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Precision observables for ultracold quantum gases

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Präzisionsobservablen für ultrakalte Quantengase:
Wir widmen uns in dieser Diplomarbeit der Frage, inwiefern ultrakalte Quantengase für Präzisionsmessungen stark korrelierter Systeme verwendet werden können. Damit soll einerseits eine Identifizierung verlässlicher Methoden zur Beschreibung solcher Systeme erzielt werden, aber auch eine systematische Verbesserung theoretischer Techniken durch Vergleich mit dem Experiment. Zu diesem Zweck diskutieren wir, wie es Zustandsgleichung und Transportkoeffizienten ermöglichen, mikroskopische Berechnungen aus ersten Prinzipien mit der Phänomenologie kalter Quantengase im Labor zu verknüpfen. Im Speziellen legen wir dar, dass Zustandsgleichung, Schwingungsfrequenzen kollektiver Anregungen, sowie die Antwortfunktion auf ein äußeres angelegtes Feld Präzisionsobservablen darstellen und erläutern die darin enthaltenen Informationen. Die Berechnung der Zustandsgleichung für ein Bosegas vermittels der Theorie von Lee und Yang wird ausführlich präsentiert. Ausgehend von Landaus Zweiflüssigkeitshydrodynamik leiten wir das Eigenwertproblem zur Bestimmung kollektiver Anregungsfrequenzen eines kalten Gases ab. Wir implementieren dies numerisch für eine isotrope harmonische Falle und diskutieren die Lösungen für ein Bosegas in Abhängigkeit von Wechselwirkungsstärke, Dimensionalität und thermischen Fluktuationen. Eine systematische Erweiterung des Eigenwertproblems durch Berücksichtigung dissipativer Terme erlaubt uns schließlich die Antwortfunktion und insbesondere die Dämpfungskonstanten kollektiver Schwingungen aus den Zweiflüssigkeitsgleichungen zu extrahieren.

Precision observables for ultracold quantum gases:
In this diploma thesis we investigate whether ultracold quantum gases can be used for precision measurements of strongly correlated systems. We aim at an identification of reliable methods to describe such systems. Moreover, this comparison with experiment should allow for a systematic improvement of theoretical techniques. For these purposes we discuss how the equation of state and transport coefficients provide a connection between microscopic calculations from first principles and the phenomenology of cold quantum gases in the laboratory. In particular, we show that the equation of state, oscillation frequencies of collective excitations and the response function to an external driving force constitute precision observables. We explain what information is contained in these quantities. The calculation of the equation of state for a Bose gas within Lee–Yang theory is presented in detail. From Landau's two-fluid hydrodynamics we derive the eigenvalue problem which determines collective oscillation frequencies of a cold gas. We suggest a numerical implementation for an isotropic harmonic trap and discuss solutions for a Bose gas taking into account interactions, dimensionality and thermal fluctuations. A systematic extension of the eigenvalue problem through inclusion of dissipative processes allows to extract the response function and damping constants of collective excitations from the two-fluid equations.

Contents

1 Introduction

The physics of strongly correlated systems constitutes a both fascinating and challenging chapter in the understanding of nature. Progress in this area of research is expected to give answers to important questions on all scales of energy: What materials show high-temperature superconductivity and what is the critical temperature? What mechanism is responsible for the confinement-deconfinement phase transition in Quantum Chromodynamics and what is its relation to the chiral phase transition? How can we describe dense nuclear matter in neutron stars or relativistic heavy-ion collisions? What is the physics of the unitary Fermi gas and how does it emerge in the BEC-BCS crossover? Numerous open questions could be added. However, a simple description of the mentioned systems is not possible. In fact, perturbative methods and mean-field theory cannot be applied when strong correlations dominate. We therefore have to use more sophisticated treatments like the Renormalization Group, lattice simulations, 2PI approaches, conformal field theory, etc.

Non-perturbative approaches to quantum field theory, if they do not provide an exact solution, are accompanied by approximations, which are quantitatively less understood than in the perturbative case, where a small parameter is present. In order to gain insight on the reliability of methods when applied to strongly correlated systems, we need to find precision observables which can resolve advancements or shortcomings of our theoretical treatment. The experimental determination of these quantities will then set stringent bounds on theory.

Experiments on ultracold quantum gases provide an ideal base for such an investigation. Indeed, measurements done so far revealed that cold atoms realize many-body Hamiltonians in the laboratory in a clean way. Additionally, the parameters of these systems can be tuned by application of an external field. This situation is very different from solid state physics, where the samples are impure and the parameters of the system are fixed by the material. In particular, by the use of a Feshbach resonance it is possible to enter the strongly interacting regime of ultracold quantum gases and even drive through the crossover from attractive to repulsive interactions. For a review on ultracold gases see Ref. [1] and references therein.

The phenomenology of cold atom experiments is mainly restricted to absorption images of the density profile. This has to be accounted for when choosing the right precision observables. In this thesis we concentrate on three quantities: the equation of state of cold atoms, the frequencies of collective oscillations ("modes") of trapped gases and the response of a quantum gas to an external driving force. This selection of observables constitutes a successive enhancement in the degree of sophistication. They share the dependence on objects that can directly be computed from (non-)equilibrium quantum field theory, namely the equation of state and the transport coefficients, and thus link microscopic theory to experiment.

The *equation of state* is given by the pressure P as a function of two independent thermodynamic variables. In this thesis we work with the grand canonical variables, i.e. chemical potential μ and temperature T. Once $P(\mu, T)$ is known, all other thermodynamic quantities can be derived from thermodynamic relations. Theoretically one obtains the equation of state from the quantum partition function. The latter can be accessed in different ways. Lattice computations, for example, directly calculate the corresponding euclidean functional integral using Monte Carlo techniques, while a Functional Renormalization Group approach solves a flow equation to yield the desired quantity. The information contained in $P(\mu, T)$ is enormous. For example, well-known observables like the Bertsch parameter or the critical temperature at unitary constitute only a small subset of the predictions of the full equation of state of a two-component Fermi gas. The function $P(\mu, T)$ itself represents a two-dimensional manifold of values which can be measured by density imaging, compare Eq. (3.82). Usually, the (μ, T)-plane can be separated into regions corresponding to different phases of the system. The location of the border lines represents the phase diagram of the system. Moreover, the dependence on additional parameters like interaction strength yields further phase diagrams. In general, both the values of $P(\mu, T)$ and the location of the phase boundaries require a non-perturbative treatment. Recent developments have shown the potential of in situ density imaging to measure the equation of state beyond mean-field [2, 3].

Collective excitations of trapped quantum gases have been a promising observable since the early experiments on cold atoms. Therefore, a great amount of literature already exists on this subject, both experimentally [4] - [17] and theoretically [18] - [36]. For a harmonic trap these collective modes typically show an oscillatory behavior with characteristic frequencies. These eigenfrequencies can be excited in several ways. One of them is to perform a sudden perturbation of the system, e.g. by a ramp of the trapping potential or a change in the scattering length via a Feshbach resonance. Then, releasing the oscillating cloud from the trap, it is possible to fit a sinus-type function to the density images obtained in free flight. Another possibility is to apply a periodic driving force with fixed frequency. Depending on whether this frequency is close to one of the eigenfrequencies of the system or not, the response will be correspondingly high or low. In this thesis, we calculate these eigenfrequencies for an arbitrary equation of state in the hydrodynamic regime. In particular, we respect that below the temperature of the superfluid phase transition, superfluid density and superfluid velocity field enter as additional macroscopic degrees of freedom in the hydrodynamic description. The corresponding set of equations has been derived by Landau and is called two-fluid hydrodynamics. For the calculations of the collective modes it is sufficient to restrict to the linearized equations, which describe small deviations from static equilibrium. When dissipative effects are neglected the corresponding linear partial differential operator depends only on the equation of state $P(\mu, T)$ and the normal fluid density $n_n(\mu, T)$, the latter supplementing the equation of state below the critical temperature. The eigenvalues of this operator are the collective oscillation frequencies of the trap gas. We show in this thesis how these eigenvalues are obtained numerically and how the measurement of collective modes can also be used for precision measurements of temperature, particle number or chemical potential.

Our considerations on collective modes imply that the trapped quantum gas can be regarded (formally) as a set of classical harmonic oscillators. Indeed, once the eigenfrequencies of the system are known, the origin from a cloud of atoms is irrelevant. We therefore formulate a linear response theory for the trapped gas with equations of motion governed by dissipative two-fluid hydrodynamics. The corresponding observables are contained in the *response function to an external driving force*. Besides the eigenfrequencies, which are slightly shifted due to the dissipative terms, we can deduce further observables from the response function. The response coefficients, a generalization of the concept of the mass of an oscillating particle, give the relative amplitude of response of distinct modes to an external force. Moreover, damping constants of the collective oscillations are derived in this formalism. In contrast to all other quantities mentioned so far, which only depend on $P(\mu, T)$ and $n_n(\mu, T)$, the damping constants explicitly require knowledge of the transport coefficients. This, of course, opens up a possibility to compare with microscopic models from non-equilibrium theory. To put it differently, measuring certain damping constants we can rule out insufficient models for the transport coefficients. We suggest that response techniques can be applied to individually address single eigenfrequencies of the system. A measurement of the response function may consist in determining the loss of energy in the applied force as a function of frequency. This could be realized electronically and thus does not rely on density images.

This thesis is organized as follows. In Sec. 2 we recapitulate the main ideas of hydrodynamics and formulate the two-fluid hydrodynamic equations in an external trapping potential for both the ideal and dissipative cases. The equation of state of a Bose gas is studied in Sec. 3 as a representative example for other ultracold quantum gases. By considering beyond mean-field corrections at zero temperature and Lee–Yang theory for nonvanishing temperatures we can derive all thermodynamic quantities. The general eigenvalue problem of collective modes is derived in Sec. 4. For concrete calculations we restrict to isotropic harmonic trapping. We derive exact results at zero temperature and suggest a method of numerical implementation. We then discuss the Bose gas beyond mean-field in three and two dimensions at $T = 0$ and finally show results from the full two-fluid hydrodynamics at nonvanishing temperature with the equation of state provided by Lee–Yang theory. A discussion of the response function of a trapped gas follows in Sec. 5. After general remarks on linear response theory and the classical harmonic oscillator we derive the corresponding perturbations to our operator eigenvalue problem. Using the unperturbed eigenfunctions of the ideal case we obtain approximate expressions for the response coefficients and damping constants. Since this analysis is not restricted to the trapped case we also show how the continuous spectrum of first and second sound arises in the homogeneous case. In Sec. 6 we conclude and give an outlook on future research. The system of units used throughout this thesis is explained in App. A. In App. B we collected thermodynamic identities which are used at several points in the thesis. A visualization of the equation of state of a Bose gas from Lee–Yang theory is given in App. C. These pictures provide a good intuition on the behavior of thermodynamic functions and the phase structure of the system. Spherical harmonics in two and three dimensions are introduced in App D.

2 Superfluid hydrodynamics

This chapter provides an introduction to the hydrodynamic description of superfluid systems. After general remarks concerning the hydrodynamic regime we present phenomenological aspects of superfluidity. The remaining part will be concerned with deriving the two-fluid equations for the superfluid and normal flow constrained by external trapping.

2.1 Hydrodynamics – an effective theory

Dealing with complex physical systems is accompanied by the impossibility of describing all internal degrees of freedom. This applies for example to many body situations with particle numbers ranging from a few thousands or millions in cold atomic gases to 10^{23} in solid state physics. Although the tremendous amount of work of book-keeping all constituents and solving all equations of motion may be argued to be solved by future computers, the realization eventually fails due to our inability of assigning the right initial conditions. We face the same problem for highly correlated systems. Furthermore, details of the underlying microphysics like the exact interaction potentials may be unknown.

However, it is an encouraging fact that these complicated situations can be handled quantitatively, if we drop the demand for including the whole physics but rather restrict ourselves to a few leading characteristic properties. The resulting theory is then called *effective*. Of course, for each system of interest, there will be a set of effective descriptions which all differ in their predictive power and range of applicability. During the process of simplification information is lost and no such theory will capture all features. They have to be used with care and keeping in mind their limitations. Nevertheless, there is one important advantage common to all effective theories: They provide an intuitive understanding of the physical setting and stimulate our *imagination*, which leads to new ideas concerning possible effects and experiments. Obviously, it has to be ruled out that we have been mislead - by further theoretical investigation or in the laboratory.

We want to consider the theory of hydrodynamics, which represents an excellent example of these properties. Besides this it will later be the starting point for our analysis of superfluid phenomena. To be concrete, we follow Ref. [37] and look at a gas in a sufficiently large container with no external field applied. If the temperature, density or average velocity of the gas particles happen to be not constant, the system is out of equilibrium and there will be transport of energy, mass and momentum, to wash out these non uniformities. We may think of the equilibration process to be realized via collisions or some other microscopic interaction. In fact, the precise details are irrelevant. It is important to note, that these processes possess characteristic time and length scales, the time τ and mean free path λ between two collisions, say. These quantities constitute the intrinsic, *microscopic scales* of the systems. Spatial non-uniformities in temperature or density, which are of order λ, will we eliminated during a time of order τ. In contrary, variations over long distances, i.e. $\gg \lambda$, will persist for a long time, $\gg \tau$. Interesting situations arise whenever the system also permits *macroscopic scales*, which can be the size of the container or the wavelength of density fluctuations. In the case of these macroscopic scales being much larger than τ and λ, we speak of the hydrodynamic regime. The gas can then be regarded as a continuum. We call the opposite case the collisionless regime. The intrinsic scales may be very different for distinct systems (cold atomic gases, superfluid Helium-II, strongly interacting quark gluon plasma), as are the macroscopic ones (trap size, container, fireball). However, in the hydrodynamic regime these systems are united by exhibiting a *separation of scales* and can therefore be described in a unified framework. We emphasize that hydrodynamics is *not* a theory of strong interactions, although the latter often provide a sufficient condition for application of the former.

In the following we construct the basic equations of hydrodynamics for an ideal fluid in two ways. First, the derivation will be guided by the presentation given in [38]. Afterwards, we state the point of view that hydrodynamics is a derivative expansion around thermal equilibrium. Non-dissipative processes are then included easily.

Let us picture a fluid in a container such that the physical conditions ensure the hydrodynamic regime. As justified above, at each given time t and point $\vec{x}$ in space the fluid can be regarded as a continuum and thermodynamics may be applied to every point separately. To fully describe the system we need information about the velocity field $\vec{v}(t, \vec{x})$ and two thermodynamic variables, $\rho(t, \vec{x})$ and $T(t, \vec{x})$, say, since the third one is encapsulated in the fluid equation of state $P(\rho, T)$. Here, the space-time point $(t, \vec{x})$ belongs to the reference frame of the box (laboratory frame) and does *not* move with the fluid. Many situations will appear in which he have to switch between this frame and the comoving one. The latter one constitutes a coordinate system attached to an infinitesimal volume element $\mathrm{d}V_0$ of the fluid. $\mathrm{d}V_0$ is small in a sense, that its volume can be neglected in comparison to the size of the whole system, but nevertheless it contains many particles such that a detailed microscopic description is not necessary. We will not adopt the frequently used terminology of calling $\mathrm{d}V_0$ a fluid particle or fluid point.

To exemplify the typical arguments in hydrodynamics we consider entropy conservation. It is worth spending some time here, as this reasoning lies at the heart of hydrodynamics. We take an infinitesimal volume element $\mathrm{d}V$ in the laboratory frame or $\mathrm{d}V_0$ moving with the flow. As it contains a macroscopic number of particles, it has a given entropy $\mathrm{d}S$ ($\mathrm{d}S_0$) and mass $\mathrm{d}M$ ($\mathrm{d}M_0$). The ratio $\mathrm{d}S/\mathrm{d}M$ is the entropy per unit mass, since both quantities are extensive ones. We will label observables per unit mass with a subscript "m" in order to distinguish them from the usual densities per unit volume. As $\mathrm{d}V = \mathrm{d}V(t, \vec{x})$ is at a fixed point in space-time, the entropy per unit mass is given at the same point via

$$s_m(t, \vec{x}) = \frac{\mathrm{d}S}{\mathrm{d}M}. \tag{2.1}$$

Contrary, at any given time t, $\mathrm{d}V_0 = \mathrm{d}V_0(t, \vec{x}(t))$ is at a different point $\vec{x}(t)$ and the dependence of s_m on space-time is modified

$$s_m(t, \vec{x}(t)) = \frac{\mathrm{d}S_0}{\mathrm{d}M_0}. \tag{2.2}$$

If s_m is multiplied by the mass density ρ, we get the entropy density

$$s(t, \vec{x}) = \rho(t, \vec{x}) s_m(t, \vec{x}) = \frac{\mathrm{d}M}{\mathrm{d}V}\frac{\mathrm{d}S}{\mathrm{d}M} = \frac{\mathrm{d}S}{\mathrm{d}V}. \tag{2.3}$$

This formula shows how $\mathrm{d}V$ has to be understood - in the actual spirit of calculus of infinitesimal quantities and on the physical grounds given above. A fluid is called *ideal* if it does not allow for dissipative processes, i.e. there is no friction or heat transfer between the particles. The absence of heat transfer results in the conservation of entropy

$$\frac{\mathrm{d}}{\mathrm{d}t} s_m(t, \vec{x}(t)) = 0 \tag{2.4}$$

in each *comoving* $\mathrm{d}V_0$ separately. We aim at an expression which holds in the laboratory frame and does not depend on the flow lines $\vec{x}(t)$. For this purpose we need to evaluate the time-differentiation

$$0 = \left(\frac{\partial}{\partial t} + \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \cdot \frac{\partial}{\partial \vec{x}}\right) s_m(t, \vec{x}) = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right) s_m(t, \vec{x}). \tag{2.5}$$

This is already one of the basic equations of ideal hydrodynamics. It has a conservation law form.

Another conservation law corresponds to mass. If we fix a volume element V_1 at given $(t, \vec{x})$ it will contain mass

$$\int_{V_1} \mathrm{d}V \rho(t, \vec{x}). \tag{2.6}$$

If $\vec{v}(t, \vec{x})$ is the velocity field of the fluid, then the flow of mass *out* of the volume V_1 will be

$$\oint_{\partial V_1} \mathrm{d}\vec{f} \cdot \rho(t, \vec{x}) \vec{v}(t, \vec{x}). \tag{2.7}$$

This quantity is positive for a decrease of the mass contained in V_1 and therefore equal to

$$-\frac{\partial}{\partial t} \int_{V_1} \mathrm{d}V \, \rho(t, \vec{x}). \tag{2.8}$$

Using Gauß's law and demanding for the result to hold independently of the choice of V_1 we arrive at the well-known form of the equation of mass conservation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{v}) = 0. \tag{2.9}$$

It can be used to rewrite the conservation law of entropy as

$$\frac{\partial (\rho s_m)}{\partial t} + \mathrm{div}(\rho s_m \vec{v}) = 0. \tag{2.10}$$

In order to derive Euler's equation, we take a volume element $\mathrm{d}V_0$ comoving with the flow. Along the border of $\mathrm{d}V_0$ will be some pressure P and the force on the volume element is therefore given by

$$-\oint_{\partial \mathrm{d}V_0} \mathrm{d}\vec{f} \cdot P = -\int_{\mathrm{d}V_0} \mathrm{d}V \cdot \nabla P. \tag{2.11}$$

We conclude $-\nabla P$ to be the force per unit volume element. In the case of the comoving part $\mathrm{d}V_0$, the mass remains constant and the force leads to an acceleration

$$\vec{a} = \frac{\mathrm{d}}{\mathrm{d}t} \vec{v} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v}. \tag{2.12}$$

Applying this to $\vec{F} = m\vec{a}$ we gain *Euler's equation*

$$\rho \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \vec{v} = -\nabla P. \tag{2.13}$$

Eqs. (2.9), (2.10) and (2.13) build the basis of ideal hydrodynamics. They hold at every space-time point $(t, \vec{x})$ in the laboratory frame and relate the quantities $\vec{v}(t, \vec{x})$, $\rho(t, \vec{x})$ and $s_m(t, \vec{x})$. Furthermore, they have to be supplemented by the (equilibrium) equation of state $P(t, \vec{x}) = P(\rho(t, \vec{x}), s_m(t, \vec{x}))$ at every point $(t, \vec{x})$.

A special case of Euler's equation arises for potential flow, i.e. a flow which is irrotational such that the velocity field can be written as

$$\vec{v}(t,\vec{x}) = \nabla\phi(t,\vec{x}). \tag{2.14}$$

By using the vector analytic identity $(\vec{v}\cdot\nabla)\vec{v} = \nabla(\frac{1}{2}v^2) - (\vec{v}\times\mathrm{curl}(\vec{v}))$ Eq. (2.13) gets for irrotational flow

$$\frac{\partial\vec{v}}{\partial t} = -\frac{1}{\rho}\nabla P - \nabla(\frac{1}{2}v^2). \tag{2.15}$$

So far the equations have been derived on intuitive arguments and they already reveal their conservation law nature. However, they can be brought into a manifestly conservative form which will then be called *constitutive*. This is the starting point for more general approaches to the hydrodynamic regime which cover also dissipative effects like heat transport or viscosity.

