theorem 30.5], we can find a field extension L/F such that the Witt index of φ_L is r and the index of the even Clifford algebra $C_0(\varphi_L)$ is still 2^{n-r}. Indeed, one may produce L by a chain of function fields of quadrics, starting with X_1, or simply take for L the function field of the variety X_r.

Since $\overline{\mathrm{CH}}(X_m) \subset \overline{\mathrm{CH}}((X_m)_L)$, where the groups $\mathrm{CH}((X_m)_{\bar{F}})$ and $\mathrm{CH}((X_m)_{\bar{L}})$ are identified by the change of field isomorphism, it suffices to show that $A_r = \overline{\mathrm{CH}}((X_m)_L)$. The inclusion $A_r \subset \overline{\mathrm{CH}}((X_m)_L)$ is obvious. We prove the equality by showing that the indexes of the subgroups A_r and $\overline{\mathrm{CH}}((X_m)_L)$ in their common overgroup $\mathrm{CH}(\bar{X}_m)$ coincide.

Recall that by [17], the Grothendieck group $K(X_m)$ is a direct sum of several copies of $K(F) = \mathbb{Z}$ and several copies of $K(C_0(\varphi)) = 2^{n-r}\mathbb{Z}$. Moreover, the number c of copies of

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[1] The abuse can be partially justified by [11, Theorem 2.1] affirming that these images satisfy the exactly same relations as the original elements.

$C_0(\varphi)$ depends on $\dim\varphi$ only. We determine c by taking a *generic* $(2n+1)$-dimensional φ, where $r=0$ and the group $\mathrm{CH}(X_m)=A_0$ is free with a basis given by all products

$$c_1^{\alpha_1}\dots c_{2n-m}^{\alpha_{2n-m}}$$

satisfying $\alpha_1+\dots+\alpha_{2n-m}\leqslant m$ and $\alpha_i\leqslant 1$ for $i\geqslant n-m+1$ (see [11, Theorem 2.1]). A basis of the group $\mathrm{CH}(\bar X_m)$ is given by the similar products with c_i replaced by e_i for all $i\geqslant n-m+1$. It follows that

$$\log_2[\mathrm{CH}(\bar X_m):\mathrm{CH}(X_m)] \;=\; 1\cdot\tbinom{n}{1}\cdot f_{m-1}+2\cdot\tbinom{n}{2}\cdot f_{m-2}+\dots+m\cdot\tbinom{n}{m}\cdot f_0,$$

where f_i is the number of monomials $c_1^{\alpha_1}\dots c_{n-m}^{\alpha_{n-m}}$ with $\alpha_1+\dots+\alpha_{n-m}\leqslant i$. Since the group $\mathrm{CH}(X_m)$ is free of torsion, the canonical surjective homomorphism of $\mathrm{CH}(X_m)$ to the graded ring associated with the topological filtration on $K(X_m)$ is an isomorphism. Therefore

$$\log_2[\mathrm{CH}(\bar X_m):\mathrm{CH}(X_m)]=\log_2[K(\bar X_m):K(X_m)]=cn$$

(cf. [7, Proposition 2]), and we conclude that

$$\begin{aligned} c=\tfrac{1}{n}\Big(1\cdot\tbinom{n}{1}\cdot f_{m-1}+2\cdot\tbinom{n}{2}\cdot f_{m-2}+\dots+m\cdot\tbinom{n}{m}\cdot f_0\Big) &= \\ \tbinom{n-1}{0}\cdot f_{m-1}+\tbinom{n-1}{1}\cdot f_{m-2}+\dots+\tbinom{n-1}{m-1}\cdot f_0&. \end{aligned}$$

For arbitrary (not necessarily generic) φ with $r=0$, we have

$$cn=[\mathrm{CH}(\bar X_m:A_0)]\geqslant[\mathrm{CH}(\bar X_m):\overline{\mathrm{CH}}(X_m)]\geqslant[K(\bar X_m):K(X_m)]=cn,$$

where the second inequality comes from [7, Proposition 2]. Therefore all the inequalities in the chain are in fact equalities. In particular, $A_0=\overline{\mathrm{CH}}(X_m)$ which is the statement of Proposition 3.1 for $r=0$.

For arbitrary r, we have $[K(\bar X_m):K(X_m)]=c(n-r)$. For $r\geqslant 1$, a direct computation of the index $[A_r:A_{r-1}]$ yields c. Using this computation and inducting on r, we show that $c(n-r)=[\mathrm{CH}(\bar X_m):A_r]$. Thus

$$c(n-r)=[\mathrm{CH}(\bar X_m):A_r]\geqslant[\mathrm{CH}(\bar X_m):\overline{\mathrm{CH}}((X_m)_L)]\geqslant[K(\bar X_m):K((X_m)_L)]=c(n-r).$$

Consequently, $A_r=\overline{\mathrm{CH}}((X_m)_L)$. □

4. INDEX REDUCTION AND NOT ONLY

Let φ be a non-degenerate anisotropic quadratic form over a field F of an odd dimension $\dim\varphi=2n+1$ with some $n\geqslant 3$ and such that the index $\operatorname{ind}C_0(\varphi)$ of its even Clifford algebra $C_0(\varphi)$ is at least the "almost highest" one 2^{n-1}. We consider the "almost highest" grassmannian X_{n-1} of φ – the grassmannian of $(n-1)$-dimensional totally isotropic subspaces. Let ψ be an anisotropic quadratic F-form of dimension $2n+2$ and of nontrivial discriminant.

The restriction on n showing up in the following result is the main reason why we are not constructing higher than 11 values of the u-invariant here:

Proposition 4.1. *If $n\leqslant 5$, then the Witt index $\operatorname{ind}(\psi_{F(X_{n-1})})$ is at most $n-1$.*

Proof. We assume the contrary:

$$\operatorname{ind}(\psi_{F(X_{n-1})}) \geqslant n, \tag{4.2}$$

i.e., the form $\psi_{F(X_{n-1})}$ is "almost split". Then $\psi_{F(X_{n-1})}$ becomes split over its discriminant (quadratic) extension field $E(X_{n-1})/F(X_{n-1})$, where E/F is the discriminant field extension of ψ. In particular, the central simple $E(X_{n-1})$-algebra $C_0(\psi_{F(X_{n-1})})$ is split and it follows from Index Reduction Formula (for the orthogonal grassmannian X_{n-1}) of [16] that the central simple E-algebra $C_0(\psi)$ is split. Actually, the application of Index Reduction Formula for the orthogonal grassmannian can be replaced by repeated application of Index Reduction Formula for quadrics.

