Magnetohydrodynamics Tutorial
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Magnetohydrodynamics: What it is, what it can be used for, and how it can be solved.

Kelvin-Helmholtz Instability in the Dayside Magnetosphere

Connecting MHD and Kinetics

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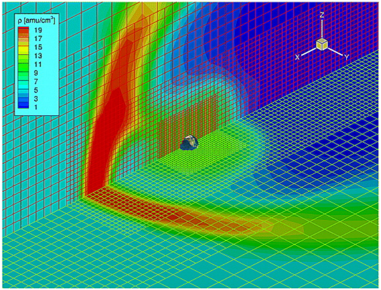
MHD: Global modeling...

- can be used when ...
- nonrelativistic, $v/c << 1$
- charge neutral, $\omega_{pe}\tau >> 1$
- collisions frequent
- example: BATS-R-US in 3D
- 20×10^6 cells, 5 levels of refinement, $1/8R_E \leq \Delta x \leq 4R_E$
- ref: Jichun Zhang, J. Geophys. Rsch., doi: 1029/2006Ja011846, 2007.



Plasma Momentum Equations

Fluid equations plus Lorentz force in red:

- Mass, momentum, and energy for species $s = e, i$

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{v}_s n = 0,$$

$$\left(\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s \right) = \frac{e_s}{m_s} \left(\mathbf{E} + \frac{\mathbf{v}_s \times \mathbf{B}}{c} \right) - \frac{\nabla p_s}{n m_s}$$

$$\frac{\partial i_s}{\partial t} + \mathbf{v} \cdot \nabla i_s = -\frac{p_s}{n m_s} \nabla \cdot \mathbf{v}_s.$$

- Equation of state for an ideal gas,

$$\frac{p_s}{n m_s} = (\gamma - 1) i_s.$$

- n … number density
- i_s … internal energy
- $\mathbf{v}_s$ … velocity.
- $\gamma = 5/3$.

Do Our Approximations Eliminate **E**?

- non-relativistic, no displacement current in Ampere's law

$$0 = \frac{c}{4\pi}\nabla \times \mathbf{B} - \mathbf{J}$$

- plasma quasi-neutral

$$\nabla \cdot \mathbf{E} = 0$$

- no **E** in center of mass momentum equation

$$n(m_e + m_i)\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \frac{\mathbf{J}\times\mathbf{B}}{c} - \nabla p$$

- ... but still need **E** in Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -c\left(\nabla \times \mathbf{E}\right)$$

E from Electron Momentum Equation

- in simulation units, can identify small terms

$$t' = t\omega_{pe}, \;\; \frac{c}{\omega_p e}\nabla' = \nabla, \;\; \mathbf{v}'_e = \frac{\mathbf{v}_e}{c}$$

- eliminate $\mathbf{v}'_e$ using Ampere's law

$$ne\left(\mathbf{v}'_i - \mathbf{v}'_e\right) = \frac{c}{\omega_{pe}}\left(\frac{c}{4\pi}\right)\nabla' \times \mathbf{B}$$

- solve electron momentum equation for **E**

$$\mathbf{E} = -\mathbf{v}'_i \times \mathbf{B} + \frac{c}{\omega_{pe}}\left(\frac{1}{ne}\right)\left(\frac{c}{4\pi}\left(\nabla' \times \mathbf{B}\right) \times \mathbf{B} - \nabla' p_e - nm_e\frac{\mathrm{d}\mathbf{v}'_e}{\mathrm{d}t'}\right)$$

- on gradient scales longer than c/ω_{pe}, blue term small

MHD Equations

- only $\mathbf{B}$
- momentum

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p$$

- Faraday's law for ideal MHD

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

- $\rho = n(m_i + m_e)$

Standard Form: System of Conservation Laws

Mass, momentum, magnetic flux, and energy are constants of the motion in closed systems $\hat{\mathbf{n}} \cdot \mathbf{v} = 0, \ \ \hat{\mathbf{n}} \times \mathbf{E} = 0 \in \partial D$.