The constitutive equations read

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\vec{g}) = 0, \tag{2.16}$$

$$\frac{\partial g_i}{\partial t} + \partial_k\Pi_{ik} = 0, \tag{2.17}$$

$$\frac{\partial\varepsilon}{\partial t} + \mathrm{div}(\vec{j_E}) = 0 \tag{2.18}$$

where i,k run over the spatial dimensions, $\vec{g}$ is the mass flow (momentum density), Π is the the energy-momentum tensor of the fluid, ε is the energy density and $\vec{j_E}$ is the energy flux. These equations represent conservation of mass, momentum and energy and are therefore *exact*. One may ask, why entropy conservation does not appear and the answer is simply that entropy will no longer be conserved for dissipative flows and hence s_m would be a bad starting point for constitutive laws. Obviously, for solving these equations we have to assign explicit expressions to $\vec{g}$, Π_{ik} and $\vec{j_E}$. Like all other physical quantities of the system, they are functions of the velocity field, two independent thermodynamic variables and derivatives thereof. The idea is to *expand* the functions around their equilibrium values in higher and higher orders of these derivatives. The lowest order will just be ideal hydrodynamics, where the expressions are found by demanding for Galilean invariance and entropy conservation to be

$$\vec{g} = \rho\vec{v}, \tag{2.19}$$

$$\Pi_{ik} = \rho v_i v_k + P\delta_{ik}, \tag{2.20}$$

$$\vec{j_E} = (\varepsilon + P)\vec{v} \tag{2.21}$$

with $\varepsilon = \varepsilon_0 + \frac{1}{2}\rho v^2$, the internal energy ε_0 in the rest frame of the fluid and again P has to be given by the equation of state $P(\rho, \varepsilon)$. If we insert these expressions into the constitutive equations, we arrive at the basic equations (2.9), (2.10) and (2.13) stated above.

We go one step beyond ideal hydrodynamics and include first order dissipative processes, namely heat transport and viscosity. They appear to be interesting observables as they measure deviations from the ideal behavior. Of course, they are only meaningful if these deviations are small, otherwise the expansion would break down anyway. We write

$$\begin{aligned} \vec{g} &= \rho\vec{v}, && (2.22)\\ \Pi_{ik} &= \rho v_i v_k + P\delta_{ik} + \delta\Pi_{ik}, && (2.23)\\ j_{E,i} &= (\varepsilon + P)v_i + v_k\delta\Pi_{ik} - \kappa_T\partial_i T. && (2.24) \end{aligned}$$

These expressions contain the correction to the energy-momentum tensor

$$\delta\Pi_{ik} = -\eta\left(\partial_i v_k + \partial_k v_i - \frac{2}{3}\mathrm{div}(\vec{v})\delta_{ik}\right) - \zeta\mathrm{div}(\vec{v})\delta_{ik} \qquad (2.25)$$

with the shear viscosity η in front of the symmetric, traceless part of $\delta\Pi$ and the bulk viscosity ζ. In the energy flow we introduced a heat current term with the thermal conductivity κ_T. The derivative expansion nature of these terms is apparent. η, ζ and κ_T have to be non-negative because of the second law of thermodynamics. A negative κ_T, for example, would result in an energy flow from regions of low temperature to higher temperatures. The insertion of Eq. (2.23) into the constitutive equation (2.20) provides us with the Navier-Stokes equation

$$\rho\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\vec{v} = -\nabla P + \eta\Delta\vec{v} + (\zeta + \frac{2}{3}\eta)\nabla(\mathrm{div}(\vec{v})). \qquad (2.26)$$

2.2 Signs of superfluidity

Superfluidity is an interesting manifestation of quantum many-body physics exhibiting unusual and surprising features. It was discovered independently by Kapitza ([39], 1938) and by Allen and Misner ([40], 1938), who found that ^{4}He at temperatures below the lambda point $T_\lambda \simeq 2K$ flows through narrow capillaries without showing any friction, i.e. it seems to have vanishing shear viscosity. Shortly after that Landau gave an explanation of this behavior ([41], 1941). We will present phenomenological facts about superfluid motion and then derive

Landau's criterion on the excitation spectrum to allow for superfluidity. Our derivation follows Refs. [38] and [43].

^{4}He has the property of remaining liquid for zero temperature (and not too high pressures). It is a strongly interacting quantum liquid and its macroscopic behavior can be described by hydrodynamics. However, superfluidity is also found in Fermi gases or weakly interacting Bose gases. The main attribute of these substances at low temperatures is the existence of two kinds of flows, a normal and a superfluid one, both very different in their properties. These do not interact with each other and can be described separately. However, it would be wrong to regard this as a consequence of the fluid being a binary mixture. The true nature of this *two-fluid model* consists in quantum effects and is therefore beyond our imagination. Nevertheless, we can assign effective masses to the normal ('n') and superfluid ('s') motion, such that their sum equals the mass of the fluid, or in terms of the densities,

$$\rho = \rho_n + \rho_s. \tag{2.27}$$

These densities are defined to be the coefficients in the expansion

$$\vec{g} = \rho_n \vec{v}_n + \rho_s \vec{v}_s \tag{2.28}$$

for the momentum density, where $\vec{v}_n(t, \vec{x})$ and $\vec{v}_s(t, \vec{x})$ represent the velocity fields of the normal and superfluid flow, respectively. We note the superfluid fraction ρ_s/ρ to be a function of temperature T such that it goes to 1 for $T \to 0$, i.e. the whole substance is superfluid at $T = 0$ and $\rho_s/\rho = 0$ for $T \geq T_c$, i.e. only the normal component survives above a critical temperature. This already shows us, that the notion of superfluidity is not in 1–1-correspondence with Bose–Einstein condensation, although they share a lot of features. In fact, even at zero temperature not all particles will be Bose condensed because of depletion effects. For ^{4}He the condensate fraction is around 8% at $T = 0$.

As stated above, the flow of liquid ^{4}He through capillaries or small slits is found to be inviscid for $T < T_c$. In apparent contradiction the torque on a rotating disk or cylinder is found to be finite which can only be explained with a nonzero viscosity. The resolution to this puzzle is the following. When passing surfaces, the superfluid motion shows no friction and has vanishing viscosity. This does not hold for the normal flow behaving like a typical fluid. When giving liquid helium the opportunity to pass through a narrow slit, both flows will take part in this process, but the time-scale for the superfluid motion will be far smaller than for the normal fluid part. Therefore, the latter can be neglected and is invisible in experiment. We directly see the importance of scales in hydrodynamics. More suitable for measuring the viscosity of the normal component is a rotating disk. Via its rotation it generates a normal flow in the liquid and hence produces friction, which damps its motion. Another manifestation of this complementary behavior is found in a rotating cylindrical container. Initially at rest, the superfluid component will not be able to follow the motion of the container, but the normal one can. The moment of inertia associated with this motion will be proportional to the normal fluid fraction ρ_n/ρ and is therefore smaller than the classically expected value for the full rotating mass.

The collective behavior of quantum liquids at low temperatures can be described in terms of elementary excitations (quasi-particles), which behave like particles with a given energy, momentum and dispersion relation. Based on the above experimental observations we conclude that this notion only applies to the normal component. In fact, the latter constitutes the gas of thermally excited quasi-particles which follow Bose statistics. Interactions between them or with the walls of the container lead to heat transfer and viscosity. One may ask how the superfluid fraction manages to circumvents these dissipative processes. The answer to this will be Landau's criterion. First, we investigate its consequences. We already discussed the absence of friction. Moreover, as the entropy production in the liquid is related to the creation of quasi-particles, but there is no coupling mechanism between normal and superfluid part, the superfluid flow will be entropy conserving and thus cannot transfer heat. It is counterintuitive that the mass transport of superfluid and normal flow can compensate but nevertheless there is transport of heat. Finally, the superfluid flow has to be irrotational

$$\mathrm{curl}(\vec{v}_s)(\vec{x},t) = 0 \text{ for all } (\vec{x},t) \tag{2.29}$$

which is the macroscopic expression of the fact that the long wavelength excitations of the fluid are sound waves because of a phonon-type spectrum (see below). There is another remarkable fact related to the phonon degrees of freedom. Below the critical temperature T_c there exist two of them, namely a second velocity of sound c_2 which can be shown to be smaller than the first velocity of sound c_1. The second sound constitutes a temperature fluctuation, as opposed to the first sound being a pressure wave. Above T_c only c_1 survives and is given by the expression $c_1^2 = (\partial P/\partial \rho)_S$.

In the last part of this section we will derive Landau's criterion explaining the mentioned lack of dissipative processes for the superfluid fraction of the fluid. It states: *For temperatures below T_c there exists a critical velocity v_c such that for relative velocities $|\vec{v}_n - \vec{v}_s| < v_c$ no thermal excitations of the superfluid part are possible and the superfluid flow remains entropy conserving and inviscid. This critical velocity is given by*

$$v_c = \min_{\vec{q}} \frac{\varepsilon(\vec{q})}{q}. \tag{2.30}$$

The crucial point of the proof is best understood at zero temperature where the normal fraction and hence the normal flow vanish, $\vec{v_n} = 0$. We think of a capillary at rest in the laboratory frame and a superfluid motion with velocity $\vec{v_s}$ relative to it. Let K_0 be the rest frame of the superfluid. As stated above, no elementary excitations are present there and the total energy and momentum in K_0 are given by

$$E_0 = e_0, \tag{2.31}$$

$$\vec{P}_0 = 0 \tag{2.32}$$

where e_0 is the internal energy, whose explicit form is irrelevant, and the total momentum vanishes per definition of the rest frame. If we go to a frame K moving relative to K_0 with velocity $\vec{V}$ the energy and momentum in K will be given by (see for example [44])

$$E = E_0 - \vec{P}_0 \cdot \vec{V} + \frac{1}{2} M V^2, \tag{2.33}$$

$$\vec{P} = \vec{P}_0 - M\vec{V} \tag{2.34}$$

with the total mass of the system being M. In our case, of course, the velocity of the laboratory frame K relative to K_0 is $\vec{V} = -\vec{v}_s$, M corresponds to the mass of the fluid. Now we go one step ahead. Again in the rest frame of the superfluid we assume exactly one quasi-particle of momentum $\vec{q}$ and energy $\varepsilon(\vec{q})$ to be excited. We do not ask for the mechanism of its creation. The energy and momentum will change according to

$$E_0 = e_0 + \varepsilon(\vec{q}), \tag{2.35}$$

$$\vec{P}_0 = \vec{q} \tag{2.36}$$

and so will the quantities in the laboratory frame (using $\vec{V} = -\vec{v}_s$)

$$E = e_0 + \varepsilon(\vec{q}) + \vec{q} \cdot \vec{v}_s + \frac{1}{2} M v_s^2, \tag{2.37}$$

$$\vec{P} = \vec{q} + M\vec{v}_s. \tag{2.38}$$

We see that creation of a quasi-particle increases the energy by an amount of $\varepsilon(\vec{q}) + \vec{q} \cdot \vec{v}_s$. This will be *favored* if it corresponds to a loss of energy. The best we can do to create such an excitation is choosing $\vec{v}_s$ anti-parallel to $\vec{q}$, $\varepsilon(\vec{q}) - q v_s < 0$. If even that *fails* no elementary excitations will occur and the behavior is superfluid. This is Landau's criterion: Below the lambda point, there is a critical velocity (2.30) such that the condition for creation of a quasi-particle can never be fulfilled for $v_s < v_c$. By inspecting (2.30) it is obvious that phonon spectra will satisfy this condition, while an ideal Bose gas will not.

At nonzero temperature, allowing for nontrivial superfluid and normal densities, our reasoning stays the same. The former may move with the velocity field $\vec{v}_s$ and the latter with $\vec{v}_n$. We take a narrow capillary in the laboratory frame, go to the rest frame K_0 of the superfluid motion and excite exactly one quasi-particle with momentum $\vec{q}$. We assume the normal component to be in equilibrium with the capillary due to collisions. This is equivalent to giving the capillary an additional velocity shift of $\vec{v}_n$. For this reason the relative motion between the resting superfluid and the capillary is not given by $\vec{V} = -\vec{v}_s$ but $\vec{v}_n - \vec{v}_s$. The creation of the quasi-particle will therefore cost an energy $\varepsilon(\vec{q}) + \vec{q} \cdot (\vec{v}_s - \vec{v}_n)$. However, we have to modify our argument from above at finite temperature and apply a consistency condition. We already mentioned the quasi-particles to be bosons thus having an equilibrium momentum distribution

$$n_{\vec{q}} = \left[\exp\left(\frac{\varepsilon(\vec{q}) - \vec{q} \cdot (\vec{v}_n - \vec{v}_s)}{k_B T}\right) - 1\right]^{-1} \tag{2.39}$$

where k_B is Boltzmann's constant. For $n(\vec{q})$ to be non-negative we necessarily have to ensure $|\vec{v}_s - \vec{v}_n|$ to be smaller than v_c given by Eq.(2.30). Thus, our above description in terms of a normal and a superfluid part only holds if Landau's criterion is fulfilled.

2.3 The Landau equations of two-fluid hydrodynamics

The last two sections have set the stage for developing the equations of two-fluid hydrodynamics first derived by Landau (1941, [41, 42]). Some related quantities have already been given in the phenomenological part but will be introduced again.

We recall that the macroscopic behavior between zero temperature and the critical temperature can be described in terms of a superfluid and normal fluid flow. Given their velocity fields $\vec{v}_s$ and $\vec{v}_n$, respectively, the momentum density will be

$$\vec{g} = \rho_s \vec{v}_s + \rho_n \vec{v}_n \tag{2.40}$$

wherein the coefficients are given such that their sum

$$\rho = \rho_s + \rho_n \tag{2.41}$$

constitutes the density of the fluid. We will call them superfluid density and normal density, respectively. ρ_s and ρ_n are functions of two independent thermodynamic variables, e.g. pressure P and temperature T, and the relative velocity $\vec{v}_s - \vec{v}_n$. Of course, they cannot depend on any absolute velocities. Mass is conserved during the flow resulting in

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\vec{g}) = 0, \tag{2.42}$$

and so is momentum

$$\frac{\partial g_i}{\partial t} + \partial_k \Pi_{ik} = 0. \tag{2.43}$$

The superfluid density $\rho_s = \rho_s(\rho, T)$ has to be provided by an underlying microscopic theory or phenomenological approach. It plays a similar role like the equation of state. According to the observations in the previous section, the superfluid motion does not transport heat and the heat current is given in terms of the normal flow $\vec{v}_n$ only. We have

$$\vec{q} = Ts\vec{v}_n \tag{2.44}$$

for the heat flux with entropy density s. For ideal hydrodynamics entropy will be conserved yielding

$$\frac{\partial s}{\partial t} + \mathrm{div}(\vec{j}_S) = \frac{\partial s}{\partial t} + \mathrm{div}(s\vec{v}_n) = 0. \tag{2.45}$$

These conservation laws have to be supplemented by another one ensuring the irrotationality of the superfluid flow,

$$\frac{\partial \vec{v}_s}{\partial t} + \nabla \left(\frac{v_s^2}{2} + \mu_m \right) = 0, \tag{2.46}$$

with μ_m being a scalar which turns out to be the chemical potential per unit mass. This is motivated by Euler's law for potential flow given in Eq. (2.15). It represents Josephon's relation.

Further restrictions on the equations have to be applied. First of all, energy conservation shall hold, leading to

$$\frac{\partial \varepsilon}{\partial t} + \mathrm{div}(\vec{j}_E) = 0 \tag{2.47}$$

with energy density ε and energy flux $\vec{j}_E$. Thus we are left with Π_{ik}, μ_m and $\vec{j}_E$ which have to be determined in a consistent manner. Moreover, we demand for

Galilean invariance of the equations, i.e. we restrict ourselves to the homogeneous situation now. (For the equations later to be implemented into a trap, Galilean invariance will of course be broken. However, in the framework of a local density approximation the equations derived for the homogeneous case still hold as we assume the scales for local interaction to be much smaller than the characteristic length of the trapping potential.)

Considering ideal hydrodynamics, where entropy is conserved, we find [38] that μ_m is indeed the chemical potential per unit mass and

$$\vec{j}_S^{(\mathrm{id})} = s\vec{v}_n, \tag{2.48}$$

$$\vec{j}_E^{(\mathrm{id})} = \left(\mu_m + \frac{v_s^2}{2}\right)\vec{g} + Ts\vec{v}_n + \rho_n\vec{v}_n(\vec{v}_n \cdot (\vec{v}_n - \vec{v}_s)), \tag{2.49}$$

$$\Pi_{ik}^{(\mathrm{id})} = \rho_n v_{n,i} v_{n,k} + \rho_s v_{s,i} v_{s,k} + P\delta_{ik}, \tag{2.50}$$

$$P = -\varepsilon_0 + Ts + \mu n + \rho_n(\vec{v}_n - \vec{v}_s)^2, \tag{2.51}$$

where ε_0 is the energy density in the frame of vanishing superfluid velocity $\vec{v}_s$. It is just given by e_0 from (2.31) divided by the unit volume. Eqs. (2.42) - (2.46) together with the given expression for Π_{ik} are called the Landau equations of ideal two-fluid hydrodynamics. We note that they are highly non-linear since besides the terms quadratic in the velocities the quantities ρ_s, ρ_n, μ, s not only depend on the thermodynamic quantities but also on $\vec{v}_s - \vec{v}_n$.

We can include dissipative effects into the hydrodynamic equations by the introduction of transport coefficients, see for example Refs. [45, 46]. These are the coefficients appearing in a further expansion of energy and entropy flux and stress tensor in derivatives of the velocity fields. Entropy is no longer conserved and we have

$$\frac{\partial s}{\partial t} + \mathrm{div}(\vec{j}_S) = \Sigma \tag{2.52}$$

with entropy production rate Σ. In Eq. (2.46) for the superfluid velocity we introduce a field H according to

$$\frac{\partial \vec{v}_s}{\partial t} + \nabla\left(\frac{v_s^2}{2} + \mu_m + H\right) = 0. \tag{2.53}$$

Since higher derivatives are expected to be small, the contribution from the first transport coefficients on the hydrodynamics will be dominant. For a one-component fluid one introduces shear viscosity η, bulk viscosity ζ and heat conductivity κ_T like in Eqs. (2.22) - (2.25). In the two-fluid description we need to introduce five transport coefficients η, ζ_1, ζ_2, ζ_3 and κ_T according to

$$\vec{j}_S = \vec{j}_S^{\,\mathrm{(id)}} + \frac{1}{T}\vec{q}, \tag{2.54}$$

$$j_{E,i} = j_{E,i}^{\mathrm{(id)}} + v_{n,k}\delta\Pi_{ik} + w_i H + q_i, \tag{2.55}$$

$$\Pi_{ik} = \Pi_{ik}^{\mathrm{(id)}} + \delta\Pi_{ik} \tag{2.56}$$

with

$$\vec{q} = -\,\kappa_T \nabla T, \tag{2.57}$$

$$\delta\Pi_{ik} = -\,\eta\sigma_{ik} - \delta_{ik}\left(\zeta_1 \mathrm{div}(\vec{w}) + \zeta_2 \mathrm{div}(\vec{v}_n)\right), \tag{2.58}$$

$$H = -\,\zeta_3 \mathrm{div}(\vec{w}) - \zeta_1 \mathrm{div}(\vec{v}_n). \tag{2.59}$$

Here, $\vec{q}$ is the heat current and

$$\sigma_{ik} = \partial_i v_{n,k} + \partial_k v_{n,i} - \frac{2}{3}\mathrm{div}(\vec{v}_n)\delta_{ik}, \tag{2.60}$$

$$\vec{w} = \rho_s(\vec{v}_s - \vec{v}_n). \tag{2.61}$$

The entropy production rate is given by

$$\Sigma = -\frac{\vec{q}\cdot\nabla T}{T} - \delta\Pi_{ik}\partial_k v_{n,i} - H\mathrm{div}(\vec{w}). \tag{2.62}$$

Every tensor T_{ij} of second rank can be decomposed into a diagonal, symmetric-traceless, and antisymmetric part such that

$$T_{ij} = \frac{1}{3}T_{kk}\delta_{ij} + T_{ij}^{\mathrm{(S)}} + T_{ij}^{\mathrm{(A)}}. \tag{2.63}$$

Therein, $T_{ij}^{\mathrm{(S)}} = \frac{1}{2}(T_{ij} + T_{ji} - \frac{2}{3}T_{kk}\delta_{ij})$ and we assumed the underlying vector space to be three-dimensional. Given another tensor V_{ij} we have

$$T_{ij}V_{ij} = \frac{1}{3}T_{kk}V_{ii} + T_{ij}^{\mathrm{(S)}}V_{ij}^{\mathrm{(S)}} + T_{ij}^{\mathrm{(A)}}V_{ij}^{\mathrm{(A)}}. \tag{2.64}$$

We observe the tensor σ_{ij} appearing together with η in Eq. (2.58) to fulfill

$$(\partial_i v_{n,j})^{\mathrm{(S)}} = \frac{1}{2}\sigma_{ij} = \frac{1}{2}\sigma_{ij}^{\mathrm{(S)}}. \tag{2.65}$$

From this we conclude

$$(\sigma_{ij})(\partial_i v_{n,j}) = \sigma_{ij}^{\mathrm{(S)}}(\partial_i v_{n,j})^{\mathrm{(S)}} = \frac{1}{2}\sigma_{ij}\sigma_{ij} = \frac{1}{2}\mathrm{tr}(\sigma^2). \tag{2.66}$$

Here, tr denotes the trace.

Inserting expressions (2.57) - (2.59) into the entropy production rate Σ we find

$$\Sigma = \frac{\kappa_T}{T}|\nabla T|^2 + \frac{\eta}{2}\mathrm{tr}(\sigma^2) + 2\zeta_1 \mathrm{div}(\vec{w})\mathrm{div}(\vec{v}_n) + \zeta_2[\mathrm{div}(\vec{v}_n)]^2 + \zeta_3[\mathrm{div}(\vec{w})]^2. \quad (2.67)$$

This expression has to be strictly positive for all choices of T, σ, $\vec{w}$ and $\vec{v}_n$. Setting all expect for one of these quantities to zero we successively obtain $\eta, \kappa_T, \zeta_2, \zeta_3 > 0$. With $\nabla T = 0 = \mathrm{tr}(\sigma^2)$ we can write

$$\Sigma = (\mathrm{div}(\vec{v}_n), \mathrm{div}(\vec{w})) \begin{pmatrix} \zeta_2 & \zeta_1 \\ \zeta_1 & \zeta_3 \end{pmatrix} \begin{pmatrix} \mathrm{div}(\vec{v}_n) \\ \mathrm{div}(\vec{w}) \end{pmatrix} > 0, \quad (2.68)$$

from which we conclude $\zeta_2 > 0$ and $\zeta_1^2 < \zeta_2\zeta_3$. The equality of the off-diagonal elements in Eq. (2.68) is due to the Onsager symmetry principle.

