FIELDS OF ANY u-INVARIANT BUT A 2-POWER MINUS 1 AND 3

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Abstract. The u-invariant of a field is the highest dimension of a non-degenerate anisotropic quadratic form over this field. Let u be a natural number not of the form a 2-power minus 1 or a 2-power minus 3. Given any field, we find its overfield of the u-invariant u.

Contents

1. Introduction

The u-invariant of a field (which is defined as the highest dimension of a non-degenerate anisotropic quadratic form over the field), takes the value 1 (on, say, any algebraically closed field) and never takes the values $3, 5, 7$. Due to Alexander Merkurjev, it is known since the last century that all even natural numbers are taken, see [1, Theorem 38.4]. Here we show (see Theorem 1.1) that for any natural number u which is neither a 2-power minus 1 nor a 2-power minus 3, there exists a field of u-invariant u. Aside from Merkurjev's result just cited, this extends a recent author's result [3] on $u = 11$ along with a 20 years old result [5] of Alexander Vishik on $u = 2^r + 1$, the latter being extending a previous result [2] on $u = 9$ by Oleg Izhboldin.

The core result here is Theorem 5.2 which for odd $u \geqslant 17$ we deduce from Proposition 4.3, whose proof goes by induction on u starting with the already available base of $u = 9$ and $u = 11$.

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The techniques showing up are mostly in the spirit of [5] and [3]; we refer to [3] for more detailed description of the tools involved, the most important one being the Steenrod operations on the modulo 2 Chow groups; for their construction and properties in characteristic not 2 and 2 we refer to [1, Chapter XI] and [4] respectively. In contrast to [5], Index Reduction Formula for quadrics [1, Theorem 30.5], which was the main tool for constructing fields of even u-invariants, is also multiply applied.

By a standard argument (as in [1, §38] or in [5, §4]), Theorem 5.2 implies (for odd u) the title result:

Theorem 1.1. *For any given field F_0 and for any natural number u not of the form a 2-power minus 1 or a 2-power minus 3, there is an overfield $F \supset F_0$ of the u-invariant u.*

Notation and results we are using are introduced "on the go".

2. Isotropic forms

Here we prove Proposition 2.2 which will be used later on to prove Proposition 4.3 and Theorem 5.2. It has also been already used during step (ii) in the proof of [3, Theorem 5.4].

Let F be a field and let V be a vector F-space of odd dimension $2n+1$. Let $\varphi\colon V \to F$ be a non-degenerate isotropic quadratic form over F. We fix an isotropic line $L \subset V$ and let φ' be the quadratic form on $V' := L^\perp/L$ induced by φ, where $L^\perp$ is the orthogonal complement of L with respect to the polar bilinear form of φ. Thus φ' is a Witt-equivalent to φ non-degenerate quadratic form of dimension $2n-1$ over F.

For $m \in \{0, \ldots, n\}$, let X_m be the mth grassmannian of φ. In particular, $X_0 = \operatorname{Spec} F$ and X_1 is the projective quadric of φ.

For $m \in \{1, \ldots, n\}$, let $X_m^1 \subset X_m$ be the closed subvariety of m-dimensional totally isotropic subspaces in $L^\perp$ and let $X_m^2 \subset X_m^1$ be the closed subvariety of those of them that contain L. The variety X_m^2 is identified with the $(m-1)$st grassmannian X'_{m-1} of φ' via $U \mapsto U/L$.

Let us consider the homomorphism of graded groups

$$i^*\colon \operatorname{CH}^*(X_m) \to \operatorname{CH}^*(X'_{m-1})$$

given by the pull-back with respect to the closed embedding

$$i\colon X'_{m-1} = X_m^2 \hookrightarrow X_m,$$

or, in slightly different terms, induced by the correspondence

$$\iota^t\colon X_m \rightsquigarrow X'_{m-1}$$

given by the transpose of the graph ι of i.

The incidence correspondence $\alpha\colon X'_1 \rightsquigarrow X_1$ is defined as the variety of pairs $(U'/L, U)$ of totally isotropic lines in V' and V with $U \subset U'$. By [1, Lemma 72.3], the homomorphism

$$\alpha^*\colon \operatorname{CH}(\bar{X}_1) \to \operatorname{CH}(\bar{X}'_1),$$

given by α^t, maps l_i to l'_{i-1} for all i, where $l'_{-1} := 0$. The notation $\bar{X}_1$ stands for the variety X_1 with scalars extended to an algebraic closure of the base field; $l_i \in \operatorname{CH}_i(\bar{X}_1)$ (and $l'_i \in \operatorname{CH}_i(\bar{X}'_1)$) is the class of an i-dimensional projective space lying on $\bar{X}_1$ (respectively on $\bar{X}'_1$).