Since the discriminant of ψ is nontrivial, the highest Witt index $\operatorname{ind}_h(\psi)$ of ψ is 1. Assume that the almost highest Witt index $\operatorname{ind}_{h-1}(\psi)$ is greater than 1. Then by [4, Theorem 73.26(2)] the 1-primordial cycle for the *almost leading* form of ψ, i.e., the almost last quadratic form in the generic splitting tower of ψ – the one of dimension

$$2(\operatorname{ind}_{h-1}(\psi) + \operatorname{ind}_h(\psi)),$$

is binary. It follows then by [4, Corollary 80.10] that the integer

$$2(\operatorname{ind}_{h-1}(\psi) + \operatorname{ind}_h(\psi)) - \operatorname{ind}_{h-1}(\psi)$$

is a 2-power, i.e., $\operatorname{ind}_{h-1}(\psi) = 2^r - 2$ for some $r \geqslant 2$. Since

$$2(\operatorname{ind}_h(\psi) + \operatorname{ind}_{h-1}(\psi)) \leqslant \dim\psi = 2n + 2 \leqslant 12,$$

we have $r = 2$.

We conclude that $\operatorname{ind}_{h-1}(\psi)$ is either 1 or 2 and we consider these two cases below separately.

In the case with $\operatorname{ind}_{h-1}(\psi) = 1$, replacing F by the function field of the $(n-1)$st grassmannian Y_{n-1} of ψ and than replacing ψ by its anisotropic part, we come to the situation with $\dim\psi = 4$ whereas the index of $C_0(\varphi)$ is still at least 2^{n-1}. Replacing φ by its anisotropic part, we therefore get that $\dim\varphi \geqslant 2n-1$ for the new φ. Besides, all higher Witt indexes of the new φ (as well as of the old one) are 1.

Now we replace the current F by $F(\varphi)$ and replace φ by its anisotropic part. By [4, Theorem 76.1(1)], the 4-dimensional form ψ is still anisotropic (we recall that $n \geqslant 3$). We repeat the same procedure until φ becomes 3-dimensional. At this point, anisotropy of ψ contradicts assumption (4.2).

In the case with $\operatorname{ind}_{h-1}(\psi) = 2$, replacing F by the function field of Y_{n-2} and than replacing ψ by its anisotropic part, we come to the situation with $\dim\psi = 6$ whereas the index of $C_0(\varphi)$ is still at least 2^{n-1}. Replacing φ by its anisotropic part, we therefore get once again that $\dim\varphi \geqslant 2n-1$ and all higher Witt indexes of φ are 1.

If $\dim\varphi \geqslant 7$ for the current φ, we replace the current F by $F(\varphi)$ and replace φ by its anisotropic part. By [4, Theorem 76.1(1)], the 6-dimensional form ψ is still anisotropic. We repeat the same procedure until φ becomes 5-dimensional. At this point, the quadratic forms φ and ψ satisfy hypotheses of Lemma 4.3 below and therefore $\psi_{F(\varphi)}$ is anisotropic, contradicting (4.2). □

Lemma 4.3. *Let φ be a non-degenerate quadratic form over a field F of an odd dimension at least 5 and such that $C_0(\varphi)$ is a division algebra. Let ψ be a non-degenerate anisotropic quadratic form of dimension $\dim\psi = \dim\varphi + 1$ over F and such that $\operatorname{ind}_1(\psi) \geqslant 2$.*

Then $\psi_{F(\varphi)}$ is anisotropic.

Proof. Assume that $\psi_{F(\varphi)}$ is isotropic. Then $\psi'_{F(\varphi)}$ is isotropic for a codimension 1 non-degenerate subform $\psi' \subset \psi$. It follows by [4, Theorem 76.1(2)] that the quadratic form $\varphi_{F(\psi')}$ isotropic. By [9, Corollary 2.15], this implies that the upper motives $U(X)$ and $U(Y')$ of the quadrics X of φ and Y' of ψ' are isomorphic. Here we talk about the Chow motives [4, §64] with coefficients $\mathbb{Z}/2\mathbb{Z}$. Since $C_0(\varphi)$ is a division algebra, the total motive $M(X)$ of X is indecomposable (see, e.g., [8, Proposition 3.6]), i.e., $U(X) = M(X)$. Since $\operatorname{ind}_1(\psi) \geqslant 2$, the total motive $M(Y)$ of the quadric Y of ψ contains the direct sum

$$U(Y') \oplus U(Y')\{1\} \simeq M(X) \oplus M(X)\{1\}$$

as a direct summand (see [19, Theorem 4.13]), where $\{1\}$ stands for the Tate shift. This is however not possible because

$$\dim \operatorname{Ch}^1(\bar{X}) + \dim \operatorname{Ch}^0(\bar{X}) = 2 > 1 = \dim \operatorname{Ch}^1(\bar{Y}). \qquad \square$$

Remark 4.4. We actually need Lemma 4.3 only in the case with $\dim\psi = 6$ here. In this case it is easy to show that ψ is a Pfister neighbour. On the other hand, the condition on $C_0(\varphi)$ ensures that the 5-dimensional φ is not and thus implies anisotropy of $\psi_{F(\varphi)}$.

5. PROOF OF THEOREM 1.1

Recall that the construction of a field with even u-invariant $2n$, provided in [15] (see [4, Theorem 38.4] for a proof which includes characteristic 2), is based on the following consequence of Index Reduction Formula for quadrics (discovered in [15] on the occasion):

Theorem 5.1 ([15])**.** *Let φ be a non-degenerate $2n$-dimensional quadratic form over a field F such that the index of $C_0(\varphi)$ equals 2^{n-1} (the highest possible value from the dimension prospective); then φ is anisotropic and for any ($2n+1$)-dimensional non-degenerate quadratic form ψ over the same field, the index of $C_0(\varphi_{F(\psi)})$ over the function field $F(\psi)$ of the projective quadric $\psi = 0$ is still the same (implying that φ remains anisotropic over $F(\psi)$). (One may ask the discriminant of φ to be nontrivial to ensure that $C_0(\varphi)$ is a central simple algebra (over the discriminant extension field).)*

The similar statement for odd $u = 2n+1$ is false: if $\operatorname{ind} C_0(\varphi)$ has its highest possible value 2^n, it lowers to 2^{n-1} over $F(\psi)$ for ψ of trivial discriminant containing φ as a subform.