Given a unit of mass m, which in our cases will always be associated with the mass of the atoms, we introduce n_s and n_n by $\rho_s = mn_s$ and $\rho_n = mn_n$, respectively. One could also think of m as the effective mass m^* of the collective motion because superfluid hydrodynamics is a macroscopic phenomena and it is in general not true, that m^* coincides with the atomic mass m. However, as the densities $\rho_{s,n}$ themselves, the precise meaning of $n_{s,n}$ is just given by this definition without any further interpretation. It follows $\mu_m = \mu/m$ for the chemical potential per unit mass.

2.4 Linearized equations in an external potential

Our focus will now be on implementing a time-independent potential V_{ext} in the ideal two-fluid equations. We start with a simple motivation of the necessary modifications. Combining Eq. (2.42) with Eq. (2.43) and the explicit form of the stress tensor Π_{ik} in Eq. (2.50) we arrive at

$$\rho_n \left(\frac{\partial}{\partial t} + \vec{v}_n \cdot \nabla\right)\vec{v}_n + \rho_s \left(\frac{\partial}{\partial t} + \vec{v}_s \cdot \nabla\right)\vec{v}_s = -\nabla P, \quad (2.69)$$

which generalizes Euler's equation (2.13) of a one-component fluid. Since this equation represents (except for the factors of mass density) Newton's law

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{1}{m}\vec{F} = -\frac{1}{m}\nabla V_{\mathrm{ext}} \quad (2.70)$$

for the force $\vec{F} = -\nabla V_{\mathrm{ext}}$, we need an additional term $-\frac{1}{m}\rho\nabla V_{\mathrm{ext}} = -n\nabla V_{\mathrm{ext}}$ on the right-hand side of Eq. (2.69). The same argument can be applied to Eq.

(2.46) for the superfluid velocity,

$$\frac{\partial \vec{v}_s}{\partial t} + \nabla \left(\frac{v_s^2}{2} + \frac{1}{m}\mu \right) = \left(\frac{\partial}{\partial t} + \vec{v}_s \cdot \nabla \right) \vec{v}_s + \frac{1}{m} \nabla \mu = 0. \tag{2.71}$$

In the first equality we used $(\vec{v}_s \cdot \nabla)\vec{v}_s = \nabla(\frac{1}{2}v_s^2) - (\vec{v}_s \times \mathrm{curl}(\vec{v}_s))$ and $\mathrm{curl}(\vec{v}_s) = 0$. Again the external potential will modify Eq. (2.71) by an additional term $-\frac{1}{m}\nabla V_{\mathrm{ext}}$ on the right-hand side.

Our result can be formulated in the framework of constitutive equations like before. Since the external potential breaks translational invariance, the law of momentum conservation has to account for the force and therefore the right-hand sides of Eqs. (2.43) and (2.46) are not zero but

$$\partial_t g_i + \partial_k \Pi_{ik} = -n \partial_i V_{\mathrm{ext}}, \tag{2.72}$$

$$\partial_t \vec{v}_s + \frac{1}{m} \nabla \mu = -\frac{1}{m} \nabla V_{\mathrm{ext}}. \tag{2.73}$$

Together with $\partial_t \rho + \mathrm{div}(\vec{g}) = 0$, $\partial_t s + \mathrm{div}(s\vec{v}_n) = 0$ and expressions (2.40), (2.41) and (2.50) this set of equations corresponds to the ideal two-fluid equations in an external potential. The dissipative case is obtained by replacing entropy conservation by energy conservation (or introducing the entropy production rate Σ) and the corresponding form of the stress tensor.

An external potential ("trap") provides a macroscopic length scale of the hydrodynamic system. If this scale is much larger than the characteristic distance over which equilibration takes place in the confined substance (e.g. the mean free path of the particles), we can assume the system to be in local equilibrium at each point $\vec{x}$ in the trap $V_{\mathrm{ext}}(\vec{x})$. The static configuration which emerges in this situation can be obtained by virtue of a local density approximation. We first derive this prescription from thermodynamic arguments. We will then see that it also arises naturally in the static limit of the hydrodynamic equations. Consider a gas to be contained in an external potential $V_{\mathrm{ext}}(\vec{x})$ such that we have thermal equilibrium at each point of space. If we take two neighboring small but yet macroscopic parts of the gas their energies E_1 and E_2 and particle numbers N_1 and N_2 will adjust in a manner maximizing the entropy $S = S_1 + S_2$ under the constraints $E_1 + E_2 = E$ and $N_1 + N_2 = N$, respectively. This implies temperature and the *full* chemical potential to be constant inside the trap. However, the full chemical potential is given by the Gibbs free energy per particle and thus reads $\mu_{\mathrm{full}}(\vec{x}) = \mu_{\mathrm{hom}}(P(\vec{x}), T) + V_{\mathrm{ext}}(\vec{x})$, where $\mu_{\mathrm{hom}}(P, T)$ is the equilibrium chemical potential of the homogeneous system as a function of pressure P and temperature T. We conclude that a system where the chemical potential μ in the homogeneous equilibrium functions is substituted according to $P_{\text{hom}}(\mu, T) \to P_{\text{hom}}(\mu - V_{\text{ext}}, T)$ etc. behaves like a system trapped in an external potential of large spatial extend. This rule is called local density approximation. Furthermore, we see that an equation of state of the form $P(\mu, T)$ is most well-suited for transferring it into a trap. Dealing with thermodynamic functions of the independent variables density n and temperature T one has to invert $\mu(n)$ for $n(\mu)$ for applying the local density approximation, which may be accompanied by numerical errors.

The hydrodynamic equations derived so far are nonlinear. This is obvious for the velocity fields $\vec{v}_s$ and $\vec{v}_n$, e.g. due to terms $\propto v_i v_k$ in the stress tensor Π_{ik}. However, even higher nonlinearities arise from the thermodynamic quantities. One can choose for example $\mu(\vec{x}, t)$ and $T(\vec{x}, t)$ as independent variables. Then, since the equation of state $P(\mu, T)$ and $\rho_n(\mu, T)$ will in general be a function of higher order in μ and T, or even nonanalytic, further nonlinear terms appear in the partial differential equations.

If we are only concerned with small deviations from the static equilibrium solution with small velocity fields, we can use the linearized hydrodynamic equations. Since superfluidity is expected to break down for too high velocities, this is not a bad approximation, but effects due to the nonlinearity (e.g. backreaction of the flow) will not be captured. We first derive the static solution to the two-fluid equations, which corresponds to vanishing velocity field $\vec{v}_{s,0} = \vec{v}_{n,0} = 0$. Ideal and dissipative equations yield the same result in this situation, because the latter consist in an expansion in derivatives of the (vanishing) velocities. The static solution will be denoted by a subscript "0". The stress tensor reduces towards $\Pi_{ik,0} = P_0 \delta_{ik}$ and we find for Eqs. (2.42), (2.72), (2.73) and (2.45)

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{2.74}$$

$$\nabla P_0 = -n_0 \nabla V_{\text{ext}}, \tag{2.75}$$

$$\nabla \mu_0 = -\nabla V_{\text{ext}}, \tag{2.76}$$

$$\frac{\partial s_0}{\partial t} = 0. \tag{2.77}$$

We conclude that number density n_0 and entropy density s_0 are time-independent in equilibrium. Moreover, since V_{ext} is not a function of time we find P_0 and μ_0 to be time-independent. The static profile satisfies the Gibbs-Duhem relation $\mathrm{d}P = n\mathrm{d}\mu + s\mathrm{d}T$ at each point in space $\vec{x}$, i.e. $\nabla P_0 = n_0 \nabla \mu_0 + s_0 \nabla T_0$. Therefore, temperature T_0 is also constant in time. We further read off the local density approximation $\mu_0(\vec{x}) + V_{\text{ext}}(\vec{x}) = \bar{\mu}$ to arise naturally. The parameter $\bar{\mu}$ can be adjusted in order to vary the particle number N. The pressure satisfies $\nabla P_0 = -n_0 \nabla V_{\text{ext}} = n_0 \nabla \mu_0$, which implies $s_0 \nabla T_0 = 0$ because of the Gibbs-

Duhem relation. As $s_0 \neq 0$ for all points in space we deduce $\nabla T_0 = 0$. Two independent thermodynamic variables describe the system. We choose chemical potential and temperature and obtain the right equilibrium behavior: $T_0(\vec{x}, t) = \bar{T}$ is constant in time and space, while $\mu_0(\vec{x}, t) = \bar{\mu} - V_{\text{ext}}(\vec{x})$ is time-independent and follows the external potential according to the local density approximation. This investigation shows that thermodynamics is included in ideal hydrodynamics and higher derivative expansions with dissipative terms. The other quantities can be derived from μ_0 and T_0 by the use of the equation of state $P(\mu, T)$, e.g. $P_0(\vec{x}, t) = P(\mu_0(\vec{x}), T_0(\vec{x})) = P(\bar{\mu} - V_{\text{ext}}, \bar{T})$.

The static solution allows for a systematic expansion of the time-dependent variables in the hydrodynamic equations. As before, we choose chemical potential and temperature to be our independent variables. We linearize

$$\mu(\vec{x}, t) = \mu_0(\vec{x}) + \delta\mu(\vec{x}, t), \tag{2.78}$$

$$T(\vec{x}, t) = \bar{T} + \delta T(\vec{x}, t), \tag{2.79}$$

$$\vec{v}_s(\vec{x}, t) = \delta\vec{v}_s(\vec{x}, t), \tag{2.80}$$

$$\vec{v}_n(\vec{x}, t) = \delta\vec{v}_n(\vec{x}, t), \tag{2.81}$$

where the corrections to the static solution are assumed to be small. Since the velocity fields vanish in equilibrium we cannot quantify “small” in this case. However, superfluidity provides a natural velocity for comparison. Indeed, the superfluid flow is expected to break down above a critical velocity v_c, and therefore we will work with $|\delta\vec{v}_{s,n}|/v_c \ll 1$. In the same way we linearize $P = P_0 + \delta P, n = n_0 + \delta n, s = s_0 + \delta s$. Eqs. (2.42), (2.72), (2.73) and (2.45) in the *ideal case* now read

$$\partial_t \delta\rho + \text{div}(\delta\vec{g}) = 0, \tag{2.82}$$

$$\partial_t \delta\vec{g} + \nabla\delta P + \delta n \nabla V_{\text{ext}} = 0, \tag{2.83}$$

$$m\partial_t \delta\vec{v}_s + \nabla\delta\mu = 0, \tag{2.84}$$

$$\partial_t \delta s + \text{div}(s_0 \delta\vec{v}_n) = 0 \tag{2.85}$$

with $\delta\vec{g} = \rho_{s,0}\delta\vec{v}_s + \rho_{n,0}\delta\vec{v}_n$ containing only first order small quantities. This set of equations will be the starting point for further investigation. Using the relation $\nabla P_0 = s_0 \nabla T_0 + n_0 \nabla\mu_0$ we get $\nabla\delta P = s_0\nabla\delta T + \delta n \nabla\mu_0 + n_0\nabla\delta\mu = s_0\nabla\delta T - \delta n\nabla V_{\text{ext}} + n_0\nabla\delta\mu$, since $\nabla T_0 = 0$ and $\nabla\mu_0 = -\nabla V_{\text{ext}}$. Together with Eq. (2.84) this can be used to cast Eq. (2.83) into the form

$$0 = \rho_{n,0}\partial_t \left(\delta\vec{v}_n - \delta\vec{v}_s\right) + s_0\nabla\delta T. \tag{2.86}$$

In the derivation of Eqs. (2.82) - (2.85) we used the fact that the ideal stress tensor $\Pi_{ik}^{(\text{id})} = P\delta_{ik} + \rho_s v_{s,i} v_{s,k} + \rho_n v_{n,i} v_{n,k}$ is of second order in the velocity fields and these corrections can thus be neglected in linearized equations. Including dissipative processes to first order, we have to apply formula (2.56) for Π_{ik}. This yields

$$\Pi_{ik} = (P_0 + \delta P)\delta_{ik} - \eta\delta\sigma_{ik} - \delta_{ik}(\zeta_1 \mathrm{div}(\delta\vec{w}) + \zeta_2 \mathrm{div}(\delta\vec{v}_n)) \tag{2.87}$$

with

$$\delta\sigma_{ik} = \partial_i \delta v_{n,k} + \partial_k \delta v_{n,i} - \frac{2}{3}\delta_{ik}\mathrm{div}(\delta\vec{v}_n), \tag{2.88}$$

$$\delta\vec{w} = \rho_{s,0}(\delta\vec{v}_s - \delta\vec{v}_n). \tag{2.89}$$

The H-field from Eq. (2.51) is given in linearized form by

$$H = \delta H = -\zeta_3 \mathrm{div}(\delta\vec{w}) - \zeta_1 \mathrm{div}(\delta\vec{v}_n) \tag{2.90}$$

with $H_0 = 0$. The continuity equation is not modified by dissipation. An additional term arises in the entropy flux (2.54)

$$\delta\vec{j}_S = s_0 \delta\vec{v}_n - \frac{1}{T}\kappa_T \nabla\delta T \tag{2.91}$$

which enters $\partial_t s + \mathrm{div}(\vec{j}_S) = \Sigma$. The entropy production rate Σ from Eq. (2.67) is quadratic in the perturbations and thus can be neglected to linear order. The linearized equations including the five transport coefficients of *dissipative* two-fluid hydrodynamics therefore read

$$\partial_t \delta\rho + \mathrm{div}(\delta\vec{g}) = 0, \tag{2.92}$$

$$\partial_t \delta g_i + \partial_i \delta P + \delta n \partial_i V_{\mathrm{ext}} = \partial_k(\eta\delta\sigma_{ik}) + \partial_i(\zeta_1 \mathrm{div}(\delta\vec{w}) + \zeta_2 \mathrm{div}(\delta\vec{v}_n)), \tag{2.93}$$

$$m\partial_t \delta\vec{v}_s + \nabla\delta\mu = m\nabla(\zeta_3 \mathrm{div}(\delta\vec{w}) + \zeta_1 \mathrm{div}(\delta\vec{v}_n)), \tag{2.94}$$

$$\partial_t \delta s + \mathrm{div}(s_0 \delta\vec{v}_n) = \frac{1}{T}\mathrm{div}(\kappa_T \nabla\delta T). \tag{2.95}$$

The linearized ideal hydrodynamic equations (2.82) - (2.85) will be used in Sec. 4 to calculate the eigenfrequencies of collective oscillations of a harmonically trapped quantum gas. This is achieved by numerical solution of the corresponding linear system of equations. The result will then be applied in Sec. 5 to obtain the response of a trapped gas to a variation of the external trapping potential. It will be necessary to include dissipation for these purposes and we therefore use Eqs. (2.92) - (2.95).

3 Equation of state

This chapter provides approximate solutions for the equation of state of a weakly interacting Bose gas in three spatial dimensions. For such a system the interaction can be described in terms of two-body collisions. Since we are only concerned with low-energy properties interaction is completely characterized in terms of the scattering length a. The dimensionless gas parameter a^3n, where n is the density, provides an expansion parameter allowing perturbative treatment in the regimes of dilute (small n) or weakly interacting (small a) systems. In order to ensure the applicability of a hydrodynamic description the density has to be sufficiently large such that local equilibration is given. This is why we are dealing with a weakly interacting Bose gas. For a detailed discussion of the Bose gas see also Ref. [47]

In Sec. 3.3 we show how the equation of state can be extracted from in situ absorption images of the density profile.

3.1 Zero temperature Bose gas: Mean field theory and beyond

The properties of the weakly interacting Bose gas at zero temperature have been discussed by Bogoliubov [48] in a mean-field treatment. This leading order equation of state describes many aspects of the system and provides a solid base to go to higher order corrections. These beyond mean-field effects have been calculated by Lee, Huang and Yang. Both theoretical descriptions rely on a small gas parameter a^3n.

In order to introduce our notation we first consider a *non-interacting* ideal quantum gas. For simplicity, we deal with spinless identical particles in the following. For bosons this is reached when dealing with spin-0 particles whereas spin less (spin-polarized) fermions can be realized in a strong magnetic field with all spins pointing in the same direction. Assuming no further internal degrees of freedom the state of each single particle is described by its momentum $\vec{p}$. More precisely, given the system in a box of finite volume $V = L^3$ with periodic boundary conditions momentum is quantized according to the rule ($\hbar = 1$)

$$\vec{p}_{\vec{\nu}} = \frac{2\pi}{L}\vec{\nu}, \tag{3.1}$$

where $\vec{\nu} \in \mathbb{Z}^3$ is a triple of quantum numbers associated with the single particle state. Obviously, the vectors $\vec{\nu}$ can be used to address and distinguish all states of the single particle Hilbert space. Given more than one non-interacting particles of this kind each state of the many particle Hilbert space can be described in terms of occupation numbers $n_{\vec{\nu}}$ where $n_{\vec{\nu}}$ is the number of particles with momentum $\vec{p}_{\vec{\nu}}$. We write $|\{n_{\vec{\nu}}\}\rangle$ or simply $\{n_{\vec{\nu}}\}$ for such a state.

If we choose other boundary conditions the concrete quantization rule will differ. However, in the infinite volume limit the boundaries will not play a role and the unique continuum limit is reached. Indeed, in the thermodynamic limit, where particle number N and volume V go to infinity while density $n = N/V$ is kept constant, the summation over discrete $\vec{\nu}$ can be replaced by integration over continuous $\vec{\nu} = (2\pi/L)\vec{p}$. For the particle number this reads

$$N = \sum_{\vec{\nu}\in\mathbb{Z}^3} n_{\vec{\nu}} \simeq \int_{\mathbb{R}^3} \mathrm{d}^3\nu n_{\vec{\nu}} = V \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 p}{(2\pi)^3} n_{\vec{p}}. \tag{3.2}$$

It is convenient to write $\sum_{\vec{p}}$ implying that this expression has to be understood as the sum over $\vec{\nu}$ if one is dealing with a finite system or as the integral in the infinite volume limit. In all cases $\sum_{\vec{p}}$ is a dimensionless quantity.

For an *interacting* system the single particle states can still be used to formulate the Hamiltonian of the interacting theory but their products will in general not correspond to eigenstates. Introducing the creation and annihilation operators for bosons of momentum $\vec{p}$, $a^+_{\vec{p}}$ and $a_{\vec{p}}$, respectively, the Hamiltonian of a Bose system interacting only through two-body collisions is given by

$$H = \sum_{\vec{p}} \frac{p^2}{2m} a^+_{\vec{p}} a_{\vec{p}} + \frac{1}{2V} \sum_{\vec{p},\vec{p}',\vec{q}} \tilde{V}(\vec{q}) a^+_{\vec{p}+\vec{q}} a^+_{\vec{p}'-\vec{q}} a_{\vec{p}} a_{\vec{p}'}. \tag{3.3}$$

$\tilde{V}(\vec{q}) = \int \mathrm{d}^3 r V(r) e^{-i\vec{q}\cdot\vec{r}}$ is the Fourier transform of the two-body interaction potential $V(r)$. The Bogoliubov approximation assumes that the main contribution to the particle number comes from the zero momentum state,

$$N \simeq \left\langle a_0^+ a_0 \right\rangle, \tag{3.4}$$

and it is therefore justified to replace the operator a_0 by $\sqrt{N}$ and set all $a_{\vec{p}}$ with $\vec{p} \neq 0$ to zero. This results in a ground state energy

$$E_0 = \frac{1}{2V} \tilde{V}(0) N^2. \tag{3.5}$$

Applying the Born approximation, i.e. treating the interaction potential as a perturbation to the free system, we can replace $\tilde{V}(0) = g = 4\pi a/m$, where a is the scattering length. Note that a has the same sign as $\tilde{V}(0)$ and thus a positive a corresponds to repulsive interactions. Using $\mu = \partial E/\partial N$ we get

$\mu(n) = gN/V = gn$. Since we are interested in the relation $P = P(\mu)$ we integrate $\mathrm{d}P = n\mathrm{d}\mu$ and arrive at

$$P(\mu) = \frac{\mu^2}{2g}. \tag{3.6}$$

The next-to-leading order correction to the equation of state can be obtained by virtue of a Bogoliubov transformation and a renormalization of the coupling g. This yields the ground state energy

$$E_0 = \frac{1}{2V} gN^2 + \frac{1}{2} \sum_{\vec{p}\neq 0} \left(\varepsilon(p) - gn - \frac{p^2}{2m} + \frac{m(gn)^2}{p^2} \right). \tag{3.7}$$

Replacing the sum with an integral this becomes (Lee, Huang, Yang [49, 50])

$$E_0 = \frac{1}{2V} gN^2 \left(1 + \frac{128}{15\sqrt{\pi}} (a^3 n)^{1/2} \right). \tag{3.8}$$

The chemical potential as a function of density is obtained through $\partial E_0/\partial N$ and reads

$$\mu(n) = gn \left(1 + \frac{32}{3\sqrt{\pi}} (a^3 n)^{1/2} \right), \tag{3.9}$$

whereas inverting this relation iteratively we get the density as a function of chemical potential,

$$n(\mu) = \frac{\mu}{g} - \frac{32}{3\sqrt{\pi}} \left(\frac{\mu a}{g} \right)^{3/2}. \tag{3.10}$$

The pressure can now easily be found by integrating $\mathrm{d}P = n\mathrm{d}\mu$ and is given by

$$P(\mu) = \frac{\mu^2}{2g} - \frac{64}{15\sqrt{\pi}} \left(\frac{a}{g} \right)^{3/2} \mu^{5/2}. \tag{3.11}$$

We prove this formula below. It will be used in Sec. 4.4.2 to calculate beyond mean-field shifts of the collective oscillation frequencies of a Bose gas confined to a harmonic trapping potential.