The incidence correspondence $\gamma\colon X_1 \rightsquigarrow X_m$ is defined as the variety of pairs (U_1, U_m) with $U_1 \subset U_m$. The homomorphism

$$\gamma_*\colon \mathrm{CH}(\bar{X}_1) \to \mathrm{CH}(\bar{X}_m)$$

maps the elements $l_{n-1}, \dots, l_0$ respectively to $e_{n-m+1}, \dots, e_{2n-m}$. This is the definition of the latter set of elements; they are indexed by codimension: $e_i \in \mathrm{CH}^i(\bar{X}_m)$ for all i, whereas the elements of the former set are indexed by their dimensions: $l_i \in \mathrm{CH}_i(\bar{X}_1)$.

Similarly, the homomorphism

$$\gamma'_*\colon \mathrm{CH}(\bar{X}'_1) \to \mathrm{CH}(\bar{X}'_{m-1})$$

induced by the incidence correspondence $\gamma'\colon X_1 \rightsquigarrow X_m$ maps the elements $l'_{n-2}, \dots, l'_0$ respectively to $e'_{n-m+1}, \dots, e'_{2n-m-1}$.

Lemma 2.1 (c.f. [1, Lemma 86.7])**.** *For any $m \in \{1, \dots, n\}$, the square of correspondences*

$$\begin{array}{ccc} X_1 & \overset{\alpha^t}{\rightsquigarrow} & X'_1 \\ \gamma \wr\downarrow & & \wr\downarrow \gamma' \\ X_m & \overset{\iota^t}{\rightsquigarrow} & X'_{m-1} \end{array}$$

is commutative.

Lemma 2.1 results in the following proposition which is [1, Corollary 86.8] if $m = n$:

Proposition 2.2. *For any $m \in \{2, \dots, n\}$, the homomorphism*

$$i^*\colon \mathrm{CH}^*(\bar{X}_m) \to \mathrm{CH}^*(\bar{X}'_{m-1})$$

maps the elements $e_{n-m+1}, \dots, e_{2n-m-1}$ to $e'_{n-m+1}, \dots, e'_{2n-m-1}$ (whereas $e_{2n-m} \mapsto 0$). □

3. A computation

For a smooth projective quadric Q over a field F of dimension $2n$ or $2n+1$, recall (from, e.g., [1]) that the Chow group $\mathrm{CH}(\bar{Q})$ of the variety $\bar{Q}$ (which is Q with the scalars extended to an algebraic closure of F) is free with a basis

$$h^0, \dots, h^n, l_n, \dots, l_0.$$

The element $h \in \mathrm{CH}^1(\bar{Q})$ is the hyperplane section class, $h^i \in \mathrm{CH}^i(\bar{Q})$ is its ith power, and $l_i \in \mathrm{CH}_i(\bar{Q})$ is the class of an i-dimensional projective space lying on $\bar{Q}$.

We recall that by $\bar{X}$ for a smooth quasi-projective F-variety X, we mean X with the base field extended to an algebraic closure of F. We write $\mathrm{Ch}(X) := \mathrm{CH}(X)/2\,\mathrm{CH}(X)$ for the modulo 2 Chow group of X; $\overline{\mathrm{Ch}}(X)$ stands for the image of the change of field homomorphism $\mathrm{Ch}(X) \to \mathrm{Ch}(\bar{X})$; similarly, $\overline{\mathrm{CH}}(X)$ is the image of $\mathrm{CH}(X) \to \mathrm{CH}(\bar{X})$.

For $i \geqslant 0$, we work with the cohomological Steenrod operations

$$\mathrm{St}^i\colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^{*+i}(X)$$

on the modulo 2 Chow groups. Their history, construction, and basic properties are explained in [1, Chapter XI] for characteristic not 2 and in [4] for characteristic 2.