If we replace the condition on φ by

$$\operatorname{ind} C_0(\varphi) \geqslant 2^{n-1}, \tag{5.2}$$

then it is preserved over $F(\psi)$ for all $(2n+2)$-dimensional ψ – as we want – but the issue is that this modified condition does not imply anisotropy of φ anymore. So, one looks for another condition which ensures anisotropy and at the same time can be shown to be preserved when going from F to $F(\psi)$.

The condition used in [21] included

$$e_{n+1} \notin \overline{\operatorname{Ch}}(X_{n-1}), \tag{5.3}$$

where X_{n-1} is the almost maximal grassmannian of φ and $e_{n+1} \in \mathrm{Ch}(\bar{X}_{n-1})$ is the element defined in §3. Because of the relation between e_{n+1} and the rational point class $l_0 \in \mathrm{Ch}(\bar{X}_1)$, condition (5.3) alone already implies the condition $l_0 \notin \overline{\mathrm{Ch}}(X_1)$ meaning anisotropy of φ. However, (5.3) alone is not preserved under the passage to $F(\psi)$.

The combination of (5.2) and (5.3) however leads to the success in the situation we are treating:

Theorem 5.4. *Let φ be a non-degenerate quadratic form over a field F of dimension* 11*. Let ψ be a non-degenerate quadratic form over F of dimension* 12*. Assume that φ satisfies (5.2) and (5.3) (with $n = 5$). Then $\varphi_{F(\psi)}$ also satisfies (5.2) and (5.3).*

Proof. We may assume that ψ is anisotropic and we only need to check that $\varphi_{F(\psi)}$ satisfies (5.3). Assume that it doesn't, i.e., $e_6 \in \overline{\mathrm{Ch}}((X_4)_{F(Y_1)})$, where Y_1 is the quadric of ψ. Lifting e_6 to the product $Y_1 \times X_4$, we get some $\alpha \in \overline{\mathrm{Ch}}(Y_1 \times X_4)$. We have

$$\alpha = h^0 \times e_6 + h^1 \times \alpha_1 + \cdots + h^5 \times \alpha_5 + l_5 \times \beta + a l_4 \times [X_4]$$

for some homogeneous $\alpha_1, \dots, \alpha_5, \beta \in \mathrm{Ch}(\bar{X}_4)$ and $a \in \mathbb{Z}/2\mathbb{Z}$.

Here is the plan:

(i) Using Proposition 4.1, we reduce to the case with $a = 0$. (As explained in [21, Proof of Theorem 5.1], this is "the most delicate part of the proof".)

(ii) Using Proposition 3.1 as well as related to the u-invariant 9 Theorem 6.1 below, we narrow (even more) the shape of α.

(iii) Using the shape of α, obtained in (ii), and Proposition 5.6 below, we show that $e_6 \in \overline{\mathrm{Ch}}(X_4)$ achieving a contradiction.

(i) For ψ of nontrivial discriminant, execution of (i) is instant: if $a \neq 0 \in \mathbb{Z}/2\mathbb{Z}$, then the image of α in $\overline{\mathrm{Ch}}((Y_1)_{F(X_4)})$ equals l_4 so that $\mathrm{ind}(\psi)_{F(X_4)} \geqslant 5$ contradicting Proposition 4.1.

For ψ of trivial discriminant, if $a \neq 0$, then the quadratic form $\psi_{F(X_4)}$ is split, i.e., $\mathrm{ind}(\psi_{F(X_4)}) = 6$, and we have $l_5 \in \overline{\mathrm{Ch}}((Y_1)_{F(X_4)})$. Lifting l_5 to the group $\overline{\mathrm{Ch}}(Y_1 \times X_4)$, we get some α' of the form

$$\alpha' = h^0 \times \alpha'_0 + h^1 \times \alpha'_1 + \cdots + h^4 \times \alpha'_4 + l_5 \times [X_4].$$

Subtracting from α the product

$$(h^1 \times [X_4]) \cdot \alpha' = h^1 \times \alpha'_0 + h^2 \times \alpha'_1 + \cdots + h^5 \times \alpha'_4 + l_4 \times [X_4] \in \overline{\mathrm{Ch}}(Y_1 \times X_4),$$

we get rid of the term $l_4 \times [X_4]$ in α.

(ii) By Index Reduction Formula,

$$\mathrm{ind}\, C_0((\varphi)_{F(Y_i)}) \geqslant 2^{5-i} \quad \text{for } i \in \{1, \dots, 5\}.$$

Viewing α as a correspondence $Y_1 \rightsquigarrow X_4$ and applying it to $l_i \in \overline{\mathrm{Ch}}((Y_1)_{F(Y_{i+1})})$, we get

$$\alpha_i \in \overline{\mathrm{Ch}}^{6-i}((X_4)_{F(Y_{i+1})}).$$

Therefore by Proposition 3.1 we may assume that $\alpha_i = a_i e_{6-i}$ for some $a_i \in \mathbb{Z}/2\mathbb{Z}$.