$$\frac{\partial}{\partial t} \int_D \rho \mathrm{d}\, V = - \int_{\partial D} \hat{\mathbf{n}} \cdot \mathbf{v} \rho \mathrm{d}\, \mathbf{S}.$$

$$\frac{\partial}{\partial t} \int_D \mathbf{B} \mathrm{d}\, V = -c \int_{\partial D} \hat{\mathbf{n}} \times \mathbf{E} \mathrm{d}\, \mathbf{S},$$

$$\frac{\partial}{\partial t} \int_D \rho \mathbf{v} \mathrm{d}\, V = - \int_{\partial D} \hat{\mathbf{n}} \cdot (\mathbf{v} \rho \mathbf{v} + \mathbf{T} - p \mathbf{I}) \,\mathrm{d}\, \mathbf{S}$$

Because $\nabla \cdot \mathbf{B} = 0$

$$\mathbf{J} \times \mathbf{B} = \nabla \cdot \mathbf{T} \equiv \nabla \cdot \left(- \frac{\mathbf{B} \cdot \mathbf{B}}{2} \mathbf{I} + \mathbf{B}\mathbf{B} \right)$$

Lagrangean Form: Faraday's Law

- $\mathbf{B}$ affected only by $\mathbf{v}_\perp = \mathbf{v} - \mathbf{B}\mathbf{B}\cdot\mathbf{v}/B^2$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v}_\perp \times \mathbf{B})$$

- in fluid frame

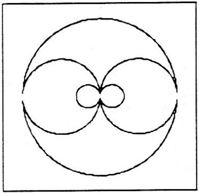
$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = -\mathbf{B}\,(\nabla \cdot \mathbf{v}_\perp) + (\mathbf{B}\cdot\nabla)\,\mathbf{v}_\perp$$

  - magnetoacoustic mode
  - Alfven mode
- magnetic energy

$$\mathbf{B}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = -\mathbf{B}\cdot\mathbf{B}\nabla\cdot\mathbf{v}_\perp$$

Fast and Slow Waves

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left(\frac{B^2}{8\pi} + \rho i \right) \mathrm{d}\, V = \int \left[- \left(\frac{B^2}{8\pi} + p \right) \nabla \cdot \mathbf{v}_\perp + \left(\frac{B^2}{8\pi} - p \right) \nabla \cdot \mathbf{v}_\| \right] \mathrm{d}\, V$$



(a)

Figure: $\hat{\mathbf{n}} \cdot \mathbf{A} = 3a\hat{\mathbf{n}}$; $a = \sqrt{\gamma p / \rho}$, and $A = \pm\sqrt{\mathbf{B}^2/\rho}$

FLIP: A Particle-in-Cell MHD Algorithm, from $\mathcal{M}$ to **B**

- each particle has a mass, m_p, a position, $\mathbf{x}_p$, velocity, $\mathbf{v}_p$, and internal energy, e_p
- each particle has also a magnetic moment, μ_p
- the magnetization is the corresponding continuum variable

$$\mathcal{M} = \sum_p \mu_p \mathcal{S}\left(\mathbf{x} - \mathbf{x}_p\right)$$

- the magnetic field, **B** is computed from $\mathcal{M}$

$$\mathcal{M} = \mathbf{B} + \nabla \phi$$

- solve Poisson's equation for ϕ

Ref: J. U. Brackbill, J. Comput. Phys. **96** 163 (1991).

FLIP: From $\mathbf{B}$ to $\mathcal{M}$

- when $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$ or $\phi = 0$ on boundary, $\mathbf{B}$ and $\nabla\phi$ are orthogonal

$$0 = \int_D \mathbf{B} \cdot \nabla\phi \mathrm{d}\, V = \int_{\partial S} \hat{\mathbf{n}} \cdot \mathbf{B}\phi \mathrm{d}\, \mathbf{S} - \int_D \phi \nabla \cdot \mathbf{B} \mathrm{d}\, V$$

- always true

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathbf{B} \cdot \nabla\phi \mathrm{d}\, V$$

- with Lagrangean equation for $\mathbf{B}$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \nabla\phi + \nabla\mathbf{v} \cdot \nabla\phi$$

- solve MHD equations for $\mathrm{d}\mathbf{B}/\mathrm{d}t$, then solve for $\mathcal{M}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B} - \nabla\mathbf{v} \cdot \nabla\phi$$