Proposition 3.1. *For an integer $n \geqslant 4$ such that neither $n+1$ nor $n+2$ is a 2-power, let Q be a smooth projective quadric of dimension $2n$ over a field F. Let V be a projective homogeneous variety over F. For an element $\alpha \in \operatorname{Ch}^{n+1}(Q \times V)$, we consider the decomposition*

$$\bar{\alpha} = h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^n \times \alpha_n + l_n \times \beta + a l_{n-1} \times [V]$$

of its image $\bar{\alpha} \in \operatorname{Ch}^{n+1}(\bar{Q} \times \bar{V})$ with $\alpha_i \in \operatorname{Ch}^{n+1-i}(\bar{V})$, $\beta \in \operatorname{Ch}^1(\bar{V})$, and $a \in \mathbb{Z}/2\mathbb{Z}$. Then

$$\alpha_0 + \sum_{i=1}^{[(n+1)/2]} a_i \operatorname{St}^i(\alpha_i) \in \overline{\operatorname{Ch}}^{n+1}(V) \tag{3.2}$$

for some $a_i \in \mathbb{Z}/2\mathbb{Z}$, where $[(n+1)/2]$ is the integral part of $(n+1)/2$.

Proof. For every $i \geqslant 0$, let s^i be the image in $\operatorname{CH}^{n+1+i}(\bar{Q} \times \bar{V})$ of an element of the group $\operatorname{CH}^{n+1+i}(Q \times V)$ representing the value $\operatorname{St}^i(\alpha) \in \operatorname{Ch}^{n+1+i}(Q \times V)$ of the Steenrod operation. We will refer to these elements s^i later in the proof.

For $d := 2^r - 1$, where 2^r is the highest 2-power not exceeding n, we choose a d-dimensional smooth subquadric P of Q. Writing in for the imbedding $P \times Y \hookrightarrow Q \times Y$ and applying [1, Theorem 61.9 and Proposition 61.10] (along with their characteristic 2 counterparts from [4]), we have

$$pr^P_* \sum_{i=0}^{d} c_i(-T_P)\, in^* \operatorname{St}^{d-i} \alpha = \operatorname{St}^d pr^P_*\, in^* \alpha \in \operatorname{Ch}^{n+1}(V), \tag{3.3}$$

where T_P is the tangent bundle on P and pr^P is the projection $P \times V \to V$. The summation on the left side in (3.3) is from 0 to d only because $c_i(-T_P) = 0$ for $i < 0$ whereas $\operatorname{St}^{d-i} \alpha = 0$ for $i > d$. If $d > n+1-d$, the right side in (3.3) vanishes because

$$\operatorname{St}^d pr^P_*\, in^* \alpha \in \operatorname{St}^d \operatorname{Ch}^{n+1-d}(V)$$

and $\operatorname{St}^d \operatorname{Ch}^{n+1-d}(V) = 0$ (see [1, Theorem 61.13] and [4, Corollary 7.5]). However the relation $d > n+1-d$ fails if (and only if) n has the highest possible value with respect to $d = 2^r - 1$. Since n is not a 2-power minus 1 / minus 2, the highest possible value of n is $n = 2^{r+1} - 3$, in which case $d = n+1-d$ and the operation St^d on $\operatorname{Ch}^{n+1-d}(V)$ is the squaring (by [1, Theorem 61.13] and [4, Corollary 7.3]).

Rewriting the left side of (3.3) with a help of the projection formula [1, Proposition 56.9], we obtain

$$pr_* \sum_{i=0}^{d} in_*(c_i(-T_P)) \operatorname{St}^{d-i} \alpha,$$

where pr is the projection $Q \times V \to V$. By the computation [1, Lemma 78.1] of the Chern classes of the tangent bundle, it follows from (3.3) that

$$\sum_{i=0}^{d} \binom{-d-2}{i} \cdot pr_*(h^{2n-d+i} \cdot s^{d-i}) \in 2\,\overline{\operatorname{CH}}^{n+1}(V) \subset \operatorname{CH}^{n+1}(\bar{V}) \tag{3.4}$$

provided that $n \neq 2^{r+1} - 3$. For $n = 2^{r+1} - 3$, we obtain relation (3.4) with $2\,\overline{\operatorname{CH}}^{n+1}(V)$ on the right replaced by $4\operatorname{CH}^{n+1}(\bar{V})$. So, in both cases the following combined relation holds:

$$\sum_{i=0}^{d} \binom{-d-2}{i} \cdot pr_*(h^{2n-d+i} \cdot s^{d-i}) \in 2\,\overline{\mathrm{CH}}^{n+1}(V) + 4\,\mathrm{CH}^{n+1}(\bar V) \subset \mathrm{CH}^{n+1}(\bar V). \tag{3.5}$$