We are going to show that $\alpha_1 = 0$. For this, let us consider the anisotropic part φ' of the quadratic form $\varphi_{F(X_1)}$. We have $\dim \varphi' = 9$ and $C_0(\varphi')$ is a division algebra. Let X'_3 be the almost highest grassmannian of φ', and let $e'_5 \in \mathrm{Ch}^5(\bar{X}'_3)$ be the corresponding generator. Note that $e'_5 \notin \overline{\mathrm{Ch}}^5(X'_3)$ by Proposition 3.1. It follows by Theorem 6.1 that $e'_5 \notin \overline{\mathrm{Ch}}((X'_3)_{F(X_1)(Y_2)})$. Since the motive of X'_3 is a direct summand in the motive of $(X_4)_{F(X_1)}$ (see [2]), the group $\mathrm{Ch}^5(\bar{X}'_3)$ is a direct summand in $\mathrm{Ch}^5(\bar{X}_4)$; moreover, the element $e'_5 \in \mathrm{Ch}^5(\bar{X}'_3)$ corresponds to the element $e_5 \in \mathrm{Ch}^5(\bar{X}_4)$. It follows that $e_5 \notin \mathrm{Ch}^5((X_4)_{F(X_1)(Y_2)})$ and, in particular, $e_5 \notin \mathrm{Ch}^5((X_4)_{F(Y_2)})$. Consequently, $a_1 = 0$ and $\alpha_1 = 0$.

(iii) By Proposition 5.6, taking into account that $\alpha_1 = 0$, we have

$$e_6 + \mathrm{St}^2(\alpha_2) \in \overline{\mathrm{Ch}}(X_4). \tag{5.5}$$

Recall that α_2 is 0 or e_4. Since $e_6 \in \overline{\mathrm{Ch}}((X_4)_{F(X_1)})$, the element $\mathrm{St}^2(\alpha_2)$ is also in $\overline{\mathrm{Ch}}((X_4)_{F(X_1)})$. Therefore, by Proposition 3.1, we are only interested in the coefficient at e_6 in the decomposition of $\mathrm{St}^2(e_4)$ given in [21]. This coefficient is equal to $\binom{4}{2}$ and is even. It follows that $e_6 \in \overline{\mathrm{Ch}}(X_4)$ as desired. □

The following proposition, applied in the above proof, is an analogue of [10, Proposition 5.3]. It makes use of cohomological Steenrod operations St^i on the modulo 2 Chow groups, see [4, Chapter XI] for characteristic not 2 and [18] for characteristic 2.

Proposition 5.6. *Let Q be a smooth projective quadric of dimension 10 over a field F. Let V be a projective homogeneous variety over F. Let $\alpha \in \mathrm{Ch}^6(Q \times V)$ be an element such that in the decomposition*

$$\bar{\alpha} = h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^5 \times \alpha_5 + l_5 \times \beta + a l_4 \times [V]$$

of its image $\bar{\alpha} \in \mathrm{Ch}^6(\bar{Q} \times \bar{V})$ with $\alpha_i \in \mathrm{Ch}^{6-i}(\bar{V})$, $\beta \in \mathrm{Ch}^1(\bar{V})$, and $a \in \mathbb{Z}/2\mathbb{Z}$, the element a is trivial.

Then $\alpha_0 + \alpha_1\beta + \mathrm{St}^2(\alpha_2) \in \overline{\mathrm{Ch}}^6(V)$.

Proof. For every $i = 0, 1, \ldots, 5$, let s^i be the image in $\mathrm{CH}^{6+i}(\bar{Q} \times \bar{V})$ of an element of $\mathrm{CH}^{6+i}(Q \times V)$ representing $\mathrm{St}^i(\alpha) \in \mathrm{Ch}^{6+i}(Q \times V)$. We also set $s^i := 0$ for $i > 6$ as well as for $i < 0$. Finally, we define s^6 to be the square $(s^0)^2$ of s^0 (related to $\mathrm{St}^6(\alpha)$ by [4, Theorem 61.13]).

Note that we have

$$s^0 := h^0 \times \boldsymbol{\alpha}_0 + h^1 \times \boldsymbol{\alpha}_1 + \cdots + h^5 \times \boldsymbol{\alpha}_5 + l_5 \times \boldsymbol{\beta} + b l_4 \times [\bar{V}] \in \mathrm{CH}^6(\bar{Q} \times \bar{V})$$

with some $\boldsymbol{\alpha}_i \in \mathrm{CH}^{6-i}(\bar{V})$ for $i = 0, 1, \ldots, 5$, $\boldsymbol{\beta} \in \mathrm{CH}^1(\bar{V})$, and an integer b. Since

$$s^0 \mod 2\,\mathrm{CH}^6(\bar{Q} \times \bar{V}) = \bar{\alpha},$$

the integer b is even. Since the element $2l_4 = h^6$ is in $\overline{\mathrm{CH}}^6(Q) \subset \mathrm{CH}^6(\bar{Q})$, we remove the last summand from the above decomposition of s^0.

Let d be any integer with $4 \leqslant d \leqslant 10$. For a d-dimensional smooth subquadric P of Q, writing *in* for the imbedding $P \times Y \hookrightarrow Q \times Y$ and applying [4, Theorem 61.9 and Proposition 61.10]), we have

$$pr^P_* \sum_{i=d-6}^{d} c_i(-T_P)\, in^* \,\mathrm{St}^{d-i} \alpha = \mathrm{St}^d \, pr^P_* \, in^* \alpha \in \mathrm{Ch}^6(V),$$

where T_P is the tangent bundle on P and pr^P is the projection $P \times V \to V$. The summation on the left side is from $d-6$ to d only because $\mathrm{St}^{d-i}\,\alpha = 0$ for i outside of this interval, see [4, Theorem 61.13]. The right side is actually zero since

$$\mathrm{St}^d\, pr^P_*\, in^*\, \alpha \in \mathrm{St}^d\, \mathrm{Ch}^{6-d}(V)$$

and $\mathrm{St}^d\,\mathrm{Ch}^{6-d}(V) = 0$ because $d > 6-d$. Rewriting the left side with a help of the projection formula [4, Proposition 56.9], we obtain the relation

$$pr_* \sum_{i=d-6}^{d} in_*(c_i(-T_P))\,\mathrm{St}^{d-i}\,\alpha = 0 \in \mathrm{Ch}^6(V),$$

where pr is the projection $Q \times V \to V$. By the computation [4, Lemma 78.1] of the Chern classes of the tangent bundle, it follows that

$$\sum_{i=d-6}^{d} \binom{-d-2}{i} \cdot pr_*(h^{10-d+i} \cdot s^{d-i}) \in 2\,\overline{\mathrm{CH}}^6(V) \subset \mathrm{CH}^6(\bar{V}). \tag{5.7}$$

We are going to use (5.7) for various values of d. Note that when d varies, the sum showing up in (5.7) is a linear combination of always the same elements

$$pr_*(h^4 s^6),\ pr_*(h^5 s^5),\ \ldots,\ pr_*(h^{10} s^0) \in \overline{\mathrm{CH}}^6(V).$$

However, the coefficients of this linear combination vary with d.