Reconnection on the Dayside Magnetosphere

- On average, the rate of reconnection must be the same on the dayside and magnetotail
  - intermittent reconnection in the magnetotail causes substorms
  - dayside, reconnection occurs in small scale events or diffuse, continuous process
- One contributor to reconnection may be the Kelvin-Helmholtz instability

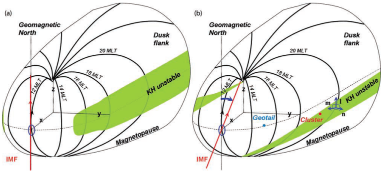
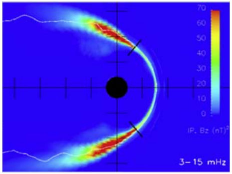


Figure: Unstable flow region in green for northward IMF (left). Foullon et al., JGR **113** A11203 (2008).

Kelvin-Helmholtz Instability Observed in Global MHD Results

- Top: ULF wave activity in equatorial plane occurs in unstable flow shear layer in low-latitude boundary layer
- Bottom: mixing layer extends from $20^\circ S$ to $20^\circ N$ latitude
- Figures from Claudepierre et al., JGR **113** A05218 (2008)



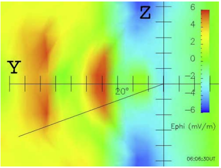


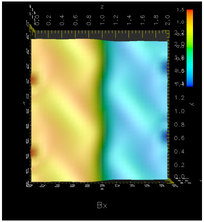
Figure 8. The extent of the KH waves out of the equatorial plane, for the inner KH mode in the 800 km/s simulation. The origin of the coordinate system is located at the KH wave appearance region (WAR), 1624 LT along the magnetopause. The Y axis lies along the IEBL in the equatorial plane with the positive direction tailward. The Z axis is parallel to the GSM z axis. E_ϕ is on the color scale from −6 to 6 mV/m. The axes ticks are spaced at 1 R_E. Note that the KH waves are generated near the equatorial plane and propagate in both the Y and Z directions.

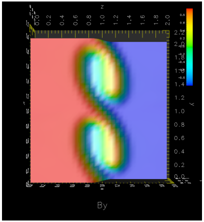
Dispersion

- dispersion for a thin shear layer in a constant density plasma,

$$\omega = -\mathbf{k}\cdot(\mathbf{U}_1+\mathbf{U}_2) \pm \sqrt{\frac{(\mathbf{k}\cdot\mathbf{B})^2}{4\pi\rho} - (\mathbf{k}\cdot(\mathbf{U}_1-\mathbf{U}_2))^2}$$

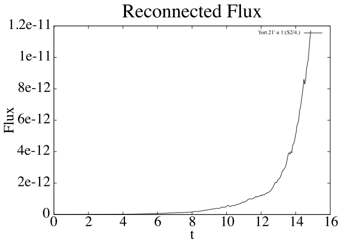
... when Alfven speed and shear are equal





... parallel **B** and **U** (left), perpendicular **B** and **U** (right)

- when **B** and **U** are perpendicular and no variation along field direction, there is no reconnection
- 16^3 cells, 110600 particles, 1:03 min (1.65 speedup), 75% particle routines, 12% implicit solvers



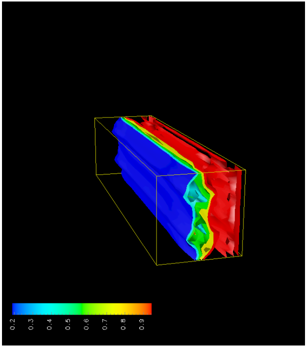
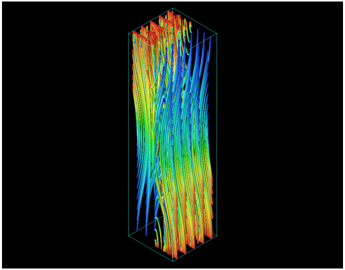
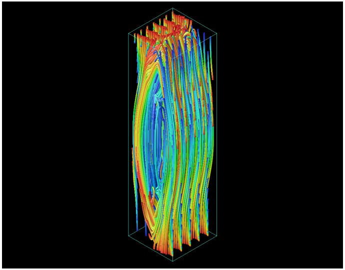


Figure: Left: Absolute flux crossing current sheet. Right: Density of upward flowing plasma.