The sum showing up in (3.5) is a linear combination of the elements

$$pr_*(h^{2n-d}s^d),\ pr_*(h^{2n-d+1}s^{d-1}),\ \dots,\ pr_*(h^{2n}s^0) \in \overline{\mathrm{CH}}^{n+1}(V) \subset \mathrm{CH}^{n+1}(\bar V).$$

Let us compute the ith element $pr_*(h^{2n-d+i}\cdot s^{d-i}) \in \mathrm{CH}^{n+1}(\bar V)$ modulo $4\,\mathrm{CH}^{n+1}(\bar V)$. Since $2n-d+i > n$, the factor $h^{2n-d+i} \in \mathrm{CH}(\bar Q)$ is divisible by 2. The other factor modulo 2 is $\mathrm{St}^{d-i}(\bar\alpha)$ and it follows that

$$pr_*(h^{2n-d+i} \cdot s^{d-i}) \equiv 2\sum_{k\geqslant 0} \binom{k}{d-i-k} \varepsilon_k \pmod{4\,\mathrm{CH}^{n+1}(\bar V)},$$

where $\varepsilon_k \in \mathrm{CH}^{n+1}(\bar V)$ is an integral representative of $\mathrm{St}^k(\alpha_k) \in \mathrm{Ch}^{n+1}(\bar V)$ which in the case of $k > n+1-k$ we choose to be 0 taking into account that $\mathrm{St}^k(\alpha_k) = 0$ for such k because $\alpha_k \in \mathrm{Ch}^{n+1-k}(\bar Q \times \bar V)$. So, the sum over k runs up to the integral part of $(n+1)/2$ only.

Since the binomial coefficient $\binom{-d-2}{i}$ is odd for every $i \in \{0,1,\dots,d\}$ (the binomial coefficient $\binom{-d-2}{i} = \binom{d+1+i}{i}$ is easy to compute modulo 2 using [1, Lemma 78.6]), we deduce from (3.5) that

$$2\sum_{i=0}^{d} \sum_{k=0}^{[(n+1)/2]} \binom{k}{d-i-k} \varepsilon_k \equiv 2\gamma \pmod{4\,\mathrm{CH}^{n+1}(\bar V)} \tag{3.6}$$

for some $\gamma \in \overline{\mathrm{CH}}^{n+1}(V)$. The coefficient at ε_0 is 2. Therefore

$$2\varepsilon_0 + 2\sum_{k=1}^{[(n+1)/2]} a_k\varepsilon_k \equiv 2\gamma \pmod{4\,\mathrm{CH}^{n+1}(\bar V)}$$

for some integers a_k. Dividing by 2 (we have the right to divide by 2 because the group $\mathrm{CH}^{n+1}(\bar V)$ for a projective homogeneous variety V is free of torsion), we obtain the congruence

$$\varepsilon_0 + \sum_{k=1}^{[(n+1)/2]} a_k\varepsilon_k \equiv \gamma \pmod{2\,\mathrm{CH}^{n+1}(\bar V)}$$

meaning that

$$\alpha_0 + \sum_{k=1}^{[(n+1)/2]} \mathrm{St}^k\,\alpha_k \quad \in \quad \overline{\mathrm{Ch}}^{n+1}(V). \qquad \square$$

4. Highest Schur index

Given a non-degenerate quadratic form φ of dimension $2n+1$ over a field F, we consider the following two conditions:

$$\operatorname{ind} C_0(\varphi) = 2^n, \tag{4.1}$$

where $\operatorname{ind} C_0(\varphi)$ is the Schur index of the even Clifford algebra $C_0(\varphi)$ of φ (note that the index always is a 2-power, 2^n is its highest possible value for quadratic forms of such dimension), and

$$e_{n+1} \notin \overline{\mathrm{Ch}}^{n+1}(X_{n-1}), \tag{4.2}$$

where X_{n-1} is the almost highest grassmannian of φ and $e_{n+1} \in \mathrm{Ch}^{n+1}(\bar{X}_{n-1})$ is the defined in §2 element, corresponding to the rational point class $l_0 \in \overline{\mathrm{Ch}}(\bar{X}_1)$ on the quadric $\bar{X}_1$ of φ (over an algebraic closure of F).

Note that by [3, Proposition 3.1], condition (4.1) implies condition (4.2).