The next-to-next-to-leading order (NNLO) correction to (3.8) has been calculated by Wu [51] and is given by

$$E_0 = \frac{1}{2V} gN^2 \left(1 + \frac{128}{15\sqrt{\pi}} (a^3 n)^{1/2} + 8 \left(\frac{4\pi}{3} - \sqrt{3} \right) a^3 n \right) (\log(a^3 n) + B). \tag{3.12}$$

The constant B depends on the three-body physics.

We now derive (3.11) and also the NNLO correction to $P(\mu) = \mu^2/2g$. We start from formula (3.12) for the energy density $\varepsilon = E/V$ but will neglect the term containing the logarithm. It is then given by

$$\begin{aligned}\varepsilon(n,a) &= \frac{4\pi a n^2}{2}\left(1 + \frac{128}{15\sqrt{\pi}}(na^3)^{1/2} + 8\left(\frac{4\pi}{3} - \sqrt{3}\right)Bna^3\right) && (3.13)\\ &= 2\pi a n^2(1 + c_1(na^3)^{1/2} + c_2 na^3). && (3.14)\end{aligned}$$

In the second line we defined dimensionless constants c_1 and c_2. The chemical potential is obtained by a partial derivative with respect to n. We find

$$\mu(n,a) = 4\pi a n(1 + c_1(na^3)^{1/2} + c_2 na^3) + 2\pi a n^2\left(\frac{c_1}{2}a^{3/2}n^{-1/2} + c_2 a^3\right). \quad (3.15)$$

This equation has to be solved for $n(\mu)$. The pressure will then be given by $P(\mu) = \int \mathrm{d}\mu n(\mu)$. Since (3.15) is only true to order na^3 we can invert the relation $\mu = \mu(n)$ perturbatively for n. The corresponding small parameter is $(\mu a^2)^{1/2}$. Our ansatz for the density reads

$$n(\mu) = \frac{\mu}{4\pi a}(1 + \epsilon_1(\mu a^2)^{1/2} + \epsilon_2\mu a^2). \quad (3.16)$$

Inserting this expression into (3.15) we arrive at

$$\begin{aligned}\mu &= 4\pi a\left[\frac{\mu}{4\pi a}(1 + \epsilon_1(\mu a^2)^{1/2} + \epsilon_2\mu a^2)\right]\\ &\quad + 5\pi c_1 a^{5/2}\left[\frac{\mu}{4\pi a}(1 + \epsilon_1(\mu a^2)^{1/2} + \epsilon_2\mu a^2)\right]^{3/2}\\ &\quad + 6\pi a^4 c_2\left[\frac{\mu}{4\pi a}(1 + \epsilon_1(\mu a^2)^{1/2} + \epsilon_2\mu a^2)\right]^2\\ &= \mu(1 + \epsilon_1(\mu a^2)^{1/2} + \epsilon_2\mu a^2)\\ &\quad + \mu\frac{5\pi c_1}{(4\pi)^{3/2}}(\mu a^2)^{1/2}(1 + \epsilon_1(\mu a^2)^{1/2} + \epsilon_2\mu a^2)^{3/2}\\ &\quad + \mu\frac{6\pi c_2}{(4\pi)^2}\mu a^2(1 + \epsilon_1(\mu a^2)^{1/2} + \epsilon_2\mu a^2)^2. && (3.17)\end{aligned}$$

We neglect terms of higher order than μa^2 and obtain

$$\mu = \mu\left[1 + \left(\epsilon_1 + \frac{5\pi c_1}{(4\pi)^{3/2}}\right)(\mu a^2)^{1/2} + \left(\epsilon_2 + \frac{15\pi c_1\epsilon_1}{2(4\pi)^{3/2}} + \frac{6\pi c_2}{(4\pi)^2}\right)\mu a^2\right], \quad (3.18)$$

which has to be true for any small value of $(\mu a^2)^{1/2}$. We conclude that

$$\begin{aligned}\epsilon_1 &= -\frac{5\pi c_1}{(4\pi)^{3/2}} = -\frac{16}{3\pi}, && (3.19)\\ \epsilon_2 &= -\frac{15\pi c_1\epsilon_1}{2(4\pi)^{3/2}} - \frac{6\pi c_2}{(4\pi)^2} = \frac{128}{3\pi^2} - \frac{3}{\pi}\left(\frac{4\pi}{3} - \sqrt{3}\right)B. && (3.20)\end{aligned}$$

This results in

$$n(\mu) = n_{LO}(\mu) + n_{NLO}(\mu) + n_{NNLO}(\mu)$$

for the density with

$$\begin{aligned} n_{LO}(\mu) &= \frac{\mu}{4\pi a}, && (3.21)\\ n_{NLO}(\mu) &= \frac{\mu}{4\pi a}\left(-\frac{16}{3\pi}\right)(\mu a^2)^{1/2} = -\frac{4}{3\pi^2}\mu^{3/2}, && (3.22)\\ n_{NNLO}(\mu) &= \frac{\mu}{4\pi a}\left(\frac{128}{3\pi^2} - \frac{3}{\pi}\left(\frac{4\pi}{3} - \sqrt{3}\right)B\right)\mu a^2,\\ &= \left(\frac{32}{3\pi^2} - \frac{3}{4\pi^2}\left(\frac{4\pi}{3} - \sqrt{3}\right)B\right)\mu^2 a. && (3.23) \end{aligned}$$

Via integration $P = \int \mathrm{d}\mu n$ we obtain

$$\begin{aligned} P_{LO}(\mu) &= \frac{\mu^2}{8\pi a}, && (3.24)\\ P_{NLO}(\mu) &= -\frac{8}{15\pi^2}\mu^{5/2}, && (3.25)\\ P_{NNLO}(\mu) &= \left(\frac{32}{9\pi^2} - \frac{1}{4\pi^2}\left(\frac{4\pi}{3} - \sqrt{3}\right)B\right)\mu^3 a. && (3.26) \end{aligned}$$

3.2 Lee–Yang theory for Bose gas at nonzero temperature

The calculation of the full thermodynamics of a quantum system requires sophisticated methods because both quantum and thermal fluctuations have to be taken into account. Moreover, the possibility of phase transitions and strong correlations necessarily rules out perturbative treatments. Nevertheless, to get an intuitive understanding of the thermodynamic properties it is wise to first investigate systems in a mean-field or perturbative approach, which of course will not give exact or even reasonable results in all regimes.

The equation of state $P(\mu, T)$ of a three-dimensional weakly interacting Bose gas has been calculated by T.D. Lee and C.N. Yang in the late 1950s [52, 53]. This work is based on zero temperature calculations of low-energy properties of bosonic systems with short-range or contact interactions. It is interesting that different models give similar results which is a manifestation of universality. Indeed, the precise form of the interaction potential is of no importance on large scales (small momenta). We will focus on references [52, 53] where the thermodynamic properties for small gas parameter are deduced for a system of hard sphere bosons using the pseudo-potential method. The diameter of the hard spheres can be translated into a scattering length a and thus Lee–Yang theory describes weakly interacting cold atomic Bose gases. The authors also derive non-equilibrium properties like the two-fluid hydrodynamics from the microscopic model. These equations include that the mass m^* of the collective excitations of the normal fluid component does not have to coincide with the mass m of the atoms. For the present work we neglect such details and only give a summary of the equation of state which enters the Landau two-fluid hydrodynamics.

The mentioned zero temperature treatment of the hard sphere Bose gas reveals that the eigenstates of the interacting N-body system for small gas parameter are approximately given in terms of plane waves with associated momentum $\vec{p}$ but non-trivial energy spectrum, i.e. quasi-particles. As a results, the many body system can be described in terms of occupations numbers giving the number of quanta in the particular state $|\vec{p}\rangle$. At this stage we have to incorporate the bosonic nature of the system which is represented by a macroscopic occupation of the ground state at low temperatures. Therefore, we will describe states by a set of occupation numbers $|\xi, \{m_p\}\rangle$ such that ξN is the number of quasi-particles in the ground state and m_p is the number of quasi-particles with momenta $|\vec{p}| = p > 0$. These numbers fulfill

$$\sum_{\vec{p}\neq 0} m_p = (1-\xi)N. \tag{3.27}$$

ξ and $\{m_p\}$ are stochastic variables and will eventually be replaced by their expectation values in the thermodynamic limit. At zero temperature the occupation of the interacting ground state will be complete, $\xi = 1$, due to missing thermal fluctuations. However, at nonzero temperature some excitations will be present and $\xi < 1$. This behavior is characteristic for superfluidity and indeed we will associate the expectation value $\bar{\xi}$ with the superfluid fraction.

We consider an N-body state $|\xi, \{m_p\}\rangle$. The energy of this configuration is found to have two contributions,

$$E(\xi, \{m_p\}, N) = E_0 + \sum_{\vec{p}\neq 0} m_p \omega_p, \tag{3.28}$$

where

$$E_0(\xi, N) = \frac{gn^2}{2} V \left(1 + (1-\xi)^2 + \frac{128}{15\sqrt{\pi}} (na^3\xi^3)^{1/2} + O(na^3) \right), \tag{3.29}$$

is the energy of the ground state with occupation ξN and the excited states are subject to the energy dispersion relation

$$\omega_p(\xi) = \sqrt{\varepsilon_p(\varepsilon_p + 2gn\xi)}. \tag{3.30}$$

Here, $g = 4\pi\hbar^2 a/m$ is the coupling constant and $\varepsilon_p = p^2/2m$ is the free particle energy. Setting $\xi = 1$ we immediately recover expression (3.8). In the following we will neglect the term proportional to $(na^3)^{1/2}$ in (3.29) because it represents a minor contribution to the equation of state as compared to thermal fluctuations. Furthermore, we will use units where $\hbar = k_B = m = 1$.

To highlight the main idea in calculating the equation of state under the constraint (3.27) on the excited particles $\{m_p\}$ we recall the different approaches to a quantum system in the canonical and the grand canonical formalism. We then apply this technique and introduce a parameter $\tilde{\mu}$ which will, however, not coincide with the chemical potential. We then discuss the phase structure of the system and give explicit formulae for the equation of state in both phase. In App. C we visualize the corresponding hypersurfaces of thermodynamic observables.

3.2.1 On the canonical and grand canonical partition function

We assume to have a bosonic gas of quasi-particles with dispersion relation ω_p. A state of this system is then described by occupation numbers $\{n_p\}$. We emphasize that n_p is a stochastic variable for each p. The number operator $\hat{n}_p$ shall act via $\hat{n}_p \left|\{n_p\}\right\rangle = n_p \left|\{n_p\}\right\rangle$ on the state $\left|\{n_p\}\right\rangle$.

In order to implement the constraint that there are N particles in the system we have two possibilities. The first one is the *canonical* approach where we demand each set of occupation numbers $\{n_p\}$ to satisfy

$$\sum_p n_p \overset{!}{=} N. \tag{3.31}$$

Thus, the Hamilton operator $\hat{H} = \sum_p \omega_p \hat{n}_p$ appearing in the partition function only acts on states $\left|\{n_p\}\right\rangle$ which are subject to the constraint (3.31). The partition function is then given by

$$Z_C(N) = \mathrm{tr}' e^{-\beta \hat{H}} = \sum_{\substack{\{n_p\} \\ \sum_p n_p = N}} e^{-\beta \sum_p \omega_p n_p}. \tag{3.32}$$

In the thermodynamic limit, the number of particles will automatically be given by N and the density is found to be $n = N/V$. However, for practical purposes the *grand canonical* approach is more convenient. Here we implement the constraint only on the expectation values of $\{n_p\}$,

$$\left\langle \sum_p n_p \right\rangle = \sum_p \bar{n}_p \overset{!}{=} N. \tag{3.33}$$

Such a constraint can be included in the description by the introduction of a parameter μ which constructs a family of systems. The system of N particles is then found to be one of these systems with a particular choice of μ. The grand canonical partition function as a function of μ is given by

$$Z_G(\mu) = \mathrm{tr} e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_{\{n_p\}} e^{-\beta\sum_p(\omega_p-\mu)n_p}, \tag{3.34}$$

where $\hat{N} = \sum_p \hat{n}_p$ is the number operator. Eq. (3.33) can now be reformulated as

$$\frac{1}{\beta}\left.\frac{\partial \log Z_G}{\partial\mu}\right|_{\mu=\mu(N)} \stackrel{!}{=} N. \tag{3.35}$$

The crucial point is that Eq. (3.34) can be evaluated because the summation is over all states. With this expression for the partition function at hand, Eq. (3.35) can then be solved, e.g. numerically. For a gas of quasi-particles we have

$$\begin{aligned} Z_G(\mu) &= \sum_{\{n_p\}} e^{-\beta\sum_p(\omega_p-\mu)n_p} = \sum_{\{n_p\}}\prod_p e^{-\beta(\omega_p-\mu)n_p} \\ &= \prod_p \sum_{n_p=0}^{\infty} e^{-\beta(\omega_p-\mu)n_p} = \prod_p \frac{1}{1-e^{-\beta(\omega_p-\mu)n_p}}. \end{aligned} \tag{3.36}$$

In the last step we must require $\mu \leq 0$.

3.2.2 Canonical partition function and free energy density

In this section we work – like Lee and Yang – in the canonical ensemble. Thus we have temperature T, volume V and particle number N as external parameters. We define density via $n = N/V$. As described in Eqs. (3.27) - (3.30) we consider microscopic configurations $|\xi, \{m_p\}\rangle$ with energy

$$E_0(\xi, \{m_p\}) = \frac{g}{2}n^2V(1+(1-\xi)^2) + \sum_{p\neq 0} m_p\omega_p(\xi) \tag{3.37}$$

The dispersion relation $\omega_p(\xi)$ is given in Eq. (3.30) and depends explicitly on ξ and n. In addition to this difficulty we have to implement the constraint $\sum_p m_p = (1-\xi)N$. As we have learned in the previous part of this section such a constraint can be enforced microscopically on the occupation numbers itself or only on their expectation values. In order to compute the partition function we choose the second possibility and demand the operator $\hat{O} = \sum_{p\neq 0} \hat{m}_p - (1-\hat{\xi})N$ to have expectation value zero,

$$\left\langle \hat{O} \right\rangle = \left\langle \sum_{p\neq 0} \hat{m}_p - (1-\hat{\xi})N \right\rangle \overset{!}{=} 0. \tag{3.38}$$

This is achieved by the introduction of a parameter $\tilde{\mu}$ and the corresponding partition function reads

$$Z = \mathrm{tr} e^{-\beta(\hat{H}-\tilde{\mu}\hat{O})}. \tag{3.39}$$

Eq. (3.38) becomes

$$\left.\frac{\partial \log Z}{\partial \tilde{\mu}}\right|_{\tilde{\mu}(n,T),\xi(n,T)} \overset{!}{=} 0. \tag{3.40}$$

Here, the Hamiltonian is given by

$$\hat{H} = \frac{g}{2} n^2 [1 + (1-\hat{\xi})^2] V + \sum_{p\neq 0} \hat{m}_p \omega_p(\xi). \tag{3.41}$$

The partition function can now readily be computed. We find

$$\begin{aligned} Z &= \sum_{|\xi, m_p\rangle} e^{-\beta\left[\frac{g}{2}n^2[1+(1-\xi)^2]V + \sum_{p\neq 0} m_p(\omega_p(\xi)-\tilde{\mu}) + \tilde{\mu}N(1-\xi)\right]} \\ &= \int_0^1 \mathrm{d}\xi e^{-\beta\left[\frac{g}{2}n^2[1+(1-\xi)^2]V + \tilde{\mu}N(1-\xi)\right]} \sum_{|m_p\rangle} e^{-\beta\sum_{p\neq 0} m_p(\omega_p(\xi)-\tilde{\mu})} \\ &= \int_0^1 \mathrm{d}\xi \exp\left\{-\beta V \left[\varepsilon_0(\xi) + \tilde{\mu}n(1-\xi) + T\int_p \log\left(1 - e^{-\beta(\omega_p(\xi)-\tilde{\mu}}\right)\right]\right\} \\ &= \int_0^1 \mathrm{d}\xi e^{-\beta V \varphi(\xi)}. \end{aligned} \tag{3.42}$$

In the last line we defined the function $\varphi(\xi)$. The contribution to the integral from the *most probable value* $\bar{\xi}$, which satisfies

$$\varphi(\bar{\xi}) \overset{!}{=} \min, \tag{3.43}$$

will be overwhelming in the thermodynamic limit $V \to 0$. The most probable value of the distribution of the excitations, $\bar{m}_p$, is found from Eq. (3.40) to be

$$n(1-\bar{\xi}) = \int_p \bar{m}_p = \int_p \frac{1}{e^{\beta(\omega_p(\bar{\xi})-\tilde{\mu})} - 1}. \tag{3.44}$$

This equation determines $\tilde{\mu}$. We identify $n(1-\bar{\xi})$ with the normal fluid density n_n. If $\bar{\xi}$ and $\tilde{\mu}$ are known, the free energy density, which is related to the partition function via $Z = e^{-\beta V f}$, becomes

$$\begin{aligned} f(n,T) &= \varphi(\bar{\xi}) \\ &= \frac{g}{2}(n^2 + n_n^2) + \tilde{\mu} n_n + T \int_p \log\left(1 - e^{-\beta(\omega_p - \tilde{\mu})}\right). \end{aligned} \tag{3.45}$$

The system possesses two distinct phases. A normal one, where $\bar{\xi} = 0$, and a superfluid one with nonvanishing $\bar{\xi}$. We define the critical density

$$n_c = \int_p \bar{m}_p|_{\tilde{\mu}=0} = \int_p \frac{1}{e^{\beta\varepsilon_p} - 1} = \zeta(3/2)\left(\frac{T}{2\pi}\right)^{3/2}. \tag{3.46}$$

The relation between n and n_c is important for the equations which determine the functions $n_n(n,T)$ and $\tilde{\mu}(n,T)$ entering $f(n,T)$. For $n \leq n_c$, n_n satisfies

$$n_n = \int_p \frac{1}{e^{\beta(\omega_p - \tilde{\mu})} - 1} \tag{3.47}$$

and $n_n = n$. Thus, only $\tilde{\mu}(n,T)$ has to be calculated. The gas is in the normal phase. For $n \geq n_c$ the gas is in the superfluid phase. n_n is still subject to Eq. (3.47), but in addition we have

$$\tilde{\mu} = g \int_p \bar{m}_p \left(\frac{\varepsilon_p}{\omega_p} - 1\right). \tag{3.48}$$

The latter condition ensures that $\bar{\xi}$ maximizes $\varphi(\xi)$, see Eq. (3.43).

Using $\mathrm{d}f = -s\mathrm{d}T + \mu \mathrm{d}n$ we find the chemical potential to be

$$\mu(n,T) = \left(\frac{\partial f}{\partial \mu}\right)_T = \tilde{\mu} + g(n + n_n). \tag{3.49}$$

This relation holds in both phases. In particular we have

$$\mu_c(T) = \mu(n_c, T) = 2gn_c = 2g\zeta(3/2)\left(\frac{T}{2\pi}\right)^{3/2} \tag{3.50}$$

for the critical chemical potential corresponding to $n_n = n$ and $\tilde{\mu} = 0$. It will play an important role in the following. From the inversion of Eq. (3.51) we find $n(\mu, T)$ which provides the expression

$$P(\mu,T) = \mu n(\mu,T) - f(n(\mu,T),T) \tag{3.51}$$

for the pressure. The pressure is the Legendre transform of the free energy density, because $\mathrm{d}(\mu n - f) = n\mathrm{d}\mu + s\mathrm{d}T$ satisfies the Gibbs-Duhem relation, and thus has to be the pressure. We have

$$n(\mu, T) = \begin{cases} \frac{\mu - \tilde{\mu}}{g} - n_n, & \mu > \mu_c, \\ \frac{\mu - \tilde{\mu}}{2g}, & \mu \leq \mu_c, \end{cases} \tag{3.52}$$

where $\mu > \mu_c$ corresponds to the superfluid and $\mu \leq \mu_c$ to the normal phase, respectively. This can now be applied to the pressure

$$P(n, T) = \begin{cases} n_s\tilde{\mu} + \frac{g}{2}n^2 + gn_n(n - \frac{n_n}{2}) - T\int_p \log(1 - e^{-\beta(\omega_p - \tilde{\mu})} & \mu > \mu_c, \\ gn^2 - T\int_p \log(1 - e^{-\beta(\omega_p - \tilde{\mu})}), & \mu \leq \mu_c, \end{cases} \tag{3.53}$$

with $n_s = n - n_n$. $n_n(\mu, T)$ and $\tilde{\mu}(\mu, T)$ can be regarded as functions of the grand canonical variables μ and T and are determined by the same Eqs. as before, (3.47) and (3.48), but with the appropriately changed dispersion relation

$$\omega_p = \begin{cases} \sqrt{\varepsilon_p(\varepsilon_p + 2(\mu - \tilde{\mu}) - 4gn_n)}, & \mu > \mu_c \\ \varepsilon_p, & \mu \leq \mu_c. \end{cases} \tag{3.54}$$

The equation of state is *discontinuous* at the phase transition in this framework: When approaching the critical line from the normal phase, we have $n \to n_c$ with the critical density from Eq. (3.46) as it should be. In the superfluid phase, however, we stay above this value and the relative size of the jump increases with higher interaction parameter a. To see how this behavior emerges we need to consider Eqs. (3.47) and (3.48) for $n_n(\mu, T)$ and $\tilde{\mu}(\mu, T)$. This set of equations can be solved by iteration. We start with $n_n = \tilde{\mu} = 0$ in Eq. (3.47) and then continue to insert the outcome into the respective other equation and so on. (We find that the result does not depend on the choice of initial values, one could as well start for example from $n_n = n_c$. The function $\tilde{\mu}$ has to be non-positive in order to satisfy Eq. (3.47) with $n_n \geq 0$.) After several iterations one can check the result by plotting the differences of the left- and right-hand sides of Eqs. (3.47) and (3.48), respectively. The exact result would then be constantly zero. The numerical approximation to this value can be made arbitrary small by a large number of iterations. The normal fluid density n_n is then found to be discontinuous at the phase transition and hence all other thermodynamic quantities because they depend on n_n. Since n_n satisfies Eq. (3.47) we expect this (rather unsatisfactory) behavior to be intrinsic to our approach to the equation of state. At the second order superfluid phase transition the density should be continuous.