Differential Rotation

Unstable flow twists the magnetic field and causes it to reconnect.



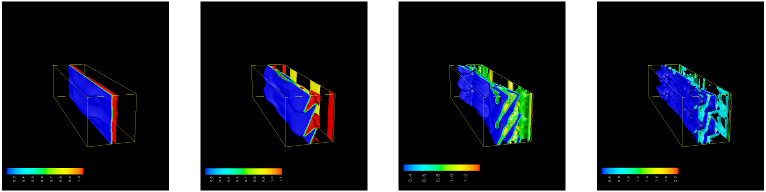


Field lines at t=1, left, and at t=2., right. (Time in eddy turnover times.) Run data: 32^3 cells, 885000 particles, 2800 s run time, 22% particle routines, 10% implicit solver, 45% $\nabla \cdot \mathbf{B}$ projection

Ref. J. U. Brackbill, D. A. Knoll, Phys. Rev. Lett. **86** 2329 (2001).

Instability Spreads along Field Lines

Flow downward on left, upward on right, maximum in the center and zero at the ends.



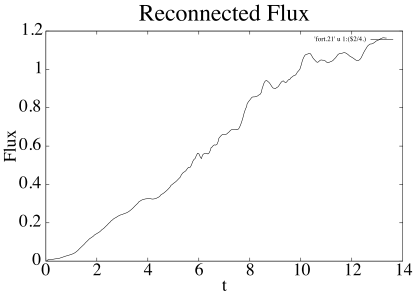
Interface at t=2, 4, 6, and 8 eddy turnover times.

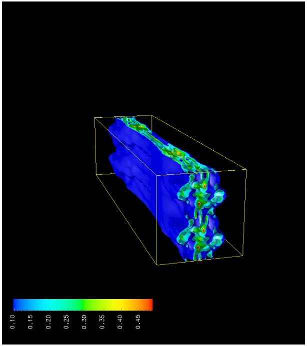
Reconnection and Mixing

- Lagrangean particles support flux calculation on deformed interface

$$\Psi = \int_D |\mathbf{B} \cdot \nabla \rho|/[\rho] \mathrm{d}\, V$$

- ... and a measure of mixing $\rho_1 \rho_2$





The Value of Small-Scale Computations

- even very modest calculations give surprisingly complex results
- opportunity to explore further in lab

Can MHD Model Plasmas on Kinetic Scales?

- one possibility is that some extension of MHD could replace kinetic simulations
  - simulations are expensive and noisy
  - open systems a challenge
- can MHD model the physics?
  - MHD is less noisy, but is it cheaper?
  - what physics must be incorporated in the closure of the moment equations?

How Can MHD be Extended?

On small scales, the blue terms in Ohm's law do different things

- decouple ion and electron motion ... Hall term
- reflect finite plasma equilibration rate ... tensor pressure
- interchange vorticity and magnetic field ... inertia term

$$\mathbf{E} = -\mathbf{v}_i' \times \mathbf{B} + \frac{c}{\omega_{pe}} \left(\frac{1}{ne} \right) \left(\frac{c}{4\pi} \left(\nabla' \times \mathbf{B} \right) \times \mathbf{B} - \nabla' \cdot \mathbf{P}_e - n m_e \frac{\mathrm{d}\mathbf{v}_e'}{\mathrm{d}t'} \right)$$

Tensor pressure equation because plasma equilibration not instantaneous

$$\begin{aligned} \frac{\mathrm{d}\mathbf{P}_s}{\mathrm{d}t} &= -\mathbf{P}_s \nabla \cdot \mathbf{v}_s - \mathbf{P}_s \cdot \nabla \mathbf{v}_s - \left(\mathbf{P}_s \cdot \nabla \mathbf{v}_s \right)^T \\ &+ \left[\mathbf{P}_s \times \mathbf{\Omega}_s + \left(\mathbf{P}_s \times \mathbf{\Omega}_s \right)^T \right] - \textit{higher order terms}, \end{aligned}$$

$$\mathbf{\Omega}_s \equiv e\mathbf{B}/m_s c$$

Ref.: O. Buneman, Phys. Fluids **4** 669 (1961).