Let us compute the ith element $pr_*(h^{10-d+i} \cdot s^{d-i}) \in \mathrm{CH}^6(\bar{V})$ modulo $4\,\mathrm{CH}^6(\bar{V})$, where $i \in \{d-6,\ d-5,\ \ldots,\ d\}$. For $i \geqslant d-4$, we have $10-d+i \geqslant 6$ so that the factor $h^{10-d+i} \in \mathrm{CH}(\bar{Q})$ is divisible by 2. The other factor modulo 2 is $\mathrm{St}^{d-i}(\bar{\alpha})$ and it follows that

$$pr_*(h^{10-d+i} \cdot s^{d-i}) \equiv 2 \sum_{k \geqslant 0} \binom{k}{d-i-k} \varepsilon_k \pmod 4 \qquad \text{for } i \geqslant d-4,$$

where $\varepsilon_k \in \mathrm{CH}^6(\bar{V})$ is an integral representative of $\mathrm{St}^k(\alpha_k) \in \mathrm{Ch}^6(\bar{V})$ which in the case of $k > 6-k$ we choose to be 0 taking into account that $\mathrm{St}^k(\alpha_k) = 0$ for such k because $\alpha_k \in \mathrm{Ch}^{6-k}(Q \times V)$. (So, the above sum over $k \geqslant 0$ runs up to $k = 3$ only.) Besides, we choose $\varepsilon_0 = \alpha_0$.

For $i = d-6$, the ith summand is

$$\begin{aligned} pr_*(h^4 s^6) &= pr_*\Big(h^4 \cdot (h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^5 \times \alpha_5 + l_5 \times \beta)^2\Big) \\ &\equiv pr_*\Big(h^4 \cdot \big(2(h^1 \times \alpha_1) \cdot (l_5 \times \beta) + (h^3 \times \alpha_3)^2\big)\Big) = 2\alpha_1\beta + 2\alpha_3^2, \end{aligned}$$

where the congruence is modulo $4\,\mathrm{CH}^6(\bar{V})$.

There is no need to compute the last remaining summand $pr_*(h^5 s^5)$ which occurs for $i = d-5$.

With the above computations in hand, we are going to use (5.7) for the following two values of d: for $d = 7$ and for $d = 5$. For the first choice of d, since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i = 0, 1, \ldots, d$ (the binomial coefficient $\binom{-d-2}{i} = \binom{d+1+i}{i}$ is easy to compute modulo 2 using [4, Lemma 78.6]), we get that

$$2\alpha_1\beta + 2\alpha_3^2 + pr_*(h^5 s^5) + 2\sum_{i=3}^{7} \sum_{k=0}^{3} \binom{k}{7-i-k} \varepsilon_k \equiv 2a \pmod{4\,\mathrm{CH}^6(\bar{V})}$$

for some $a \in \overline{\mathrm{CH}}^6(\bar{V})$. For $k < 3$, the coefficient at ε_k is twice the sum of all binomial coefficients $\binom{k}{\cdot}$ and therefore is divisible by 4 for $0 < k < 3$. The coefficient at ε_0 is 2. The coefficient at ε_3 is twice the sum of all binomial coefficients $\binom{3}{\cdot}$ except $\binom{3}{3} = 1$ and $\binom{3}{2} = 1$ and so is congruent to 0 modulo 4. Therefore the congruence we get with the first choice of d is

$$2\alpha_1\beta + 2\alpha_3^2 + pr_*(h^5s^5) + 2\varepsilon_0 \equiv 2a \pmod{4\,\mathrm{CH}^6(\bar{V})}. \tag{5.8}$$

For the second choice of d (namely, $d = 5$), the binomial coefficient $\binom{-d-2}{i}$ with $i \in \{0, 1, \dots, d\}$ is odd for $i \in \{0,\ 1\}$ and is even for $i \in \{2,\ 3,\ 4,\ 5\}$. Since $d - 6 = -1$, the term with $i = d - 6$ does not show up and we get that

$$pr_*(h^5s^5) + 2\sum_{k=0}^{3}\binom{k}{4-k}\varepsilon_k \equiv 2b \pmod{4\,\mathrm{CH}^6(\bar{V})}$$

for some $b \in \overline{\mathrm{CH}}^6(\bar{V})$. Therefore the congruence we obtain with our second choice of d is

$$pr_*(h^5s^5) + 2\varepsilon_2 + 2\varepsilon_3 \equiv 2b \pmod{4\,\mathrm{CH}^6(\bar{V})}. \tag{5.9}$$

Adding together (5.8) with (5.9) and taking into account that

$$\alpha_3^2 \equiv \varepsilon_3 \pmod{2\,\mathrm{CH}^6(\bar{V})},$$

we get

$$2\alpha_1\beta + 2\,pr_*(h^5s^5) + 2\varepsilon_0 + 2\varepsilon_2 \equiv 2(a+b) \pmod{4\,\mathrm{CH}^6(\bar{V})}.$$

Dividing by 2 (we have the right to do so because the group $\mathrm{CH}^6(\bar{V})$ is free of torsion), we obtain the congruence

$$\alpha_1\beta + \varepsilon_0 + \varepsilon_2 \equiv a + b - pr_*(h^5s^5) \pmod{2\,\mathrm{CH}^6(\bar{V})} \tag{5.10}$$

giving the statement of Proposition 5.6 because $a + b - pr_*(h^5s^5) \in \overline{\mathrm{CH}}^6(V)$. □

Remark 5.11. The term $pr_*(h^5s^5)$ in (5.10) can be omitted because it belongs to $2\,\mathrm{CH}^6(\bar{V})$ as follows from (5.8) as well as from (5.9).