For the scope of this thesis the critical behavior in Lee–Yang-theory is irrelevant and will not be further discussed.

All thermodynamic quantities can be deduced from the formulas given above. We found the following procedure convenient: In the superfluid phase one uses Eqs. (3.47), (3.48), and (3.52) - (3.54) to obtain $\tilde{\mu}(\mu, T)$, $n_n(\mu, T)$ and $P(\mu, T)$. The thermodynamic derivatives can then be calculated by virtue of finite difference formulas. Since the evaluation of the integrals is computationally demanding we use an expansion in Chebychev-polynomials. In the normal phase, however, analytic formulas for all thermodynamic quantities can be derived. In order to do so, we derive the derivatives of the pressure in the canonical variables n and T and then give formulas to relate them to μ and T.

For performing the numerical integrations over p from zero to infinity in the superfluid phase, we found it convenient to use an UV-cutoff $p \leq \Lambda$ of the form

$$\Lambda(\mu, T, n_n) = \Big(-(\mu - 2gn_n) + \big[(\mu - 2gn_n)^2 + (100T)^2\big]^{1/2}\Big)^{1/2}. \tag{3.55}$$

The value of $\tilde{\mu}$ is of no relevance here.

The polylogarithmic function $\mathrm{Li}_\nu(z)$ is defined by

$$\mathrm{Li}_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \mathrm{d}t \frac{t^{\nu-1}}{z^{-1}e^t - 1} = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu}. \tag{3.56}$$

It satisfies $z \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Li}_\nu(z) = \mathrm{Li}_{\nu-1}(z)$ and $\mathrm{Li}_\nu(1) = \zeta(\nu)$. We are only interested in the region $0 \leq z \leq 1$, where the polylogarithm is real.

Assume n and T are fixed such that we are in the normal phase, i.e. $n \leq n_c$. From $\tilde{\mu}(n, T)$ we define $\tilde{z} = e^{\beta\tilde{\mu}}$ such that the pressure becomes

$$P(n, T) = gn^2 + \frac{T^{5/2}}{(2\pi)^{3/2}} \mathrm{Li}_{5/2}(\tilde{z}). \tag{3.57}$$

The free energy density is given by

$$f(n, T) = gn^2 + \tilde{\mu}n - \frac{T^{5/2}}{(2\pi)^{3/2}} \mathrm{Li}_{5/2}(\tilde{z}). \tag{3.58}$$

Eq. (3.47), which determines $\tilde{\mu}(n, T)$, reads

$$n = \left(\frac{T}{2\pi}\right)^{3/2} \mathrm{Li}_{3/2}(\tilde{z}). \tag{3.59}$$

For the derivatives with respect to n and T we have

$$\left(\frac{\partial}{\partial n}\right)_T = \left(\frac{\partial \tilde{z}}{\partial n}\right)_T \frac{\partial}{\partial \tilde{z}} = \frac{1}{T}\left(\frac{\partial \tilde{\mu}}{\partial n}\right)_T \tilde{z}\frac{\partial}{\partial \tilde{z}}, \tag{3.60}$$

$$\left(\frac{\partial}{\partial T}\right)_n = \left(\frac{\partial \tilde{z}}{\partial T}\right)_n \frac{\partial}{\partial \tilde{z}} = \frac{1}{T}\left[\left(\frac{\partial \tilde{\mu}}{\partial T}\right)_n - \frac{\tilde{\mu}}{T}\right] \tilde{z}\frac{\partial}{\partial \tilde{z}}. \tag{3.61}$$

Differentiating Eq. (3.59) we can derive $\partial\tilde{\mu}/\partial n$ and $\partial\tilde{\mu}/\partial T$. This yields

$$\left(\frac{\partial \tilde{\mu}}{\partial n}\right)_T = T\left(\frac{T}{2\pi}\right)^{-3/2}\left[\mathrm{Li}_{1/2}(\tilde{z})\right]^{-1/2}, \tag{3.62}$$

$$\left(\frac{\partial \tilde{\mu}}{\partial T}\right)_n = \frac{\tilde{\mu}}{T} - \frac{3}{2}\frac{\mathrm{Li}_{3/2}(\tilde{z})}{\mathrm{Li}_{1/2}(\tilde{z})}. \tag{3.63}$$

By application of these relations we can obtain higher thermodynamic derivatives of the pressure. We have

$$\left(\frac{\partial P}{\partial n}\right)_T = n\left[\frac{(2\pi)^{3/2}T^{-1/2}}{\mathrm{Li}_{1/2}(\tilde{z})} + 2g\right], \tag{3.64}$$

$$s = -\left(\frac{\partial f}{\partial T}\right)_n = \frac{5}{2}\left(\frac{T}{2\pi}\right)^{3/2}\mathrm{Li}_{5/2}(\tilde{z}) - \frac{\tilde{\mu}n}{T}, \tag{3.65}$$

$$\frac{C_V}{V} = T\left(\frac{\partial s}{\partial T}\right)_n = \frac{15}{4}\left(\frac{T}{2\pi}\right)^{3/2}\mathrm{Li}_{5/2}(\tilde{z}) - \frac{9}{4}n\frac{\mathrm{Li}_{3/2}(\tilde{z})}{\mathrm{Li}_{1/2}(\tilde{z})}, \tag{3.66}$$

$$\left(\frac{\partial P}{\partial T}\right)_n = \frac{5}{2}\left(\frac{T}{2\pi}\right)^{3/2}\mathrm{Li}_{5/2}(\tilde{z}) - \frac{3}{2}n\frac{\mathrm{Li}_{3/2}(\tilde{z})}{\mathrm{Li}_{1/2}(\tilde{z})}. \tag{3.67}$$

We now want to relate the expressions (3.64) - (3.67) for the normal phase in the canonical variables (n, T) to the grand canonical set (μ, T). To do so, we need to know $\tilde{\mu}(\mu, T)$ and $n(\mu, T)$. Using Eq. (3.52) we have $n(\mu, T) = (\mu - \tilde{\mu})/2g$ and therefore $\tilde{\mu}(\mu, T)$ is found by solving Eq. (3.59),

$$\tilde{\mu}(\mu, T) = \mu - 2g\left(\frac{T}{2\pi}\right)^{3/2}\mathrm{Li}_{3/2}(e^{\beta\tilde{\mu}(\mu,T)}). \tag{3.68}$$

In the following, a superscript denotes partial differentiation with respect to the corresponding variable. From

$$\left(\frac{\partial P}{\partial n}\right)_T = \frac{\partial(P,T)}{\partial(n,T)} = \frac{P^\mu}{P^{\mu\mu}}, \tag{3.69}$$

$$\begin{aligned}\frac{C_V}{V} = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{V,N} &= T\left(\frac{\partial s}{\partial T}\right)_n = T\frac{\partial(T,\mu)}{\partial(T,n)}\frac{\partial(s,n)}{\partial(T,\mu)} \\ &= T\left[P^{TT} - \frac{(P^{\mu T})^2}{P^{\mu\mu}}\right],\end{aligned} \tag{3.70}$$

$$\left(\frac{\partial P}{\partial T}\right)_n = \frac{\partial(P,n)}{\partial(T,\mu)}\frac{\partial(T,\mu)}{\partial(T,n)} = P^T - \frac{P^\mu P^{\mu T}}{P^{\mu\mu}}. \tag{3.71}$$

we obtain all derivatives of $P(\mu, T)$ as

$$P^\mu = n, \tag{3.72}$$

$$P^T = s, \tag{3.73}$$

$$P^{\mu\mu} = \frac{P^\mu}{(\partial P/\partial n)_T}, \tag{3.74}$$

$$P^{\mu T} = \frac{P^{\mu\mu}}{P^\mu}\left[P^T - \left(\frac{\partial P}{\partial T}\right)_n\right], \tag{3.75}$$

$$P^{TT} = \left(\frac{\partial s}{\partial T}\right)_n + \frac{(P^{\mu T})^2}{P^{\mu\mu}}, \tag{3.76}$$

$$\det P = P^{\mu\mu}P^{TT} - (P^{\mu T})^2 = P^{\mu\mu}\left(\frac{\partial s}{\partial T}\right)_n. \tag{3.77}$$

Inserting the expressions (3.64) - (3.67), $\tilde{\mu}(\mu, T)$ and $n(\mu, T)$, we get the full grand canonical equation of state and its derivatives.

At a second order phase transition, the specific heat at constant volume C_V is expected to be finite. For constant pressure instead, C_P approaches the critical temperature T_c like $|T - T_c|^{-\alpha}$ with the critical index $\alpha = -0.0146(8)$ [54] of the three-dimensional XY-universality class. Thus, C_P possesses a cusp but will not diverge. This universal statement can be made without knowledge of the equation of state. Indeed, the two-fluid hydrodynamic description requires the order parameter to belong to the mentioned universality class. The compressibility at constant temperature or entropy, respectively, also behave differently at the phase transition. While κ_S stays finite, the isothermal compressibility diverges like $\kappa_T \sim |T - T_c|^{-\gamma}$ with $\gamma = 1.3177(5)$ [54] again evaluated for the three-dimensional XY-universality class. However, Lee–Yang theory cannot resolve the critical physics because its calculation would require the inclusion of fluctuations which are neglected in the saddle point approximation for the partition function.

Some thermodynamic relations are derived in App. B. In App. C we plot the two-dimensional functions $P(\mu, T)$ and $n_n(\mu, T)$ and corresponding thermodynamic derivatives. These pictures show that the equation of state can be calculated for the full range of temperatures. This turns out to be necessary for our analysis of collective modes in Sec. 4.

3.3 Equation of state from in situ density images

We show in the following that the equation of state is accessible in experiments from *in situ* density images. The brilliant idea of how this can be achieved for a harmonically trapped gas was put forward by Ho and Zhou [2] and has already been applied successfully, see Ref. [3] for a nice example.

Let us assume that the gas is trapped in a cylindrical potential of the form

$$V_{\mathrm{ext}}(\vec{x}) = \frac{m}{2}(\omega_r^2 x^2 + \omega_r^2 y^2 + \omega_z z^2), \tag{3.78}$$

i.e. $\omega_x = \omega_y = \omega_r$ (r for "radial".) When the system is in equilibrium at each point of space the density profile is obtained by virtue of a local density approximation from the equation of state $P(\mu, T)$ via

$$n(\vec{x}) = \frac{\partial P}{\partial \mu}(\bar{\mu} - V_{\mathrm{ext}}(\vec{x}), T), \tag{3.79}$$

where $\bar{\mu}$ is the chemical potential in the center of the trap. Sending a probe beam through the three-dimensional atomic cloud yields a two-dimensional absorption image, e.g. on a CCD-camera. If the probe beam was along the x-axis, we obtain the averaged density $\bar{n}(y, z) = \int \mathrm{d}x n(\vec{x})$. A further integration of this experimental data along the y-axis yields $\bar{n}(z) = \int \mathrm{d}x \mathrm{d}y$. We express this in cylindrical coordinates to obtain

$$\begin{aligned} \bar{n}(z) &= \int \mathrm{d}x\mathrm{d}y n(\vec{x}) \\ &= \int \mathrm{d}x\mathrm{d}y \frac{\partial P}{\partial \mu}(\bar{\mu} - \frac{m}{2}(\omega_r^2 x^2 + \omega_r^2 y^2 + \omega_z z^2), T) \\ &= 2\pi \int_0^\infty \mathrm{d}r r \frac{\partial P}{\partial \mu}(\mu_z - \frac{m}{2}\omega_r^2 r^2, T), \end{aligned} \tag{3.80}$$

where $\mu_z = \bar{\mu} - \frac{m}{2}\omega_z^2 z^2$. Note that the density vanishes for $r \to \infty$. Defining

$\xi = \mu_z - \frac{m}{2}\omega_r^2 r^2$, we have

$$\begin{aligned}\bar{n}(z) &= -\frac{2\pi}{m\omega_r^2}\int_{\mu_z}^{-\infty} \mathrm{d}\xi \, \underbrace{\frac{\partial \xi}{\partial \mu}}_{=1} \frac{\partial P}{\partial \xi}(\xi, T) \\ &= -\frac{2\pi}{m\omega_r^2}\left(P(\mu = -\infty, T) - P(\mu_z, T)\right) \\ &= \frac{2\pi}{m\omega_r^2} P(\mu_z, T). \qquad (3.81)\end{aligned}$$

In the last step we assumed that together with the density also the pressure vanishes at infinity, i.e. for very large negative chemical potential. Thus we arrive at

$$P(\mu_z, T) = \frac{m\omega_r^2}{2\pi}\bar{n}(z). \qquad (3.82)$$

If $\omega_x \neq \omega_y$, the derivation requires slight modifications which result in the replacement $\omega_r^2 \to \omega_x \omega_y$ in Eq. (3.82).

Formula (3.82) can be applied for known chemical potential in the center of the trap $\bar{\mu}$ and temperature T to obtain the full equation of state. Indeed, varying $z \in [0, z_{\max}]$, the value of $\mu_z = \bar{\mu} - \frac{m}{2}\omega_z^2 z^2$ drives through a particular range of chemical potentials $\mu \in [\bar{\mu}, \mu_{\min}]$.

4 Collective modes of trapped cold gases

In this section we show how collective oscillation frequencies of a trapped gas in the hydrodynamic regime can be derive from the knowledge of the equation of state $P(\mu, T)$ and normal fluid density $n_n(\mu, T)$. In Secs. 4.2.1, 4.2.2, 4.3, 4.4.1, 4.4.3, 4.5 we closely follow Ref. [55].

4.1 Eigenvalue problem for calculating collective frequencies

The linearized two-fluid hydrodynamic equations (2.82) - (2.85) simplify if we assume the time-dependence of the contained quantities to be a harmonic oscillation with frequency ω corresponding to a collective motion of the trapped gas as a whole. Only certain values for ω will be allowed in dependence of the equation of state. We insert

$$\delta n(t, \vec{x}) = e^{-i\omega t}\delta n(\vec{x}) \tag{4.1}$$

and similar expressions for $\delta\vec{v}_s$, $\delta\vec{v}_n$, δT, δP, $\delta\mu$ and δs into the linearized two-fluid equations in a trap and obtain

$$\begin{aligned} 0 &= -i\omega\delta\rho + \mathrm{div}(\rho_{s,0}\delta\vec{v}_s + \rho_{n,0}\delta\vec{v}_n), && (4.2)\\ 0 &= -im\omega\delta\vec{v}_s + \nabla\delta\mu, && (4.3)\\ 0 &= -i\omega\rho_{n,0}(\delta\vec{v}_n - \delta\vec{v}_s) + s_0\nabla\delta T, && (4.4)\\ 0 &= -i\omega\delta s + \mathrm{div}(s_0\delta\vec{v}_n). && (4.5) \end{aligned}$$

Since the equations of motion are linear, the complex ansatz in Eq. (4.1) is justified. We could also work with $\delta n \propto \mathrm{Re}[e^{-i\omega t}]$ etc. In this section all quantities will only depend on spatial coordinates justifying our convention of labeling the spatial part of $\delta n(t, \vec{x})$ again with $\delta n(\vec{x})$. We hope this to brighten the presentation and not to produce confusion. The static solution corresponding to vanishing velocity fields is given by local density approximation

$$\begin{aligned} \mu_0(\vec{x}) &= \bar{\mu} - V_{\mathrm{ext}}(\vec{x}), && (4.6)\\ T_0(\vec{x}) &= T, && (4.7)\\ n_0(\vec{x}) &= P^{\mu}(\mu_0(\vec{x}), T), && (4.8)\\ s_0(\vec{x}) &= P^{T}(\mu_0(\vec{x}), T), && (4.9) \end{aligned}$$

with equation of state $P = P(\mu, T)$. A superscript denotes a partial derivative of the equation of state $P(\mu, T)$ with respect to the corresponding variable.

It is apparent that Eqs. (4.2) - (4.5) constitute an eigenvalue problem for a partial differential operator acting on $(\delta\rho, \delta\vec{v}_s, \delta\vec{v}_n - \delta\vec{v}_n, \delta s)$. We can rewrite this set of equations by eliminating the velocity fields. This yields

$$0 = \omega^2 \delta n + \mathrm{div}(n_0 \nabla \delta\mu + s_0 \nabla \delta T), \tag{4.10}$$

$$0 = \omega^2 \delta s + \mathrm{div}(s_0 \nabla \delta\mu + \tilde{n}_0 \nabla \delta T), \tag{4.11}$$

where we defined $\tilde{n} = s^2/n_n$. Two thermodynamic variables describe the system. The fluctuations of particle number and entropy density, respectively, can be expressed as a linear combination of $\delta\mu$ and δT via $\delta n = P_0^{\mu\mu}\delta\mu + P_0^{\mu T}\delta T$ and $\delta s = P_0^{\mu T}\delta\mu + P_0^{TT}\delta T$. In a compact matrix notation we then find Eqs. (4.10) and (4.11) to be equivalent to

$$\begin{pmatrix} P_0^{\mu\mu} & P_0^{\mu T} \\ P_0^{\mu T} & P_0^{TT} \end{pmatrix}^{-1} \begin{pmatrix} -\mathrm{div}(n_0 \nabla \cdot) & -\mathrm{div}(s_0 \nabla \cdot) \\ -\mathrm{div}(s_0 \nabla \cdot) & -\mathrm{div}(\tilde{n}_0 \nabla \cdot) \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix} = \omega^2 \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix}. \tag{4.12}$$

The first matrix appearing in this equation is the inverse of the Hessian matrix of $P(\mu, T)$, which is, of course, given by

$$\begin{pmatrix} P_0^{\mu\mu}(\vec{x}) & P_0^{\mu T}(\vec{x}) \\ P_0^{\mu T}(\vec{x}) & P_0^{TT}(\vec{x}) \end{pmatrix}^{-1} = \frac{1}{\det P_0(\vec{x})} \begin{pmatrix} P_0^{TT}(\vec{x}) & -P_0^{\mu T}(\vec{x}) \\ -P_0^{\mu T}(\vec{x}) & P_0^{\mu\mu}(\vec{x}) \end{pmatrix} \tag{4.13}$$

for each $\vec{x}$. The Hessian matrix is positive and its inverse always exists. Indeed, we have

$$P^{\mu\mu} = \frac{\partial(n,T)}{\partial(P,T)} \frac{\partial(P,T)}{\partial(\mu,T)} = n \left(\frac{\partial P}{\partial n} \right)_T = n v_T^2 > 0 \tag{4.14}$$

with the isothermal velocity of sound v_T^2 and

$$\det P = P^{\mu\mu} P^{TT} - (P^{\mu T})^2 = \frac{P^\mu P^{\mu\mu}}{T} \frac{C_V}{N} = \frac{n^2 v_T^2}{T} \frac{C_V}{N} > 0 \tag{4.15}$$

with specific heat at constant volume per particle C_V/N. Compare Eq. (B.6) for the derivation of this formula.

4.2 Oscillations for isotropic harmonic trapping

We now restrict ourselves to a spherically symmetric harmonic trapping potential

$$V_{\mathrm{ext}}(\vec{x}) = \frac{m}{2} \omega_0^2 r^2 \tag{4.16}$$

with $r = |\vec{x}|$ in d spatial dimensions. The harmonicity condition is of experimental interest, as most of the traps used in quantum gases research are parabolic with spherical or axial symmetry. The investigation of low-dimensional systems ($d = 1, 2$) has turned out to be very fertile as they show a lot of new phenomena not apparent in three spatial dimensions. Their experimental realization consists in highly asymmetric trapping potentials wherein motion into one or two spatial directions is highly suppressed due to narrow confinement. In the following we give formulas for arbitrary d although we are particulary interested in $d \leq 3$.

The potential (4.16) provides us with two additional macroscopic scales of the system. Obviously, ω_0 is an inverse time scale. Moreover, the oscillator length $a_{ho} = \sqrt{\hbar/m\omega_0}$ defines a characteristic length, namely the width of the non-interacting Gaussian ground state in the harmonic potential. We set $\hbar = k_B = m = 1$, see App. A, such that the potential can be rewritten as

$$V_{\text{ext}}(r) = \frac{\omega_0}{2}(r/a_{ho})^2 = \frac{1}{2}r^2. \tag{4.17}$$

All physical quantities can now be expressed in terms of ω_0 or a_{ho}. Note that in this system of units a and ω are equivalent to a/a_{ho} and ω/ω_0, respectively, since $a_{ho} = 1$ and $\omega_0 = 1$. Moreover, mass density and particle number density are equivalent, $\rho = n$. Eq. 4.6 gets

$$\mu_0(r) = \left(\bar{\mu} - \frac{r^2}{2}\right). \tag{4.18}$$

Temperature T and chemical potential in the center of the trap $\bar{\mu}$ are additional parameters of the system, besides a_{ho} (or ω_0). Instead of $\bar{\mu}$ one may want to use the number of particles N. They are related by the normalization condition on the density, $N = N(\bar{\mu}) = \int \mathrm{d}^d r n(\vec{x})$. Another important length scale of the system is the radius of the static cloud defined by $n(r = R) = 0$. If $n(\mu)$ vanishes for $\mu = 0$ then $R = \sqrt{2\bar{\mu}}$. We introduce the abbreviations

$$P_0^{\mu}(r) := \left(\frac{\partial P}{\partial \mu}\right)_T (\mu_0(r), T), \tag{4.19}$$

$$P_0^{\mu\mu}(r) := \left(\frac{\partial^2 P}{\partial \mu^2}\right)_T (\mu_0(r), T) \tag{4.20}$$

with μ_0 from Eq. (4.18). In the literature one also finds the system of units $\hbar = k_B = 2m = 1$. This choice is equivalent to $a_{ho}^2 = 2\omega_0^{-1}$.