Is MHD Cheaper?

If resolve electron length scales, have to resolve electron time scales

- the Hall term adds a whistler wave to MHD dispersion (Hassam and Huba, Phys. Fluids **31** 318 (1988).)

$$\omega **2 = \left(\frac{k_\parallel^2 V_A^2}{\Omega_{ci}^2}\right)^2$$

- Courant condition requires $\omega \Delta t < \pi$
- when $k_{parallel} = 2\pi\omega_{pe}/c$, Δt must resolve electron cyclotron frequency

$$\omega \Delta t = \Omega_{ce} \Delta t <$$

- the equation for $\mathbf{P}_e$ imposes same requirement

Fluid Closure :Compare fluid and kinetic solutions

- Steady-state simulations of reconnection can reduce noise

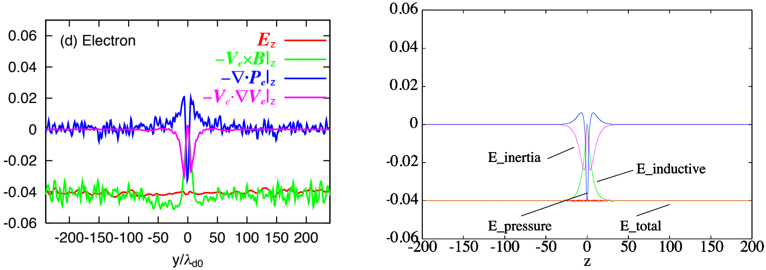


Figure: Electron contributions to the electric field. Top: 2D steady-state simulation (Ishizawa et al. for $m_i/m_e = 800$ (A. Ishizawa, R. Horiuchi, and H. Ohtani, Phys. Rev. Lett. 95, 045003, 2005. Bottom: 1D, two-fluid calculation.(Brackbill, Phys. Plasmas **18**, 032309 (2011)

Compare Fluid and Kinetic Results for Steady Reconnection: Ion Momentum

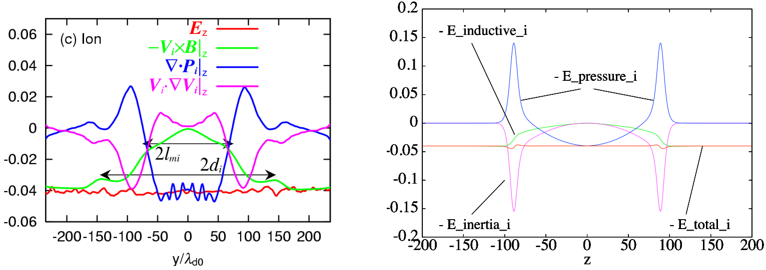


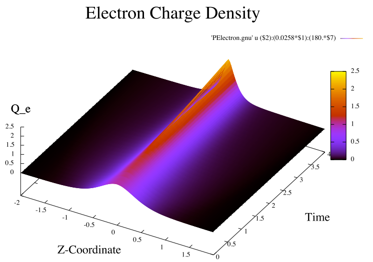
Figure: Top panel are ion contributions to the electric field from a 2D steady-state simulation by Ishizawa et al. for $m_i/m_e = 800$ (A. Ishizawa, R. Horiuchi, and H. Ohtani, Phys. Rev. Lett. 95, 045003, 2005. In the bottom panel from a 1D, two-fluid calculation.

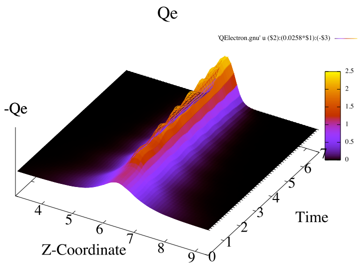
Remarks

- Closure: Isotropization of electron and ion pressure ($\tau_e = 0.1$, $\tau_i = 18.$)
- Quasi-steady. Thermal conduction along field missing.
- There are no apparent kinetic effects outside the ion and electron 'diffusion' regions.
- The fluid model assumes incompressible flow. Simulations show some ion compression.
- Solve ion and electron momentum equations. Use steady value of $\mathbf{E}$. Yields scale separation.