6. ON THE u-INVARIANT 9

Existence of fields of u-invariant 9, proved originally in [5] and reproved by a different method later in [21], follows from Theorem 6.1 here below also used here above in the proof of Theorem 5.4. In characteristic 0, Theorem 6.1 can be deduced from [21, Theorem 5.1], where the reason for the characteristic 0 assumption is [21, Proposition 3.5] whose proof makes use of the symmetric operations in algebraic cobordism [20] and thus requires resolution of singularities. Proposition 6.3 below is the same statement with a modified proof, which only makes use of Steenrod operations on modulo 2 Chow groups (available in any characteristic: see [4, Chapter XI] and [18]).

Theorem 6.1. *Let φ be a non-degenerate quadratic form over a field F of dimension* 9*. Let ψ be a non-degenerate quadratic form over F of dimension* 10*. Assume that φ satisfies (5.2) and (5.3) (with $n = 4$). Then $\varphi_{F(\psi)}$ also satisfies (5.2) and (5.3).*

Proof. We may assume that ψ is anisotropic and we only need to check that $\varphi_{F(\psi)}$ satisfies (5.3). Assume that it doesn't, i.e., $e_5 \in \overline{\mathrm{Ch}}((X_3)_{F(Y_1)}$, where Y_1 is the quadric of ψ. Lifting e_5 to the product $Y_1 \times X_3$, we get some $\alpha \in \overline{\mathrm{Ch}}(Y_1 \times X_3)$. We have

$$\alpha = h^0 \times e_5 + h^1 \times \alpha_1 + \cdots + h^4 \times \alpha_4 + l_4 \times \beta + al_3 \times [X_3]$$

for some $\alpha_1, \ldots, \alpha_4, \beta \in \mathrm{Ch}(\bar{X}_3)$ and $a \in \mathbb{Z}/2\mathbb{Z}$.

Here is the plan:

- (i) Using Proposition 4.1, we reduce to the case with $a = 0$. (As explained in [21, Proof of Theorem 5.1], this is "the most delicate part of the proof".)
- (ii) Using Proposition 3.1, we narrow the shape of $\alpha_1, \ldots, \alpha_4$.
- (iii) Using the shape of α, obtained in (ii), along with Proposition 6.3, we show that $e_5 \in \overline{\mathrm{Ch}}(X_3)$ achieving a contradiction.

(i) For ψ of nontrivial discriminant, execution of (i) is instant: if $a \neq 0 \in \mathbb{Z}/2\mathbb{Z}$, then the image of α in $\overline{\mathrm{Ch}}((Y_1)_{F(X_3)}$ equals l_3 so that $\mathrm{ind}(\psi)_{F(X_3)} \geqslant 4$ contradicting Proposition 4.1.

For ψ of trivial discriminant, if $a \neq 0$, we have $l_4 \in \overline{\mathrm{Ch}}((Y_1)_{F(X_3)}$. Lifting this element to the group $\overline{\mathrm{Ch}}(Y_1 \times X_3)$, we get some α' of the form

$$\alpha' = h^0 \times \alpha'_0 + h^1 \times \alpha'_1 + \cdots + h^3 \times \alpha'_3 + l_4 \times [X_3].$$

Subtracting from α the product

$$(h^1 \times [X_3]) \cdot \alpha' = h^1 \times \alpha'_0 + h^2 \times \alpha'_1 + \cdots + h^4 \times \alpha'_3 + l_3 \times [X_3] \in \overline{\mathrm{Ch}}(Y_1 \times X_3),$$

we get rid of the term $l_3 \times [X_3]$ in α.

(ii) By Index Reduction Formula,

$$\mathrm{ind}\, C_0((\varphi)_{F(Y_i)}) \geqslant 2^{4-i} \quad \text{for } i \in \{1,\ 2,\ 3,\ 4\}.$$

Viewing α as a correspondence $Y_1 \rightsquigarrow X_3$ and applying it to $l_i \in \overline{\mathrm{Ch}}((Y_1)_{F(Y_{i+1})})$, we get

$$\alpha_i \in \overline{\mathrm{Ch}}^{5-i}((Y_1)_{F(Y_{i+1})}).$$

Therefore by Proposition 3.1 we may assume that $\alpha_i = a_i e_{5-i}$ for $a_i \in \mathbb{Z}/2\mathbb{Z}$.

(iii) By Proposition 5.6 (in characteristic 0 one may refer to the original [21, Proposition 3.5]), we have

$$e_5 + \mathrm{St}^1(\alpha_1) + \alpha_1\beta \in \overline{\mathrm{Ch}}(X_3). \tag{6.2}$$

By Proposition 3.1, $ae_5 \in \overline{\mathrm{Ch}}(X_3)$, where a is the coefficient at e_5 in (6.2). The third summand does not provide any contribution to a. (Note that β is a multiple of c_1 because c_1 generates the group $\mathrm{Ch}^1(\bar{X}_3) \ni \beta$.) By [21, Proposition 2.9], the contribution of $\mathrm{St}^1(e_4)$ is also trivial. Therefore $a = 1$ and $e_5 \in \overline{\mathrm{Ch}}(X_3)$ as desired. □

The following proposition, used in the above proof, can potentially be used to construct fields of u-invariants congruent to 1 modulo 4 provided that the hypothesis on a can be satisfied. It is a remake of [21, Proposition 3.5] in the spirit of [10, Proposition 5.3]:

Proposition 6.3. *Let Q be a smooth projective quadric of dimension $2n$ over a field F for some even $n \geqslant 2$. Let V be a projective homogeneous variety over F. Let $\alpha \in \mathrm{Ch}^{n+1}(Q \times V)$ be an element such that in the decomposition*

$$\bar{\alpha} = h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^n \times \alpha_n + l_n \times \beta + a l_{n-1} \times [V]$$

of its image $\bar{\alpha} \in \mathrm{Ch}^{n+1}(\bar{Q} \times \bar{V})$ with $\alpha_i \in \mathrm{Ch}^{n+1-i}(\bar{V})$, $\beta \in \mathrm{Ch}^1(\bar{V})$, and $a \in \mathbb{Z}/2\mathbb{Z}$, the element a is trivial.

Then $\alpha_0 + \mathrm{St}^1(\alpha_1) + \alpha_1\beta \in \overline{\mathrm{Ch}}^{n+1}(V)$.