We define spherical harmonics in d dimensions in App. D.

4.2.1 Collective modes at zero temperature

We begin our analysis with a derivation of the eigenvalue problem for zero temperature and V_{ext} from Eq. (4.17). Choosing $\delta\mu$ and δT as independent thermodynamic variables we write

$$\delta n(r) = P_0^{\mu\mu}(r)\delta\mu \tag{4.21}$$

not taking into account temperature fluctuations. Eqs. (4.4) and (4.5) vanish at $T = 0$ because $\rho_{n,0} = 0$ and $v_n = 0$ and we are left with

$$0 = -i\omega P_0^{\mu\mu}\delta\mu + \operatorname{div}\left(P_0^{\mu}\delta\vec{v}_s\right), \tag{4.22}$$

$$0 = -i\omega\delta\vec{v}_s + \nabla\delta\mu. \tag{4.23}$$

These equations hold for $0 \leq r \leq R$. From Eq. (4.23) we see that knowledge of $\delta\mu$ is sufficient to determine $\delta\vec{v}_s$. Obviously, these equations are invariant under a rescaling $P \to cP$ with a constant c. We conclude that the eigenvalues will not depend on an overall prefactor of P. Since the trapping potential is spherically symmetric the possible solutions for $\delta\mu$ can be classified making the ansatz

$$\delta\mu(\vec{x}) = \bar{g}(r)r^l f_{lm}, \tag{4.24}$$

with f_{lm} being a spherical harmonic. When applying this ansatz to the combined equation

$$0 = \omega^2 P_0^{\mu\mu}\delta\mu + \operatorname{div}(P_0^{\mu}\nabla\delta\mu) \tag{4.25}$$

of (4.22) and (4.23) we benefit from the facts that $\mu_0(\vec{x}) = \mu_0(r) = (\bar{\mu} - r^2/2)$ such that $\nabla\mu_0 \cdot \nabla f_{lm} = 0$ and $\partial_r P(\mu_0(r)) = -rP_0^{\mu}(r)$. Using

$$\vec{e}_r \cdot \nabla\delta\mu(\vec{x}) = \left(\bar{g}'(r) + \frac{l}{r}\bar{g}(r)\right) r^l f_{lm}, \tag{4.26}$$

$$\Delta\delta\mu(\vec{x}) = \left(\bar{g}''(r) + \frac{2l + d - 1}{r}\bar{g}'(r)\right) r^l f_{lm}, \tag{4.27}$$

where $\vec{e}_r$ denotes the unit vector pointing in radial direction, we arrive at

$$0 = \omega^2\bar{g}(r) - r\bar{g}'(r) - l\bar{g}(r) + \frac{P_0^{\mu}(r)}{P_0^{\mu\mu}(r)}\left(\bar{g}''(r) + \frac{2l + d - 1}{r}\bar{g}'(r)\right) \tag{4.28}$$

where a prime denotes differentiation with respect to r. When substituting $z = r^2$ the eigenvalue problem becomes

$$Ag(z) = \omega^2 g(z) \tag{4.29}$$

for $g(z) = \bar{g}(r)$ and the differential operator

$$A = -\frac{P^{\mu}(z)}{P^{\mu\mu}(z)} \left(4z \frac{\partial^2}{\partial z^2} + 2(2l+d) \frac{\partial}{\partial z} \right) + \left(2z \frac{\partial}{\partial z} + l \right). \tag{4.30}$$

This equation has to be fulfilled on the interval $z \in [0, R^2]$ where R is the radius of the static cloud defined by $n_0(r = R) = 0$. We do not allow for fluctuations to change the cloud radius since this would be a second order small effect. Note that l enters Eq. (4.30) as a parameter. Thus the problem for general l is of the same degree of difficulty as the one for $l = 0$. Especially, for $g = \text{const.}$ we get the eigenvalues $\omega = \sqrt{l}$ which are independent of the equation of state.

4.2.2 Collective modes at nonvanishing temperature

New effects and an even richer excitation spectrum arise for nonzero temperature. Extending the notation introduced in Eqs. (4.19) and (4.20) we write

$$P_0^{\mu T}(r) := \frac{\partial^2 P}{\partial T \partial \mu}(\mu_0(r), T). \tag{4.31}$$

In the same manner $P_0^{\mu}, P_0^{\mu\mu}, P_0^{T}$ and P_0^{TT} are defined for $T > 0$. The static solutions n_0 and s_0 are connected to $\mu_0(r) = \bar{\mu} - r^2/2$ and $T_0 = T$ via $n_0 = P_0^{\mu}, s_0 = P_0^{T}$. Working again with the independent variables $\delta\mu$ and δT we get

$$\delta n = P_0^{\mu\mu} \delta\mu + P_0^{\mu T} \delta T, \tag{4.32}$$

$$\delta s = P_0^{T\mu} \delta\mu + P_0^{TT} \delta T. \tag{4.33}$$

We recall Eqs. (4.2) - (4.5),

$$0 = -i\omega \delta n + \operatorname{div}(n_{s,0} \delta \vec{v}_s + n_{n,0} \delta \vec{v}_n), \tag{4.34}$$

$$0 = -i\omega \delta \vec{v}_s + \nabla \delta\mu, \tag{4.35}$$

$$0 = -i\omega n_{n,0} (\delta \vec{v}_n - \delta \vec{v}_s) + s_0 \nabla \delta T, \tag{4.36}$$

$$0 = -i\omega \delta s + \operatorname{div}(s_0 \delta \vec{v}_n). \tag{4.37}$$

From our analysis at zero temperature we know that due to the spherically symmetry the eigenmodes are most easily obtained by eliminating the velocity fields. Defining $\tilde{n} = s^2/n_n$ we get

$$0 = \omega^2 \delta n + \operatorname{div}(n_0 \nabla \delta\mu + s_0 \nabla \delta T), \tag{4.38}$$

$$0 = \omega^2 \delta s + \operatorname{div}(s_0 \nabla \delta\mu + \tilde{n}_0 \nabla \delta T), \tag{4.39}$$

or in terms of $P(\mu, T)$,

$$0 = \omega^2 P_0^{\mu\mu} \delta\mu + \omega^2 P_0^{\mu T} \delta T + \mathrm{div}(P_0^{\mu} \nabla \delta\mu + P_0^{T} \nabla \delta T), \tag{4.40}$$

$$0 = \omega^2 P_0^{T\mu} \delta\mu + \omega^2 P_0^{TT} \delta T + \mathrm{div}(P_0^{T} \nabla \delta\mu + \tilde{n}_0 \nabla \delta T) \tag{4.41}$$

determining ω. Note that $\tilde{n}$ supplements the equation of state in the latter equation. We have chosen $\tilde{n}$ instead of n_n in order to keep the notation simple.

In their present form, Eqs. (4.40) and (4.41) do not exhibit the canonical form of an eigenvalue problem and are unsuitable for a numerical implementation. We build a more useful linear combination yielding

$$0 = \omega^2 \mathrm{det} P_0 \delta\mu + P_0^{TT} \mathrm{div}(P_0^{\mu} \nabla \delta\mu + P_0^{T} \nabla \delta T) - P^{\mu T} \mathrm{div}(P_0^{T} \nabla \delta\mu + \tilde{n}_0 \nabla \delta T), \tag{4.42}$$

$$0 = -\omega^2 \mathrm{det} P_0 \delta T + P_0^{\mu T} \mathrm{div}(P_0^{\mu} \nabla \delta\mu + P_0^{T} \nabla \delta T) - P^{\mu\mu} \mathrm{div}(P_0^{T} \nabla \delta\mu + \tilde{n}_0 \nabla \delta T), \tag{4.43}$$

where we assumed $P_0^{\mu T} = P_0^{T\mu}$ and defined $\mathrm{det} P_0 = P_0^{\mu\mu} P_0^{TT} - (P_0^{\mu T})^2$. Analogously to the zero temperature case (see Eq. (4.24)) we classify the solutions via an ansatz

$$\delta\mu(\vec{x}) = g(r) r^l f_{lm}, \tag{4.44}$$

$$\delta T(\vec{x}) = h(r) r^l f_{lm} \tag{4.45}$$

where $l = 0, 1, 2, ...$ and f_{lm} are the corresponding spherical harmonics. The resulting equations simplify on going over to the new variable $z = r^2$. We arrive at

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} g(z) \\ h(z) \end{pmatrix} = \omega^2 \begin{pmatrix} g(z) \\ h(z) \end{pmatrix}, \tag{4.46}$$

where the operators A, B, C, D are defined by

$$A = a_1(z) \left(4z \frac{\partial^2}{\partial z^2} + 2(2l + d) \frac{\partial}{\partial z} \right) + \left(2z \frac{\partial}{\partial z} + l \right) \tag{4.47}$$

$$B = b_1(z) \left(4z \frac{\partial^2}{\partial z^2} + 2(2l + d) \frac{\partial}{\partial z} \right) + b_2(z) \left(2z \frac{\partial}{\partial z} + l \right) \tag{4.48}$$

$$C = c_1(z) \left(4z \frac{\partial^2}{\partial z^2} + 2(2l + d) \frac{\partial}{\partial z} \right) \tag{4.49}$$

$$D = d_1(z) \left(4z \frac{\partial^2}{\partial z^2} + 2(2l + d) \frac{\partial}{\partial z} \right) + d_2(z) \left(2z \frac{\partial}{\partial z} + l \right) \tag{4.50}$$

with z-dependent coefficients

$$a_1 = \frac{P_0^{\mu T} P_0^{T} - P_0^{TT} P_0^{\mu}}{\det P_0}, \qquad b_1 = \frac{P_0^{\mu T} \tilde{n}_0 - P_0^{TT} P_0^{T}}{\det P_0}, \tag{4.51}$$

$$b_2 = \frac{P_0^{TT} P_0^{\mu T} - P_0^{\mu T} \tilde{n}_0^{\mu}}{\det P_0}, \qquad c_1 = \frac{P_0^{\mu T} P_0^{\mu} - P_0^{\mu\mu} P_0^{T}}{\det P_0}, \tag{4.52}$$

$$d_1 = \frac{P_0^{\mu T} P_0^{T} - P_0^{\mu\mu} \tilde{n}_0}{\det P_0}, \qquad d_2 = \frac{P_0^{\mu\mu} \tilde{n}_0^{\mu} - (P_0^{\mu T})^2}{\det P_0}. \tag{4.53}$$

Again, $\omega = \sqrt{l}$ is a solution for constant $g(z)$ and vanishing $h(z)$, which is independent of the equation of state.

An additional complication to the case of vanishing temperature arises from the fact that the normal part of the background density n_{n0} is not restricted to a region $z \leq z_{max}$ as it is the case for the superfluid density n_{s0}. In contrast it decreases exponentially for large z. In praxis, we expect that the region $z > R^2$ for a sufficiently large value of R^2 is not affecting the oscillation frequencies since only an exponentially small number of particles is in that region. We restrict our numerical treatment therefore to a finite sphere $z \leq R^2$.

Given the functions $a_1, ..., d_2$ over the whole range $z \in [0, R^2]$, the problem of determining the collective frequencies is as straightforward as it was for $T = 0$. Indeed, as we show in the next section, a simple and yet efficient method to solve for the eigenvalues will be given by replacing the operators by matrices. Thus, if the more complicated coefficient functions (4.51) - (4.53) are given to a sufficient degree of accuracy, the finite temperature collective modes are easily obtained.

4.3 Numerical implementation

Now that we have identified the ordinary differential operator(s) describing two-fluid hydrodynamic modes, we present a numerical method to determine the corresponding eigenfrequencies. We restrict ourselves to the case of $T = 0$ since the operators for $T \geq 0$ have an identical structure but different coefficient functions. We rewrite Eq. (4.29) as

$$Ag(z) = \omega^2 g(z), \tag{4.54}$$

where l and d parametrize the family of operators

$$A = A^{(l,d)} = 2z\frac{\partial}{\partial z} - \frac{P_0^{\mu}(z)}{P_0^{\mu\mu}(z)}\left(4z\frac{\partial^2}{\partial z^2} + 2(2l+d)\frac{\partial}{\partial z}\right) + l. \tag{4.55}$$

In order to solve this eigenvalue problem we discretize the interval $z \in [0, z_{max}]$ via $i = 0, 1, \ldots, M$ and $\Delta z = z_{max}/M$ such that $z_i = i\Delta z$. The function $g(z)$

becomes a vector $g = (g(z_i))_{i=0 \ldots M}$ and A is represented by a matrix (A_{ij}) having components

$$A_{ij} = 2z_i \frac{\delta_{i+1,j} - \delta_{i-1,j}}{2\Delta z} + l\delta_{ij} - \frac{P_0^{\mu}(z_i)}{P_0^{\mu\mu}(z_i)} \times \left(4z_i \frac{\delta_{i+1,j} + \delta_{i-1,j} - 2\delta_{ij}}{\Delta z^2} + 2(2l+d) \frac{\delta_{i+1,j} - \delta_{i-1,j}}{2\Delta z} \right) \tag{4.56}$$

for $i = 1, ..., M-1$ and $j = 0, ..., M$ with δ_{ij} being the Kronecker delta. The cases $i = 0$ and $i = M$ require a careful treatment. We define

$$A_{0,j} = l\delta_{0,j} - \frac{P_0^{\mu}(0)}{P_0^{\mu\mu}(0)} 2(2l+d) \frac{\delta_{1,j} - \delta_{0,j}}{\Delta z}, \tag{4.57}$$

$$A_{M,j} = 2z_{max} \frac{\delta_{M,j} - \delta_{M-1,j}}{\Delta z} + l\delta_{M,j} - \frac{P_0^{\mu}(z_{max})}{P_0^{\mu\mu}(z_{max})} 2(2l+d) \frac{\delta_{M,j} - \delta_{M-1,j}}{\Delta z} \tag{4.58}$$

having replaced the second-order difference scheme by a first order one and set the second derivative g'' at the boundary $z = z_{max}$ to zero. The eigenvalues of (A_{ij}) can now be determined using standard procedures. We found it practically convenient to first define A with all matrix elements being zero and then insert the nonvanishing values. This reduces the computation time dramatically.

Our discretization of A is closely related to the underlying physical problem. The functions $g(z)$ on which A operates describe macroscopic hydrodynamic fluctuations of the chemical potential. Thus they are implicitly assumed to be finite and free of a rich local structure including jumps and kinks. This applies in particular to the lowest lying modes with only a few radial nodes. Therefore we can safely assume $g(z)$ to be sufficiently smooth such that the approximations $g'(z_i) = (g_{i+1} - g_{i-1})/(2\Delta z) + O(\Delta z^2)$ and $g''(z_i) = (g_{i+1} - 2g_i + g_{i-1})/(\Delta z^2) + O(\Delta z^2)$ hold in the open interval $(0, z_{max})$. However, at the points $z_0 = 0$ and $z_M = z_{max}$ we run into problems as z_{-1} and z_{M+1} are not defined. A first guess might be to assign certain boundary values to g. But just looking at the breathing mode of Eq. (4.71) for $\alpha = 1$ and $d = 3$ (this choice corresponds to the functions first found by Stringari [19]), $g(z) = 1 - 5z/3$ for $z \leq z_{max}$, 0 otherwise, we recognize none of the sensible boundary conditions $g(z_{max}) = 0$ or $g'(z_{max}) = 0$ to be satisfied. This does not come as a surprise since our description necessarily breaks down in the outer regions of the cloud where the static density goes to zero. The crucial point is to circumvent the solution to diverge at the boundaries. So we demand it to be Taylor expandable even there, although the physical solution being zero for $z > z_{max}$ might have a discontinuity, and replace the second-order difference scheme in z_{-1} and z_{M+1} by a first-order one, i.e. for $z_0 = 0$

$$\frac{g_1 - g_{-1}}{2\Delta z} = \frac{g_1 - g_0}{\Delta z} + O(\Delta z). \tag{4.59}$$

$l = 0$ exact	numerically	$l = 1$	numerically	$l = 2$	numerically
1.73205	1.73205	1.00000	1.00000	1.41421	1.41421
2.70801	2.70801	2.16025	2.16025	2.51661	2.51661
3.60555	3.60555	3.10913	3.10913	3.46410	3.46410
4.47214	4.47214	4.00000	4.00000	4.35890	4.35890
5.32291	5.32290	4.86484	4.86484	5.22813	5.22813
6.16441	6.16441	5.71548	5.71548	6.08276	6.08276
7.00000	6.99999	6.55744	6.55744	6.92820	6.92820
7.83156	7.83154	7.39369	7.39369	7.76745	7.76745
8.66025	8.66023	8.22598	8.22598	8.60233	8.60233
9.48683	9.48679	9.05539	9.05539	9.43398	9.43398

Table 4.1: Eigenmodes (in units of ω_0) for $P = \mu^4$ obtained from Eq. (4.71) and using the numerical procedure described above for $M = 2000$, $d = 3$

This can be solved for g_0 and yields $g_0 \simeq (g_1 + g_{-1})/2$. The latter mean-value property is true for any function which can be linearized around $z = 0$ and a sufficiently small step size. In the same way, the approximation of $g'(z_{max})$ in Eq. (4.59) can be justified. Setting $g''(z_{max}) = 0$ in the same equation reads $g_{M+1} - 2g_M + g_{M-1} = 0$ in a discretized version, which leads us to the same mean-value property.

The efficiency of this very simple approach is quite surprising. As an example we present in table 4.1 the first 10 modes corresponding to $l = 0, 1, 2$, $d = 3$ for an equation of state $P(\mu) = \mu^4$ ($\alpha = 3$) compared to the exact results given by Eq. (4.71). Recall that the frequencies do not depend on prefactors of $\mu^{\alpha+1}$ but only on the exponent. In this case the numerical calculation was done for $M = 2000$. We find the (more interesting) lower lying frequencies to agree better than the higher ones, but of course the precision of the latter can be improved easily by further increasing M. In table 4.2 we compare our numerics with the exact solution for $l = 0$, $d = 1$ and $\alpha = 0.1, 1.8$ and $\alpha = 3.9$. (Formula (4.71) was derived under the condition $\alpha > 0$.) The values are calculcated using $M = 2000$. In figure 4.1 we show the frequency of the breathing mode $\omega_B = (2 + d/\alpha)^{1/2}$ from Eq. (4.71) for a polytropic equation of state over the polytropic index α.

For a particular physical situation it may be favorable to use a *non-uniform grid*, i. e. the points z_i shall not be equidistant. For example when we are dealing with a Bose condensate peaked in the center of the trap surrounded by a large thermal cloud an equidistant grid would underestimate the influence of the condensate because only few points of the matrix correspond to the central region.

$\alpha = 0.1$ exact	numerically	$\alpha = 1.8$	numerically	$\alpha = 3.9$	numerically
3.46410	3.46410	1.59861	1.59861	1.50214	1.50214
8.00000	7.99982	2.70801	2.70799	2.35339	2.35337
12.4900	12.4894	3.78594	3.78587	3.13786	3.13777
16.6706	16.9693	4.85341	4.85325	3.89609	3.89591
21.4476	21.4452	5.91608	5.91578	4.64095	4.64064
25.9230	25.9191	6.97615	6.97564	5.37802	5.37752
30.3974	30.3916	8.03465	8.03386	6.11010	6.10936
34.8712	34.8631	9.09212	9.09097	6.83880	6.83775
39.3446	39.3335	10.1489	10.1473	7.56510	7.56367
43.8178	43.8031	11.2052	11.2030	8.28963	8.28773

Table 4.2: Exact and numerical frequencies obtained for an equation of state $P = \mu^{\alpha+1}$ for $\alpha = 0.1, 1.8$ and 3.9 ($M = 2000$, $d = 1$, $l = 0$)

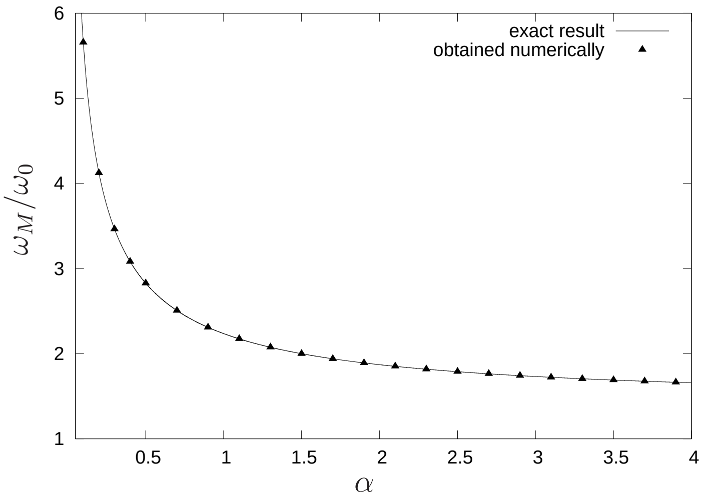

Figure 4.1: Breathing mode as a function of the polytropic index α for $T = 0$ in three dimensions ($M = 1000$)

Let $\{z_i\}_{i=0\ldots M}$ be a discretization of the interval $[0, z_{max}]$. We define

$$\Delta z_i = z_{i+1} - z_i. \tag{4.60}$$

Given an arbitrary function $f(z)$ we then expand

$$f(z_{i+1}) = f(z_i) + f'(z_i)\underbrace{(z_{i+1} - z_i)}_{\Delta z_i} + \frac{1}{2}f''(z_i)\underbrace{(z_{i+1} - z_i)^2}_{\Delta z_i^2} + O(\Delta z_i^3), \tag{4.61}$$

$$f(z_{i-1}) = f(z_i) - f'(z_i)\underbrace{(z_i - z_{i-1})}_{\Delta z_{i-1}} + \frac{1}{2}f''(z_i)\underbrace{(z_i - z_{i-1})^2}_{\Delta z_{i-1}^2} + O(\Delta z_{i-1}^3). \tag{4.62}$$

Subtracting both equations yields

$$f'(z_i) = \frac{f(z_{i+1}) - f(z_{i-1})}{\Delta z_i + \Delta z_{i-1}} + O(\Delta z). \tag{4.63}$$

Moreover, by dividing Eqs. (4.61) and (4.62) through Δz and rearranging the terms we find

$$\frac{1}{2}f''(z_i)\Delta z_i = \frac{f(z_{i+1}) - f(z_i)}{\Delta z_i} - f'(z_i) + O(\Delta z_i^2), \tag{4.64}$$

$$\frac{1}{2}f''(z_i)\Delta z_{i-1} = \frac{f(z_{i-1}) - f(z_i)}{\Delta z_{i-1}} + f'(z_i) + O(\Delta z_{i-1}^2). \tag{4.65}$$

The sum of these expressions gives

$$f''(z_i)\left(\frac{\Delta z_i + \Delta z_{i-1}}{2}\right) = \frac{f(z_{i+1}) - f(z_i)}{\Delta z_i} + \frac{f(z_{i-1}) - f(z_i)}{\Delta z_{i-1}} + O(\Delta z^2) \tag{4.66}$$

from which a formula for $f''(z_i)$ can be obtained. Eqs. (4.63) and (4.66) allow to construct the discretized operator (A_{ij}) on $[0, z_{max}]$ for an arbitrary grid.