Fluid and Kinetic Results for Plasma Compression

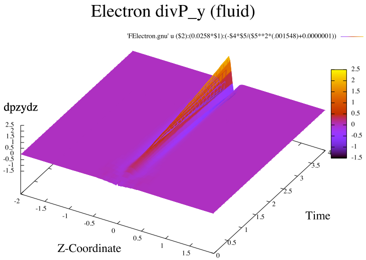
- Simplest geometry, 1D compression, $v_{inflow} = 0.4 v_{the}$
- Same compression for fluid (left) and kinetic(right) plasmas .
- Kinetic calculation performed on a 128x128 grid with 50 particles/cell, and averaged over x to reduce noise.

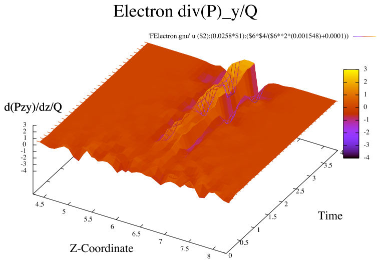




Choose Isotropization Time to Match $\nabla \cdot \mathbf{P}_y$

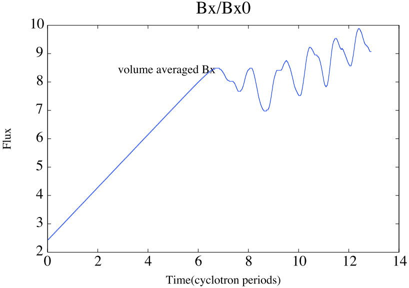
- $\tau_e = 0.2$, $\tau_i = 18.0$ in inverse electron plasma periods
- Plotted is $\nabla \cdot \mathbf{P}_y / QE_y$
- Kinetic maximum is 4, fluid 2.5, both larger than needed to sustain steady reconnection. At t=$4\Omega_{ce}^{-1}$. there is not yet any reconnection in the simulation

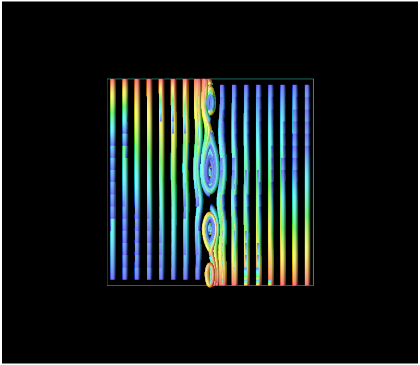




Reconnection in 2D Simulation

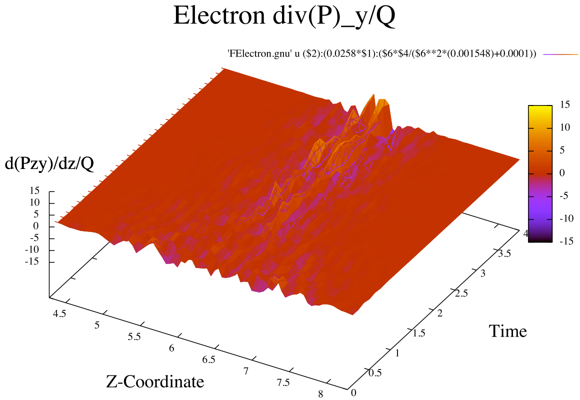
- reconnection occurs in 2D simulation
- results for $m_i/m_e = 1836$





Remark: Why Not a 1D Simulation

- Without averging, $\nabla \cdot \mathbf{P}_e$ from 1D simulation with 256 cells, 5000 particles per cell is noisy



What Matters

- The ratio, $P_{\perp}/P_{\|}$ may be more important in transient reconnection than $\nabla \cdot \mathbf{P}_y$.
- A sustained high value yields higher growth rates for the tearing mode.
- The kinetic result (right) is with a 64^2 grid. Note the small reduction in the ratio at the center of the current sheet.