Proof. For every $i = 0, 1, \ldots, n$, let s^i be the image in $\mathrm{CH}^{n+1+i}(\bar{Q} \times \bar{V})$ of an element of $\mathrm{CH}^{n+1+i}(Q \times V)$ representing $\mathrm{St}^i(\alpha) \in \mathrm{Ch}^{n+1+i}(Q \times V)$. We also set $s^i := 0$ for $i > n+1$ as well as for $i < 0$. Finally, we set $s^{n+1} := (s^0)^2$.

Note that we have

$$s^0 := h^0 \times \boldsymbol{\alpha}_0 + h^1 \times \boldsymbol{\alpha}_1 + \cdots + h^n \times \boldsymbol{\alpha}_n + l_n \times \boldsymbol{\beta} + b l_{n-1} \times [\bar{V}] \in \mathrm{CH}^{n+1}(\bar{Q} \times \bar{V})$$

with some $\boldsymbol{\alpha}_i \in \mathrm{CH}^{n+1-i}(\bar{V})$ for $i = 0, 1, \ldots, n$, $\boldsymbol{\beta} \in \mathrm{CH}^1(\bar{V})$, and an integer b. Since

$$s^0 \mod 2\,\mathrm{CH}^{n+1}(\bar{Q} \times \bar{V}) = \bar{\alpha},$$

the integer b is even. Since the element $2l_{n-1} = h^{n+1}$ is in $\overline{\mathrm{CH}}^{n+1}(Q) \subset \mathrm{CH}^{n+1}(\bar{Q})$, we remove the last summand from the above decomposition of s^0.

Let d be any integer with $n \leqslant d \leqslant 2n$. For a d-dimensional smooth subquadric P of Q, writing in for the imbedding $P \times Y \hookrightarrow Q \times Y$ and applying [4, Theorem 61.9 and Proposition 61.10]), we have

$$pr^P_* \sum_{i=d-n-1}^{d} c_i(-T_P)\, in^* \,\mathrm{St}^{d-i} \alpha = \mathrm{St}^d\, pr^P_*\, in^* \alpha \in \mathrm{Ch}^{n+1}(V),$$

where T_P is the tangent bundle on P and pr^P is the projection $P \times V \to V$. The summation on the left side is from $d-n-1$ to d only because $\mathrm{St}^{d-i} \alpha = 0$ outside of this interval, see [4, Theorem 61.13]. The right side is actually zero since

$$\mathrm{St}^d\, pr^P_*\, in^* \alpha \in \mathrm{St}^d\, \mathrm{Ch}^{n+1-d}(V)$$

and $\mathrm{St}^d\, \mathrm{Ch}^{n+1-d}(V) = 0$ because $d > 1 \geqslant n+1-d$. Rewriting the left side with a help of the projection formula [4, Proposition 56.9], we obtain the relation

$$pr_* \sum_{i=d-n-1}^{d} in_*(c_i(-T_P))\, \mathrm{St}^{d-i} \alpha = 0 \in \mathrm{Ch}^{n+1}(V),$$

where pr is the projection $Q \times V \to V$. By the computation [4, Lemma 78.1] of the Chern classes of the tangent bundle, it follows that

$$\sum_{i=d-n-1}^{d} \binom{-d-2}{i} \cdot pr_*(h^{2n-d+i} \cdot s^{d-i}) \in 2\,\overline{\mathrm{CH}}^{n+1}(V) \subset \mathrm{CH}^{n+1}(\bar{V}). \tag{6.4}$$

We are going to use (6.4) for various values of d. Note that when d varies, the sum showing up in (6.4) is a linear combination of always the same elements

$$pr_*(h^{n-1}s^{n+1}),\ pr_*(h^n s^n),\ \ldots,\ pr_*(h^{2n}s^0) \in \overline{\mathrm{CH}}^{n+1}(V).$$

However, the coefficients of this linear combination vary with d.

Let us compute the ith element $pr_*(h^{2n-d+i}\cdot s^{d-i}) \in \mathrm{CH}^{n+1}(\bar{V})$ modulo $4\,\mathrm{CH}^{n+1}(\bar{V})$. For $i \geqslant d-n+1$, we have $2n-d+i \geqslant n+1$ so that the factor $h^{2n-d+i} \in \mathrm{CH}(\bar{Q})$ is divisible by 2. The other factor modulo 2 is $\mathrm{St}^{d-i}(\bar{\alpha})$ and it follows that

$$pr_*(h^{2n-d+i}\cdot s^{d-i}) \equiv 2\sum_{k\geqslant 0}\binom{k}{d-i-k}\varepsilon_k \pmod 4 \qquad \text{for } i \geqslant d-n+1,$$

where $\varepsilon_k \in \mathrm{CH}^{n+1}(\bar{V})$ is an integral representative of $\mathrm{St}^k(\alpha_k) \in \mathrm{Ch}^{n+1}(\bar{V})$ which in the case of $k > n+1-k$ we choose to be 0 taking into account that $\mathrm{St}^k(\alpha_k) = 0$ for such k because $\alpha_k \in \mathrm{Ch}^{n+1-k}(Q\times V)$. So, the sum over k runs up to $n/2$ only (recall that n is even). Besides, we choose $\varepsilon_0 = \alpha_0$.

For $i = d-n-1$, the ith summand is

$$\begin{aligned} pr_*(h^{n-1}s^{n+1}) &= pr_*\Big(h^{n-1}\cdot(h^0\times\alpha_0 + h^1\times\alpha_1 + \cdots + h^n\times\alpha_n + l_n\times\beta)^2\Big) \\ &\equiv 2\,pr_*\Big(h^{n-1}\cdot(h^1\times\alpha_1)\cdot(l_n\times\beta)\Big) = 2\alpha_1\beta, \end{aligned}$$

where the congruence is modulo $4\,\mathrm{CH}^{n+1}(\bar{V})$. Here again we use the assumption that n is even: for odd n the answer would be $2\alpha_1\beta + 2\alpha^2_{(n+1/2)}$.

There is no need to compute the last remaining summand $pr_*(h^ns^n)$ which occurs for $i = d-n$.