4.4 Examples for oscillations at zero temperature

4.4.1 Exact results in the zero temperature limit

In this section we derive exact results on the oscillations for a polytropic equation of state at zero temperature and in the zero temperature limit for a Lee-Yang Bose gas, respectively. Due to the simplification of the equations it is possible to provide an analytic solution. These calculations provide an intuition on how solutions to our proposed eigenvalue problem should look like. Moreover, the exact eigenfrequencies allow to perform perturbation theory beyond mean-field.

We start from the zero temperature eigenvalue problem $Ag = \omega^2 g$ with A from Eq. (4.30). For a polytropic equation of state, $P(\mu) \propto \mu^{\alpha+1}$, the ratio $P_0^{\mu}(z)/P_0^{\mu\mu}(z)$ is found to be

$$\frac{P_0^{\mu}(z)}{P_0^{\mu\mu}(z)} = \frac{(\alpha+1)(\bar{\mu} - z/2)^{\alpha}}{(\alpha+1)\alpha(\bar{\mu} - z/2)^{\alpha-1}} = \frac{\bar{\mu} - z/2}{\alpha}, \tag{4.67}$$

where $\bar{\mu}$ is the chemical potential in the center of the trap. The eigenvalue problem then becomes

$$\begin{aligned} 0 = &-\omega^2 g(z) + 2zg'(z) + lg(z) \\ &- \frac{\bar{\mu} - z/2}{\alpha}(4zg''(z) + 2(2l+d)g'(z)). \end{aligned} \tag{4.68}$$

This equation has to be true for all values of z inside the cloud. The radius of the cloud R is found by solving $0 = n_0(r = R) = (\alpha + 1)(\bar{\mu} - R^2/2)^\alpha$, i.e. $z \leq R^2 = 2\bar{\mu}$. Substituting $z \mapsto \bar{z} = z/2\bar{\mu}$, the chemical potential drops out,

$$\begin{aligned} 0 = &-\omega^2 g(\bar{z}) + 2\bar{z}g'(\bar{z}) + lg(\bar{z}) \\ &- \frac{1 - \bar{z}}{2\alpha}(4\bar{z}g''(\bar{z}) + 2(2l+d)g'(\bar{z})). \end{aligned} \tag{4.69}$$

As is pointed out in Ref. [56] this is of the form of the hypergeometric differential equation. However, for arbitrary values of α, d and l we would have to distinguish many special cases of its solution [57] because the prefactors of the derivative terms in Eq. (4.69) change. Therefore, we apply a more direct method and ask for solutions of the form $g(\bar{z}) = \sum_{k\geq 0} a_k \bar{z}^k$. This yields the recursion relation

$$a_{k+1} = \frac{\alpha(l - \omega^2) + 2k(\alpha + k + l - 1 + d/2)}{2(k + l + d/2)(k + 1)} a_k \tag{4.70}$$

for the coefficients a_k. Since $Ag(z) = \omega^2 g(z)$ is invariant under a rescaling of $g(z)$ we can set $a_0 = 1$. A sufficient condition for the convergence of this series is termination and thus reduction to a polynomial of degree $n = 0, 1, ...$ This will be the case for $k \leq n$ and

$$\omega_{\alpha,n,l} = \left(\frac{2n}{\alpha}(\alpha + n + l + d/2 - 1) + l\right)^{1/2} \omega_0, \tag{4.71}$$

where we restored the unit ω_0. This formula was given in Ref. [28] for an even larger range of parameters and was also discussed in Ref. [29]. Inserting Eq. (4.71) into Eq. (4.70) we get coefficients

$$a_{k+1} = \frac{(k - n)(\alpha + k + n + l + d/2 - 1)}{(k + l + d/2)(k + 1)} a_k \tag{4.72}$$

and $a_0 = 1$.

Considering a hard-core Bose gas in the zero temperature limit, the eigenvalue problem (4.46) at nonzero temperature gets block diagonal because B and C

vanish, see also Sec. 4.5. While A approaches its zero temperature limit (4.30), the coefficient functions d_1 and d_2 of D approach

$$d_1 \overset{T\to 0}{\longrightarrow} -\frac{\tilde{n}_0}{P_0^{TT}}, \tag{4.73}$$

$$d_2 \overset{T\to 0}{\longrightarrow} \frac{\tilde{n}_0^{\mu}}{P_0^{TT}} \tag{4.74}$$

with $\tilde{n_0} = (P_0^T)^2/n_{n,0}$. For the dilute Bose gas in three dimensions these ratios can be computed analytically from the formulas given in Sec. 3.2. In the zero temperature limit nearly all values of the chemical potential obtained via $\mu(z) = \bar{\mu} - V_{ext}(z)$ satisfy the condition $T \ll \mu(z)$. This will of course not be true for $\mu = 0$ but this value can be neglected because it corresponds to the outermost points of the cloud. For $T \ll \mu$ the integrals for P and n_n receive contributions only from the phonon part of the spectrum. We have

$$P^T(\mu, T) \simeq 4T^3\pi^2/(90\mu^{3/2}), \tag{4.75}$$

$$n_n(\mu, T) \simeq 2\pi^2 T^4/(45\mu^{5/2}), \tag{4.76}$$

see for example [43]. These quantities are powers of T and will be extremely small. Nevertheless, the ratios (4.73) and (4.74) are nonvanishing because the powers of T cancel each other. We find

$$\frac{\tilde{n}_0}{P_0^{TT}} = \frac{\bar{\mu}}{3} = \frac{\bar{\mu} - z/2}{3}, \tag{4.77}$$

$$\frac{\tilde{n}_0^{\mu}}{P_0^{TT}} = -\frac{1}{6} \tag{4.78}$$

and the differential equation $D(T = 0)h(z) = \omega^2 h(z)$ for thermal fluctuations becomes

$$\begin{aligned} D(T=0)h(z) = &-\frac{1}{6}(2zh'(z) + lh(z)) \\ &- \frac{\bar{\mu} - z/2}{3}(4zh''(z) + 2(2l+3)h'(z)) \\ = &\ \omega^2 h(z). \end{aligned} \tag{4.79}$$

We already inserted $d = 3$. When we multiply both sides with -6 we find the same structure as in Eq. (4.68) but with the substitutions $\omega^2 \mapsto -6\omega^2$ and $\alpha \mapsto -1/2$. Applying these modifications to the square of both sides of Eq. (4.71),

$$\omega_{\alpha,n,l}^2 = \left(\frac{2n}{\alpha}\left(\alpha + n + l + \frac{1}{2}\right) + l\right)\omega_0^2, \tag{4.80}$$

we find the eigenvalues of Eq. (4.79) to be

$$\omega_{n,l} = \left(\frac{4n(n+l)-l}{6}\right)^{1/2} \omega_0. \tag{4.81}$$

The eigenfunctions read $h(z) = \sum_{k=0}^{n} a_k \bar{z}^k$ with $\bar{z} = z/2\bar{\mu}$, $a_0 = 1$ and

$$a_{k+1} = \frac{k(k+l)-n(n+l)}{(k+l+3/2)(k+1)} a_k. \tag{4.82}$$

4.4.2 Beyond mean-field shift for Bose gas in 3D

For a three-dimensional Bose gas at zero temperature the mean-field equation of state $P(\mu) = \mu^2/8\pi a$ is of polytropic type and the collective oscillations frequencies are therefore given by Eq. (4.71) with $\alpha = 1$ and $d = 3$. We arrive at Stringari's formula [19]

$$\omega_{n,l} = (2n(n+l+3/2)+l)^{1/2} \omega_0, \tag{4.83}$$

which is independent of the scattering length a. An explanation is provided by the fact that Eq. (4.28) is invariant under a rescaling of P by a constant. However, taking the next-to-leading order term (3.11) in the equation of state into account, there is a relative difference between the two terms and a cannot be scaled away. For small gas parameter one predicts a shift [23, 24] of the breathing mode ($n = 1$, $l = 0$)

$$\frac{\delta\omega_M}{\omega_M} = \frac{63\sqrt{\pi}}{128}\sqrt{a^3 n(0)}, \tag{4.84}$$

which is linear in the square root of the gas parameter $c = a^3 n(0)$. (Here $n(0)$ denotes the density in the center of the trap.) This formula allows us to address two questions of importance for this thesis: Given $P(\mu)$ with Lee-Huang-Yang correction from Eq. (3.11), can we numerically resolve the linear shift? Provided this is fulfilled, can the equation of state of a Bose gas calculated from Functional Renormalization Group reproduce the beyond mean-field shift of the breathing mode? The answers to both questions are positive.

In Fig. 4.2 we show the shifts beyond mean field due to the LHY-correction which we obtained by our numerical implementation described in Sec. 4.3. We reproduce the prediction to a high precision in the regime of small gas parameter, where it is valid. Although this plot only shows the breathing mode we emphasize that it is possible to calculate the shifts of the whole frequency spectrum which are found to show a similar scaling behavior with $\sqrt{n(0)a^3}$. Fig. 4.2 shows that

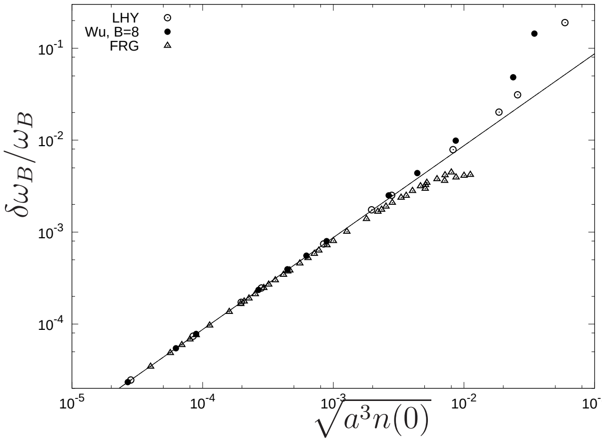

Figure 4.2: Shift of the mean-field breathing mode $\omega_B = \sqrt{5}\omega_0$ for the zero temperature Bose gas in three dimensions. We show predictions for different methods: Lee–Huang–Yang-formula (3.11), Wu-correction (3.26) with $B = 8$ and equation of state from Functional Renormalization Group (FRG). The solid line corresponds to Eq. (4.84), a is the scattering length and $n(0)$ the density in the center of the cloud.

an experimental accuracy for frequency shifts better than 10^{-2} would be useful for exploring the regime of small gas parameters. An inclusion of the Wu-Term (3.26) does not change the behavior significantly.

For large values of the gas parameter perturbation theory is no longer applicable and more sophisticated methods are necessary. The Functional Renormalization Group (FRG) for the effective average action [58] does not rely on an expansion in a small parameter and therefore the most striking effects are expected in strongly interacting systems. It is still under debate whether the regime $a \to \infty$ can be reached for a Bose gas at very low temperatures. One expects the condensate to get unstable then because of increasing importance of three-body losses.

In Fig. 4.2 we show the frequency shift obtained from the equation of state calculated with FRG at zero temperature [59, 60, 61]. We see that higher values of the gas parameter $n(0)a^3$ can no longer be identified with a unique $\delta\omega_B/\omega_B$. This is a fully non-perturbative effect. We recall that one of the assumptions in the beginning was that the interaction can be approximated to be point-like. However, this is only valid if the scattering length a is much larger than the microscopic distance Λ^{-1} where one can resolve the details of the interaction potential. For cold bosonic atoms this length is given by the typical range of the Van der Waals potential. An effective ultraviolet cutoff Λ is necessary in any field theoretic treatment of the system where the contact interaction appearing in the Lagrangian of the non-relativistic system is not renormalizable in three dimensions. Therefore, the introduction of this scale is no relict of approximations but has a clear physical meaning. In perturbation theory one assumes Λ to be infinity and thus the equation of state of the homogeneous system only involves one dimensionless parameter, the gas parameter na^3. A proper treatment has to account for the appearance of an additional length scale Λ^{-1} such that two possible combinations, na^3 and $a\Lambda$, describe the system. As a result the shift of the breathing mode should rather be plotted as a two-dimensional surface depending on these two parameters. For small gas parameter the additional parameter $a\Lambda$ does not play a role – if we take equations of state $P(\mu, a)$ for different values of a and then vary $n(0)$, all obtained shifts will lie on a line in the double-logarithmic plot. This is the expected scaling behavior. For higher densities it makes a difference whether we calculate the frequencies from $P(\mu, a, \Lambda_1)$ or $P(\mu, a, \Lambda_2)$ with $\Lambda_1 \neq \Lambda_2$, as reflected by the spread of the points in Fig 4.2.

From this example we can deduce a method to make high precision measurements using collective modes. If we assume the shifts of the frequencies $\omega_{n,l}$ to be continuous in $n(0)a^3$ or $\bar{\mu}a^2$, where $\bar{\mu}$ is the chemical potential in the center of the trap, then for small gas parameters, $n(0)a^3 \ll 1$ or $\bar{\mu}a^2 \ll 1$, the shift will be proportional to some power of the gas parameter. In the case of Eq. (4.84) we found this power to be 1/2. However, this will depend on the system under consideration. Nevertheless, driving the system through a certain range of the gas parameter, e.g. by the use of a Feshbach Resonance, and then collecting the result in a double-logarithmic plot one will find the results to lie on a straight line for small gas parameter. This simple scaling behavior can be used for very precise measurements of $n(0)a^3$ after a proper calibration. The accuracy of this method is directly related to the accuracy in measuring the frequency shifts.

4.4.3 Bose gas from Functional Renormalization in 2D

With regard to formula (4.71) we may wonder whether the cases $d = 2$ and $d = 1$ refer to highly anisotropic traps. These are of great relevance for experiments where one is often not dealing with a spherical symmetric potential. For example, the early experiments in the JILA group were performed in a disk-shaped confinement [4, 6] while the MIT group used a cigar-shaped trap [5, 7].

A really lower-dimensional system is obtained in a trap where the quantum gas is in its ground state in one or two directions. This will be the case for very tight confinement. When calculating the equation of state one can then neglect quantum and thermal fluctuations in these directions. The mean field Bose gas in this scenario is described by Eq. (4.71) when inserting $d = 2$ or $d = 1$ and $\alpha = 1$. In these cases the system is isotropic in a lower-dimensional geometry.

A different situation arises if the system is not in its ground state in the directions of tight confinement and thus still three-dimensional or in a crossover between three and lower dimensions. Formula (4.71) cannot be applied in this case because the assumption of spherical symmetry is not justified. The solution of the hydrodynamic equations for these anisotropic cases is more involved.

We now consider the two-dimensional Bose gas. Its mean field equation of state is $\mu = g_{2D}n$ and the frequency of the breathing mode is found from Eq. (4.71) to be $\omega_B = 2\omega_0$. The equation of state beyond mean field can be obtained by a Functional Renormalization Group approach [58, 59, 60, 61]. The coupling constant g_{2D}, which satisfies $\mu = g_{2D}n$ for small g_{2D}, is dimensionless in two dimensions and shows a logarithmic running with the physical scale on which the experiment is performed. It will vanish for an infinitely large system. However, realistic experiments are always performed in traps so that in a harmonic potential the oscillator length $\ell_0 = \sqrt{\hbar/m\omega_{xy}}$ provides the largest possible length scale of the physics under considerations. Therefore, when calculating the equation of state one only has to include quantum fluctuations with momenta bigger than ℓ_0^{-1}, which acts as an infrared cutoff. This is also the reason why Bose–Einstein condensation is observed experimentally in two dimensions for small temperatures $0 < T < T_c$. For an infinitely extended system the long-range order would be destroyed by fluctuations for all nonvanishing temperatures as required by the Mermin-Wagner theorem [62, 63].

In Figs. 4.3 and 4.4 we plot the shift of the breathing mode with respect to $\omega_B = 2\omega_0$ for the two-dimensional Bose gas at zero temperature with equation of state $P(\mu, g_{2D})$ from Functional Renormalization Group calculations [59, 60, 61]. Since the coupling constant is dimensionless there is no gas parameter for such a system. As we mentioned already in the three-dimensional case one always has a physical ultraviolet cutoff Λ when dealing with a contact interacting in $d \geq 2$ dimensions. Since Λ has dimension of inverse length, the dimensionless variable involving the chemical potential (or similar for the density) is $\bar{\mu}\Lambda^{-2}$, where $\bar{\mu}$ is the chemical potential of the gas in the center of the trap. A good choice for Λ^{-1} is the oscillator length of tight trapping. We observe from Fig. 4.3 that the frequencies for small interactions depend only weakly on $\bar{\mu}\Lambda^{-2}$. For larger coupling g_{2D} we find deviations from this behavior in Fig. 4.4.

We arrive at two interesting experimental scenarios to be investigated. On the one hand, by measuring the collective modes one can distinguish whether one is working with a system which is still three-dimensional (disk, cigar) or a truly lower dimensional system. On the other hand, if the latter regime is reached

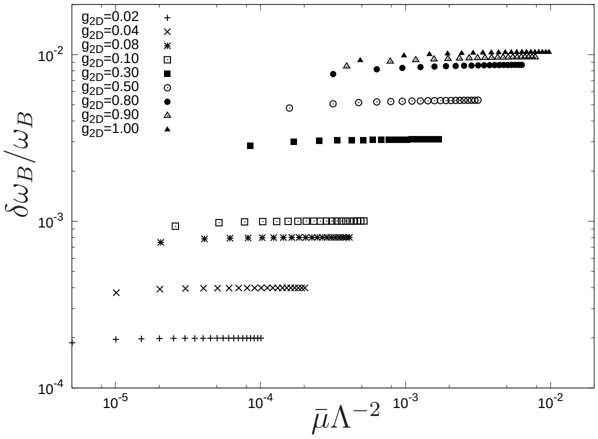

Figure 4.3: Shift of the breathing mode relative to $\omega_B = 2\omega_0$ for a two-dimensional dilute Bose gas for different values of the coupling constant g_{2D}. We show the dependence on $\bar{\mu}\Lambda^{-2}$, with chemical potential $\bar{\mu}$ in the center of the trap and effective UV cutoff Λ. The equation of state is provided by Functional Renormalization Group.

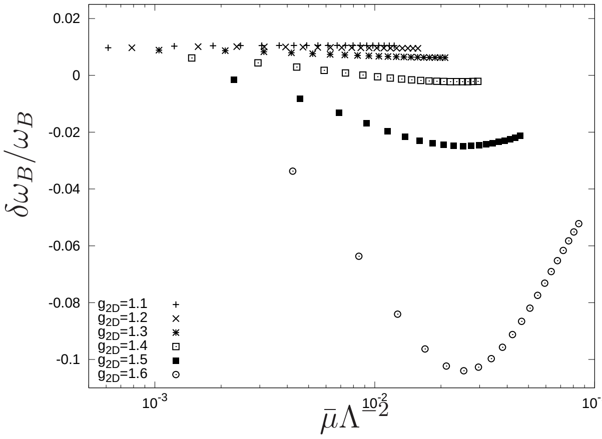

Figure 4.4: The same setting as in Fig. 4.3 but for higher values of the coupling g_{2D}. We find rather large negative shifts, which have a minimum.

experimentally it is of course tempting to verify the predictions of Fig. 4.4. The rather large negative frequency shifts can both test theoretical predictions like the occurrence of a minimum, and determine $\bar{\mu}$ or the corresponding density $n(0)$.

4.4.4 One-dimensional dipolar quantum gas

At zero temperature our treatment is applicable to any system which is described through hydrodynamic equations. Indeed, if n and $\vec{v}$ are, respectively, density and velocity field of the system, then we associate $n = n_s$ and $\vec{v} = \vec{v}_s$.