Proposition 4.3. *Let φ be a non-degenerate quadratic form of an odd dimension*

$$\dim \varphi = 2n + 1 \geqslant 9$$

over a field F and assume that $\dim \varphi$ is not a 2-power minus 1 (equivalently, n is not a 2-power minus 1). Let ψ be a non-degenerate quadratic form of dimension one higher: $\dim \psi = \dim \varphi + 1$. Assume that φ satisfies condition (4.1). Then $\varphi_{F(\psi)}$ over the function field $F(\psi)$ of the projective quadric of ψ satisfies condition (4.2).

Proof. It suffices to consider the case where ψ is anisotropic.

If the discriminant of ψ is nontrivial, the even Clifford algebra $C_0(\psi)$ is simple. In particular, any homomorphism $C_0(\psi) \to C_0(\varphi)$ is injective. Since $\dim_F C_0(\psi) > \dim_F C_0(\varphi)$, there is no such homomorphism and it follows by Index Reduction Formula for quadrics that $C_0(\varphi)_{F(\psi)}$ is still a division algebra, i.e., $\varphi_{F(\psi)}$ still satisfies (4.1). Therefore, by [3, Proposition 3.1], $\varphi_{F(\psi)}$ also satisfies (4.2).

So, we only need to consider the case where the discriminant of ψ is trivial.

We induct on $\dim \varphi$.

Concerning $\dim \varphi = 9$, we can refer to [3, Theorem 6.1]. Concerning $\dim \varphi = 11$, we refer to [3, Theorem 5.4]. (We do not need triviality of the discriminant of ψ for these two initial cases.)

Now let us assume that $\dim \varphi \geqslant 13$. Let Q be the projective quadric of ψ and let $V := X_{n-1}$ be the almost last grassmannian of φ. Assume that condition (4.2) fails for $\varphi_{F(\psi)}$ and consider a lift $\alpha \in \mathrm{Ch}^{n+1}(Q \times V)$ of an element in $\mathrm{Ch}^{n+1}(V_{F(Q)})$ representing $e_{n+1} \in \overline{\mathrm{Ch}}^{n+1}(V_{F(Q)})$. Note that $\alpha_0 = e_{n+1}$ in the decomposition

$$\bar{\alpha} = h^0 \times \alpha_0 + h^1 \times \alpha_1 + \cdots + h^n \times \alpha_n + l_n \times \beta + a l_{n-1} \times [V]$$

of the image $\bar{\alpha} \in \mathrm{Ch}^{n+1}(\bar{Q} \times \bar{V})$ of α with $\alpha_i \in \mathrm{Ch}^{n+1-i}(\bar{V})$, $\beta \in \mathrm{Ch}^1(\bar{V})$, and $a \in \mathbb{Z}/2\mathbb{Z}$. Since the discriminant of ψ is trivial, we may assume that $a = 0$ (c.f. [3, Step (i) in Proof of Theorem 6.1]).

If $\dim \varphi$ is a 2-power minus 3, we can apply [3, Proposition 6.3] and conclude that

$$\alpha_0 + \mathrm{St}^1(\alpha_1) + \alpha_1 \beta \in \overline{\mathrm{Ch}}^{n+1}(V). \tag{4.4}$$

If $\dim \varphi$ is not a 2-power minus 3, since $\dim \varphi$ is not a 2-power minus 1 neither, we can apply Proposition 3.1 and conclude that

$$\alpha_0 + \sum_{i=1}^{[(n+1)/2]} \mathrm{St}^i(\alpha_i) \in \overline{\mathrm{Ch}}^{n+1}(V). \tag{4.5}$$

For $i = 1, \ldots, n$, let X_i be the ith grassmannian of φ and let Y_i be the ith grassmannian of ψ. By Index Reduction Formula for quadrics (the function field extension $F(Y_i)/F$ in the next formula can be replaced by a chain of function fields of quadrics), we have

$$\operatorname{ind} C_0((\varphi)_{F(Y_i)}) \geqslant 2^{n-i} \quad \text{for } i \in \{1, \ldots, n\}. \tag{4.6}$$

Viewing α as a correspondence $Q \rightsquigarrow V$ and applying the homomorphism α_* to $l_i \in \overline{\mathrm{Ch}}(Q_{F(Y_{i+1})})$, we get

$$\alpha_i \in \overline{\mathrm{Ch}}^{n+1-i}(V_{F(Y_{i+1})})$$

for $i < n$. Therefore by [3, Proposition 3.1] we may assume that for every $i \in \{1, \ldots, n-1\}$, the element α_i is e_{n+1-i} or 0.