With the above computations in hand, we are going to use (6.4) for the following two values of d: for $d = 2^r-1$ and for $d = 2^r$, where $2^r \leqslant 2n$ is the highest 2-power not exceeding $2n$. For the first choice of d, since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i = 0,1,\dots,d$, we get that

$$2\alpha_1\beta + pr_*(h^ns^n) + 2\sum_{i=d-n+1}^{d}\sum_{k=0}^{n/2}\binom{k}{d-i-k}\varepsilon_k \equiv 2a \pmod{4\,\mathrm{CH}^{n+1}(\bar{V})}$$

for some $a \in \overline{\mathrm{CH}}^{n+1}(\bar{V})$. For $k < n/2$, the coefficient at ε_k is twice the sum of all binomial coefficients $\binom{k}{\cdot}$ and therefore is divisible by 4 for $0 < k < n/2$. The coefficient at ε_0 is 2. The coefficient at $\varepsilon_{n/2}$ is twice the sum of all binomial coefficients $\binom{n/2}{\cdot}$ except $\binom{n/2}{n/2} = 1$ and so is congruent to 2 modulo 4. Therefore the congruence we get with the first choice of d is

$$2\alpha_1\beta + pr_*(h^ns^n) + 2\varepsilon_0 + 2\varepsilon_{n/2} \equiv 2a \pmod{4\,\mathrm{CH}^{n+1}(\bar{V})}. \tag{6.5}$$

For the second choice of d (namely, $d = 2^r$), the binomial coefficient $\binom{-d-2}{i}$ with $i \in \{0,1,\dots,d\}$ is odd for even $i < d$ and is even for the remaining i. Since the integer $d-n-1$ is odd, we get that

$$pr_*(h^ns^n) + 2\sum_{\substack{i=d-n+2\\ i \text{ is even}}}^{d-2}\sum_{k=0}^{n/2}\binom{k}{d-i-k}\varepsilon_k \equiv 2b \pmod{4\,\mathrm{CH}^{n+1}(\bar{V})}$$

for some $b \in \overline{\mathrm{CH}}^{n+1}(\bar{V})$.

The coefficient at ε_0 is 0 here. Since for any $k \geqslant 1$, we have

$$\sum_{\text{even } l}\binom{k}{l} = 2^{k-1} = \sum_{\text{odd } l}\binom{k}{l},$$

only the coefficients at ε_1 and $\varepsilon_{n/2}$ survive modulo 4 (the coefficient at $\varepsilon_{n/2}$ survives because the binomial coefficient $\binom{n/2}{n/2}$ is missing). Therefore the congruence we get with our second choice of d is

$$pr_*(h^n s^n) + 2\varepsilon_1 + 2\varepsilon_{n/2} \equiv 2b \pmod{4\,\mathrm{CH}^{n+1}(\bar{V})}. \tag{6.6}$$

Adding together (6.5) and (6.6), we get

$$2\alpha_1\beta + 2\,pr_*(h^n s^n) + 2\varepsilon_0 + 2\varepsilon_1 \equiv 2(a+b) \pmod{4\,\mathrm{CH}^{n+1}(\bar{V})}.$$

Dividing by 2 (we have the right to do so because the group $\mathrm{CH}^{n+1}(\bar{V})$ is free of torsion), we obtain the congruence

$$\alpha_1\beta + \varepsilon_0 + \varepsilon_1 \equiv a + b - pr_*(h^n s^n) \pmod{2\,\mathrm{CH}^{n+1}(\bar{V})} \tag{6.7}$$

giving the statement of Proposition 5.6 because $a + b - pr_*(h^n s^n) \in \overline{\mathrm{CH}}^{n+1}(V)$. $\square$

Remark 6.8. The term $pr_*(h^n s^n)$ in (6.7) can be omitted because it belongs to $2\,\mathrm{CH}^{n+1}(\bar{V})$ as follows from (6.5) as well as from (6.6).

References

[1] Brosnan, P. Steenrod operations in Chow theory. *Trans. Amer. Math. Soc. 355*, 5 (2003), 1869–1903 (electronic).

[2] Brosnan, P. On motivic decompositions arising from the method of Białynicki-Birula. *Invent. Math. 161*, 1 (2005), 91–111.

[3] Buch, A. S., Kresch, A., and Tamvakis, H. Quantum Pieri rules for isotropic Grassmannians. *Invent. Math. 178*, 2 (2009), 345–405.

[4] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[5] Izhboldin, O. T. Fields of u-invariant 9. *Ann. of Math. (2) 154*, 3 (2001), 529–587.

[6] Kaplansky, I. Quadratic forms. *J. Math. Soc. Japan 5* (1953), 200–207.

[7] Karpenko, N. A. On topological filtration for Severi-Brauer varieties. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, vol. 58 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1995, pp. 275–277.

[8] Karpenko, N. A. Izhboldin's results on stably birational equivalence of quadrics. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 151–183.

[9] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math. 677* (2013), 179–198.

[10] Karpenko, N. A. Variations on a theme of rationality of cycles. *Cent. Eur. J. Math. 11*, 6 (2013), 1056–1067.

[11] Karpenko, N. A. On generic quadratic forms. *Pacific J. Math. 297*, 2 (2018), 367–380.

[12] Lam, T. Y. *Introduction to quadratic forms over fields*, vol. 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[13] Levine, M., and Morel, F. *Algebraic cobordism*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[14] Merkurjev, A. S. Kaplansky's conjecture in the theory of quadratic forms. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 175* (1989), 75–89, 163–164.

[15] Merkurjev, A. S. Simple algebras and quadratic forms. *Izv. Akad. Nauk SSSR Ser. Mat. 55*, 1 (1991), 218–224.

[16] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *K-Theory 10*, 6 (1996), 517–596.

[17] Panin, I. A. On the algebraic K-theory of twisted flag varieties. *K-Theory 8*, 6 (1994), 541–585.
[18] Primozic, E. Motivic Steenrod operations in characteristic p. *Forum Math. Sigma 8* (2020), Paper No. e52, 25.
[19] Vishik, A. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, vol. 1835 of *Lecture Notes in Math.* Springer, Berlin, 2004, pp. 25–101.
[20] Vishik, A. Symmetric operations in algebraic cobordism. *Adv. Math. 213*, 2 (2007), 489–552.
[21] Vishik, A. Fields of u-invariant $2^r + 1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.
[22] Voevodsky, V. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.*, 98 (2003), 1–57.

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