Let us choose some $i \in \{1, \ldots, n-1\}$ and consider the anisotropic part φ' of the quadratic form $\varphi_{F(X_i)}$. We have

$$\dim \varphi' = \dim \varphi - 2i = 2(n-i) + 1$$

and $C_0(\varphi')$ is a division algebra. Let X'_{n-i-1} be the almost highest grassmannian of φ', and let $e'_{n-i+1} \in \mathrm{Ch}^{n-i+1}(\bar{X}'_{n-i-1})$ be the last of the special elements from §2. Since $C_0(\varphi')$ is a division algebra, it follows by the induction hypothesis of our proof that

$$e'_{n-i+1} \notin \overline{\mathrm{Ch}}^{n-i+1}((X'_{n-i-1})_{F(X_i)(Y_{i+1})} \tag{4.7}$$

provided that $n - i + 1$ is at least 5 and not a 2-power.

By Proposition 2.2, (4.7) implies that

$$e_{n-i+1} \notin \mathrm{Ch}^{n-i+1}((X_{n-1})_{F(X_i)(Y_{i+1})}) = \mathrm{Ch}^{n-i+1}(V_{F(X_i)(Y_{i+1})})$$

for such i and, in particular,

$$e_{n-i+1} \notin \mathrm{Ch}^{n-i+1}(V_{F(Y_{i+1})}).$$

Consequently, $\alpha_1 = 0$ in (4.4) as well as $\alpha_i = 0$ in (4.5) provided that $n - i + 1$ is not a 2-power. Note that in order to handle the case of $\dim \varphi = 13$ here, we use the $\dim \varphi = 11$ base case. When handling the case of $\dim \varphi = 17$, we use the $\dim \varphi \in \{9,\ 11,\ 13\}$ cases. And so on.

If $n - i + 1$ is a 2-power, we cannot conclude that $\alpha_i = 0$. So, let us assume that $\alpha_i = e_{n-i+1}$. By [5, Proposition 2.9], providing a formula for $\mathrm{St}^i(e_{n-i+1})$, the coefficient at e_{n+1} is the binomial coefficient $\binom{n-i+1}{i}$ Note that $i < n-i+1$: otherwise $i = n-i+1$, i.e., $i = (n+1)/2$ and $n-i+1 = (n+1)/2$ is a 2-power – a contradiction with the assumption that n is not a 2-power minus 1. It follows that $\binom{n-i+1}{i} \equiv 0 \pmod 2$. Thus (4.5) yields

$$e_{n+1} = \alpha_0 \in \overline{\mathrm{Ch}}^{n+1}(V)$$

meaning that φ does not satisfy (4.5) – a contradiction. □

5. Core result

Given a non-degenerate quadratic form φ of dimension $u = 2n + 1$ over a field F, we consider the following modification of condition (4.1):

$$\operatorname{ind} C_0(\varphi) \geqslant 2^{n-1}. \tag{5.1}$$

Theorem 5.2. *Assume that an odd natural number $u = 2n + 1$ is neither a 2-power minus 1 nor a 2-power minus 3. Let ψ be a non-degenerate quadratic form over F of dimension $u + 1$. Assume that φ satisfies conditions (5.1) and (4.2). Then $\varphi_{F(\psi)}$ also satisfies conditions (5.1) and (4.2).*

Proof. Condition (5.1) is satisfied by $\varphi_{F(\psi)}$ according to Index Reduction Formula for quadrics. We just need to check condition (4.2).

For $u = 9$ and $u = 11$ we can refer to [3, Theorems 6.1 and 5.4] respectively.

For $u \geqslant 17$ (17 is the next after 11 value of u to consider), let Q be the projective quadric of ψ and let V be the almost highest grassmannian X_{n-1} of φ. Assume that condition (4.2) fails and consider a lift $\alpha \in \mathrm{Ch}^{n+1}(Q \times V)$ of an element in $\mathrm{Ch}^{n+1}(V_{F(Q)})$ representing $e_{n+1} \in \overline{\mathrm{Ch}}^{n+1}(V_{F(Q)})$. Note that $\alpha_0 = e_{n+1}$. By Proposition 3.1, relation (3.2) holds. By Index Reduction Formula, relation (4.6) holds as well. So, acting like after (4.6) but applying for getting (4.7) Proposition 4.3 instead of the induction hypothesis of the proof of Proposition 4.3, we reduce to the case with $a_i = 0$ for all i in (3.2). Then (3.2) simply states that condition (4.2) is satisfied by φ – a contradiction. □

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