In [64] Chiofalo et. al report on the collective oscillation frequencies of a one-dimensional Bose gas with dipolar interaction at zero temperature. The system consists of N bosons with mass m and nonvanishing (electric or magnetic) dipole moment arranged on a one-dimensional line. The Hamiltonian is given by

$$H/Ry^* = -(nr_0)^2 \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + (nr_0)^3 \sum_{i<j} \frac{1}{|x_i - x_j|^3}, \tag{4.85}$$

where $r_0 = mC_{dd}/(2\pi\hbar^2)$ is the effective Bohr radius corresponding to the effective Rydberg unit $Ry^* = \hbar^2/(2mr_0^2)$ and interaction strength C_{dd}. The dimensionless parameter nr_0 (n being the density) drives the behavior of the system. As is described detailed in the mentioned paper and references therein, for $nr_0 \ll 1$ the bosons are in the low-density Tonks-Girardeau (TG) regime while for $nr_0 \gg 1$ they build a quasi-ordered high density state referred to as Dipolar Density Wave (DDW). The equation of state covering this crossover is found from a Reptation Quantum Monte Carlo approach [65] and given in the form of energy per particle $E_N = E/N$ in effective Rydberg units

$$E_N/Ry^* = \frac{\zeta(3)(nr_0)^4 + a(nr_0)^e + (nr_0)^b + c(nr_0)^{2+g}}{1 + (nr_0)} + \frac{(\pi^2/3)(nr_0)^2}{1 + d(nr_0)^g}, \tag{4.86}$$

with Riemann zeta function $\zeta(\cdot)$ and fitted parameters $a = 3.1(1), b = 3.2(2), c = 4.3(4), d = 1.7(1), e = 3.503(4), f = 3.05(5)$ and $g = 0.34(4)$. In the limits of $nr_0 \ll 1$ and $nr_0 \gg 1$ Eq. (4.86) reveals the right expressions $E_N/Ry^* \sim (\pi^2/3)(nr_0)^2$ and $E_N/Ry^* \sim \zeta(3)(nr_0)^3$ for the TG and DDW regime, respectively. Using $\mathrm{d}E_N = \mathrm{d}(E/N) = \mathrm{d}E/N - E_N\mathrm{d}N/N = (\mu - E_N)\mathrm{d}N/N$ holding for fixed volume V we get the chemical potential as a function of density via

$$\mu(n) = E_N(n) + N\left(\frac{\partial E_N}{\partial N}\right)_V = \left(1 + n\frac{\partial}{\partial n}\right) E_N(n). \tag{4.87}$$

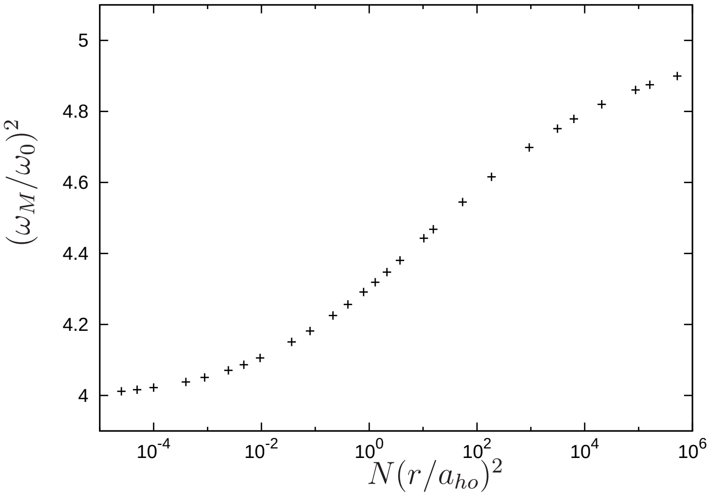

Figure 4.5: Breathing mode as a function of the dimensionless parameter $N(r_0/a_{ho})^2$. Our numerical results is in good agreement with Ref. [64].

This equation has to be inverted for $n(\mu)$. By virtue of the Gibbs-Duhem relation $E = -PV + \mu N$ $(T = 0)$ we get the pressure $P = \mu \mathrm{d}N/\mathrm{d}V - \mathrm{d}E/\mathrm{d}V = (\mu - E_N)n$ or together with Eq. (4.87)

$$P = n^2 \frac{\partial E_N}{\partial n} = r_0^{-1}(nr_0)^2 \frac{\partial E_N}{\partial (nr_0)}. \tag{4.88}$$

This relation shows us that $P \propto n^{1+x}$ for $E_N \propto n^x$, which corresponds to a polytropic index $\alpha = x^{-1}$ entering $P \propto \mu^{\alpha+1}$. We identify $\alpha = 1/2$ for the TG and $\alpha = 1/3$ for the DDW regime, respectively. In these limiting cases the frequencies are given by Eq. (4.71) as

$$\omega_{n,l}^{TG} = (4n(n+l)+l)^{1/2}\omega_0, \tag{4.89}$$

$$\omega_{n,l}^{DDW} = (n(6n+6l-1)+l)^{1/2}\omega_0. \tag{4.90}$$

However, since we are in $d = 1$ dimensions only values $l = 0, 1$ are possible and by defining $\hat{n} = 2n$ we can rewrite the above expressions and obtain

$$\omega_{\hat{n}}^{TG} = \hat{n}\omega_0, \tag{4.91}$$

$$\omega_{\hat{n}}^{DDW} = \left(\frac{\hat{n}}{2}(3\hat{n}-1)\right)^{1/2}\omega_0. \tag{4.92}$$

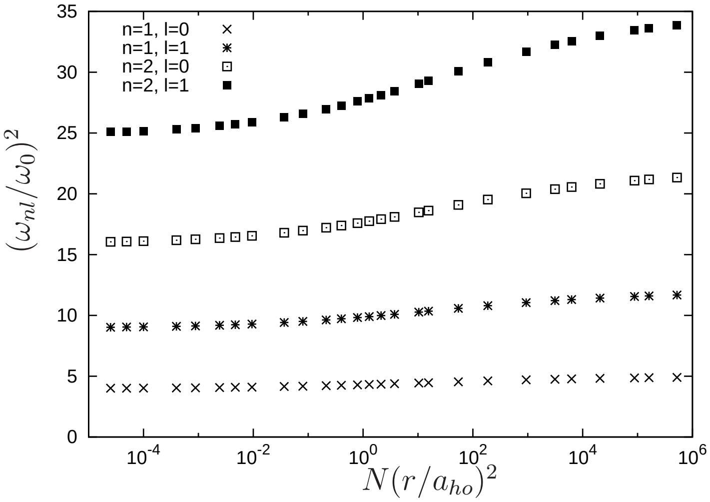

Figure 4.6: Lowest modes ω_{nl} versus $N(r_0/a_{ho})^2$.

Calculating the eigenfrequencies for the equation of state (4.86) is more challenging than in the case of the weakly interacting Bose gas desribed above because one has to invert $\mu = \mu(n)$ for $n = n(\mu)$ numerically. In order to achieve this we note that in our dimensionless units $\hbar = k_B = m = 1$ we have $\mu/Ry^* = 2r_0^2\mu$. We apply E_N from (4.86) to Eq. (4.87) and get

$$\begin{aligned}2r_0^2\mu(nr_0) =&\frac{4\zeta(3)(nr_0)^4 + ae(nr_0)^e + bf(nr_0)^f + c(2+g)(nr_0)^{2+g}}{1+nr_0}\\&+\frac{\zeta(3)(nr_0)^4 + a(nr_0)^e + b(nr_0)^f + c(nr_0)^{2+g}}{(1+(nr_0))^2}\\&+\frac{\frac{2\pi^2}{3}(nr_0)^2}{1+d(nr_0)^g}+\frac{\frac{\pi^2}{3}(nr_0)^2(1+d(1-g)(nr_0)^g)}{(1+d(nr_0)^g)^2}.\end{aligned} \tag{4.93}$$

Given this function we generate a list of 10^4 entries $(nr_0, \mu(nr_0))$ up to an $(nr_0)_{max}$, which has to be chosen such that $\mu((nr_0)_{max}) \geq 2r_0^2\bar{\mu}$. Transposing this list and interpolating the discrete point set to a smooth function ϕ we are left with the desired

$$(nr_0)(\mu) = \phi(2r_0^2\mu). \tag{4.94}$$

It is now apparent why we have chosen $(nr_0)_{max}$ as we did: With this particular choice the whole range of $\mu(z) = \bar{\mu} - z/2, z \in [0, 2\bar{\mu}]$ will be contained in the interpolating function. The presented inversion procedure is sufficient for the well-behaved $\mu(n)$ from Eq. (4.93). Inserting Eq. (4.94) into (4.88) we arrive at

$$
\begin{aligned}
r_0 P(\mu) =& (nr_0)^2 \left. \frac{\partial E_N}{\partial (nr_0)} \right|_{(nr_0)(\mu)} \\
=& \frac{4\zeta(3)(nr_0)^5 + ae(nr_0)^{e+1} + bf(nr_0)^{f+1} + c(2+g)(nr_0)^{3+g}}{1+(nr_0)} \\
& - \frac{\zeta 3(nr_0)^6 + a(nr_0)^{e+2} + b(nr_0)^{f+2} + c(nr_0)^{4+g}}{(1+(nr_0))^2} + \frac{\frac{2\pi^2}{3}(nr_0)^3}{1+d(nr_0)^g} \\
& \left. - \frac{dg\frac{\pi^2}{3}(nr_0)^{3+g}}{(1+d(nr_0)^g)^2} \right|_{(nr_0)(\mu)} . \qquad (4.95)
\end{aligned}
$$

To be as close as possible to reference [64] we will plot the crossover frequencies versus $N(r_0/a_{ho})^2$, which is obtained from

$$
\begin{aligned}
N\left(\frac{r_0}{a_{ho}}\right)^2 &= Nr_0^2 = 2r_0^2 \int_0^{\sqrt{2\bar{\mu}}} \mathrm{d}u n(\mu(u)) = 2r_0 \int_0^{\sqrt{2\bar{\mu}}} \mathrm{d}u \phi(2r_0^2(\bar{\mu} - u^2/2)) \\
&= 2\sqrt{2\bar{\mu}} r_0 \int_0^1 \mathrm{d}x \phi(2r_0^2 \bar{\mu}(1-x^2)), \qquad (4.96)
\end{aligned}
$$

where $\phi(\cdot)$ from Eq. (4.94) enters. For the numerics we set $\bar{\mu} = 1$ and adjust r_0.

The crossover of the (squared) breathing mode $n = 1, l = 0$ from $\omega_M^{TG} = \sqrt{4}$ to $\omega_M^{DDW} = \sqrt{5}$ is shown in Fig. 4.5. A selection of dipole and quadrupole modes is given in Fig. 4.6.

4.5 Oscillations of a Bose gas

We now extend our considerations of the oscillating Bose gas from Secs. 4.4.1 and 4.4.2 to nonvanishing temperature and allow for fluctuations of both density and temperature. If we stay below the critical temperature of Bose–Einstein condensation the system possesses both nonvanishing superfluid and normal fluid density and its hydrodynamic behavior has to be described by two-fluid hydrodynamics [41, 42, 45].

As explained in Sec. 4.2.2, we again find an eigenvalue problem which has to be solved in order to get the collective frequencies of the system. It has the general form

$$
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} g(z) \\ h(z) \end{pmatrix} = \omega^2 \begin{pmatrix} g(z) \\ h(z) \end{pmatrix}, \qquad (4.97)
$$

where the differential operators A, B, C and D are defined in Eqs. (4.47) - (4.50). These operators depend on the equation of state $P(\mu, T)$ and the normal

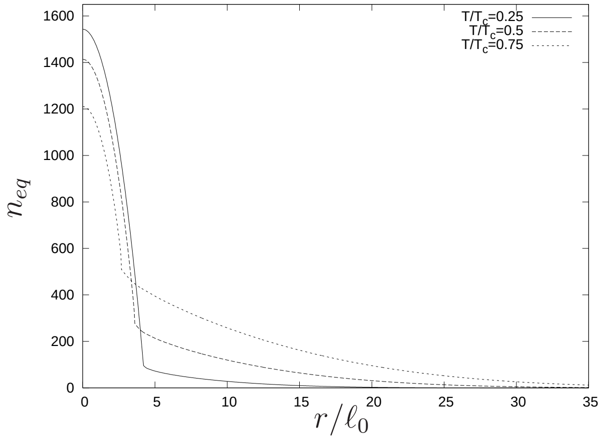

Figure 4.7: Equilibrium density profile as a function of the radius obtained from Lee–Yang theory using local density approximation for a spherical symmetric potential. We observe a peak in the center of the trap. In the outer regions the gas is in its normal phase.

fluid density $n_n(\mu, T)$. Both quantities have to be provided by an underlying microscopic theory. The structure of the eigenvalue problem is similar to the simple zero temperature case. We will show below that the zero temperature limit of Eq. (4.97) is indeed given by Eq. (4.30) and this behavior will also be found in the frequencies.

We see that calculating the frequencies at nonvanishing temperature is as straightforward as it was at $T = 0$. However, results on the temperature dependence of hydrodynamic collective modes are rare in the literature [22]. This is because the equation of state and normal fluid density are in general not known or only in certain ranges of temperature and chemical potential. But when applying the local density approximation $\mu \to \mu - V_{ext}(r)$ one drives through a wide interval of values for the chemical potential and therefore a complete equation of state in μ has to be provided. Our motivation is in applying the Functional Renormalization Group, which is capable of providing the full thermodynamics of cold quantum gases, especially the functions $P(\mu, T)$ and $n_n(\mu, T)$. For the three-dimensional weakly interacting Bose gas the equation of state has been calculated by Lee and Yang [52]. Their formulas are valid for small gas parameter na^3 for temperatures not too close to the critical temperature. We comment on the critical behavior later in this section. The Lee–Yang equation of state is summarized in Sec. 3.2. We neglect the next-to-leading order LHY-correction (3.11) because we are interested in thermal effects.

Before discussing the collective modes of a Bose gas at nonvanishing temper-

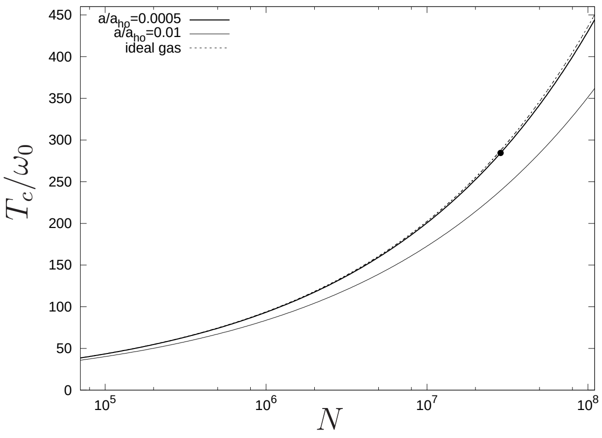

Figure 4.8: Critical temperature in a trap as a function of particle number. The point corresponds to the choice of parameters used in Figs. 4.9 and 4.10. It is very close to the ideal gas result due to the small value of the scattering length. In this plot, $a_{ho} = \ell_0$ is the oscillator length.

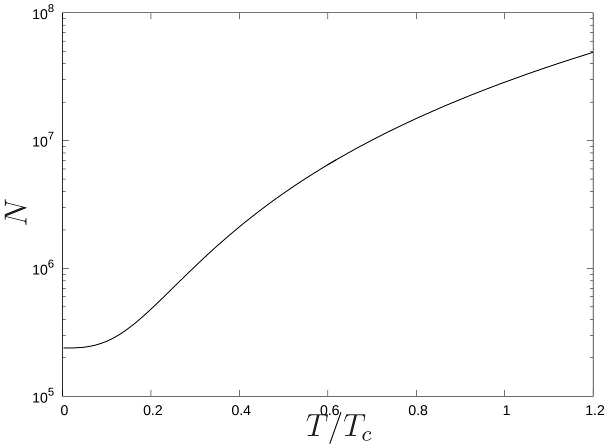

Figure 4.9: Particle number in a spherical trap as a function of T/T_c for fixed chemical potential in the center of the trap $\bar{\mu} = 10\hbar\omega_0$ and scattering length $a/\ell_0 = 0.0005$, where $\ell_0 = (\hbar/m\omega_0)^{1/2}$ is the oscillator length. These are the parameters that are also used in Figs. 4.10 and 4.11. The critical temperature is defined in Eq. (4.98).

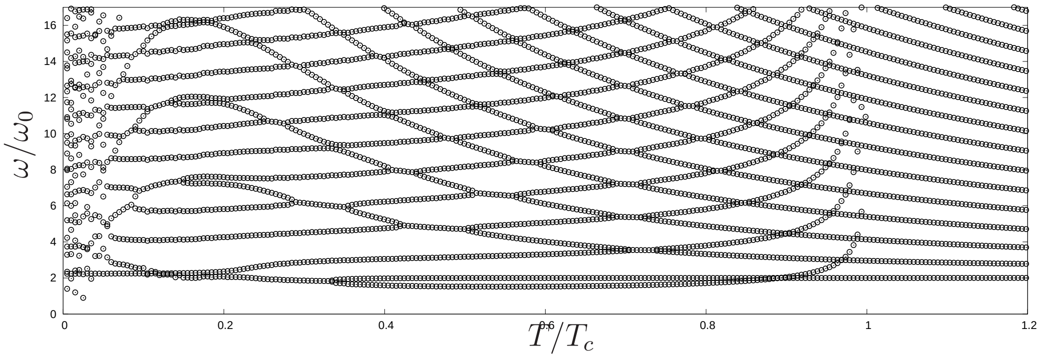

Figure 4.10: Oscillations frequencies of collective modes ($l = 0$) of both the condensate and thermal cloud. We show the temperature dependence for a fixed value of the chemical potential in the center of the trap with the same choice of parameters as in Fig. 4.9. The equation of state is provided by Lee–Yang theory.

ature we comment on some aspects of the static configuration of the trapped gas which are necessary for the interpretation of our results. For a homogeneous system with temperature T we can calculate the critical chemical potential $\mu_c(T)$ of the superfluid phase transition. Within the local density approximation in the trapped system there will be a radius r_c corresponding to the phase boundary between the superfluid and the normal regions of the cloud. This radius fulfills $\mu_c = \bar{\mu} - V_{ext}(r_c)$ with the chemical potential $\bar{\mu}$ in the center of the trap. The characteristic picture of the density profile consists in a narrow condensate peak in the center of the trap which is surrounded by a broad thermal cloud of the normal gas. Of course, there is also a nonvanishing contribution of the normal component to the inner regions. We visualize this behavior in Fig 4.7.

It is now apparent that for the description of a trapped gas at *any* nonzero temperature one has to know the equation of state for both the superfluid and normal phase. In particular the presence and dynamics of the normal gas are of importance. There can be oscillations of the thermal cloud itself and, furthermore, it provides a non-trivial background for the oscillations of the condensate. As we approach zero temperature the thermal cloud vanishes. Condensate oscillations correspond to solutions δn of the hydrodynamic equations which are only nonzero inside a sphere with radius r_c.

The critical temperature T_c of the trapped gas is defined as the temperature where the condensate peak appears in the center of the cloud. This can be reformulated as $\bar{\mu} = \mu_c$ or equivalently $r_c = 0$. For Lee–Yang theory the critical temperature is given by

$$T_c = 2\pi \left(\frac{\bar{\mu}}{2g\zeta(3/2)} \right)^{2/3} . \tag{4.98}$$

If we set $\bar{\mu} = \mu_c$ and calculate the number of particles N in the trap, we find the ideal gas prediction $T_c^{(0)}(N) = \hbar\omega_0(N/\zeta(3))^{1/3}$ to be satisfied for small gas parameter. There are several possibilities to change the parameter T/T_c. In this work we choose to keep the chemical potential in the center of the trap fixed, relating the fixed $\bar{\mu}$ to T_c by Eq. (4.98). For fixed $\bar{\mu}$, the particle number will change when decreasing the temperature. Another way of going to low temperatures is to keep the particle number N fixed. Then the chemical potential in the center of the trap has to be adjusted appropriately. In Fig. 4.9 we plot the change of N with T/T_c for fixed $\bar{\mu}$ and the parameters which are used for Fig. 4.10. In Fig. 4.8 we show the dependence of T_c on the particle number for the ideal gas and two distinct values of the scattering length.

In Fig. 4.10 the isotropic ($l = 0$) temperature dependent oscillation frequencies are shown for $\bar{\mu}/\hbar\omega_0 = 10$ and $a/\ell_0 = 0.0005$, where $\ell_0 = \sqrt{\hbar/m\omega_0}$ is the oscillator length. This corresponds to a critical temperature $T_c/\hbar\omega_0 = 284$. This particular choice of parameters is arbitrary but suffices to show the characteristics of the temperature dependent modes. We observe a rich spectrum of frequencies. The oscillations can be classified as condensate oscillations and oscillations of the thermal cloud. The condensate oscillation correspond to the branches which disappear at T_c. The computability of frequencies above the critical temperature by our method is a manifestation of the fact that the Landau two-fluid model remains formally valid for vanishing superfluid component. Of course, the frequencies for $T \geq T_c$ could be computed equivalently with a one-fluid model for the thermal liquid. For an ideal gas one expects the oscillation frequencies to be integer multiples of ω_0. We find deviations from this behavior.

For low temperatures we find the oscillation frequencies in Fig. 4.10 to scatter. This is a discretization effect related to our numerical implementation and not a short-coming of the two-fluid hydrodynamics. Indeed, with a different discretization, which puts the emphasis on the region $r < r_c$ where the condensate is present, the temperature dependence of the frequencies is found to be smooth. This is demonstrated in Fig. 4.11 where the condensate oscillations are shown for very low temperatures. This plot supplements Fig. 4.10. As T/T_c increases, there is a systematic error in Fig. 4.11 because the thermal cloud is not taken into account properly. We further comment on this issue in Sec. 4.3.

From Fig. 4.10 it is clear that there are level-crossing frequencies over wide ranges of T/T_c. These branches can be used as a thermometer by measuring a certain number of frequencies and then locating them in the plot. How this can be done experimentally by response techniques is explained in Sec. 5.

From our discussion of the equilibrium density profile we see that for all temperatures below the critical temperature T_c there is a radius r_c where the critical equation of state $P(\mu_c, T)$ has to be known. The vicinity of the point μ_c might be small but the behavior of the thermodynamic functions in this interval could nevertheless influence the frequencies significantly. Indeed, the superfluid phase transition is of second order and we expect the specific heat at constant pressure $C_P = T(\partial S/\partial T)_{P,N}$ and the isothermal compressibility $\kappa_T = -V^{-1}(\partial V/\partial P)_{T,N}$ to be singular at μ_c. Their behavior in the critical region as a function of temperature is $C_P \sim |T - T_c|^{-\alpha}$ and $\kappa_T \sim |T - T_c|^{-\gamma}$ with the critical indices [54] of the three-dimensional XY-universality class. The specific heat shows a cusp while the isothermal compressibility diverges. A systematic study of these effects on the collective modes would be very interesting. Lee–Yang theory does not allow for such an investigation.

Using the Lee-Yang formulas given Sec. 3.2 we can calculate numerically the coefficient functions appearing in the operators A, B, C, and D in Eq. (4.97). For very low temperatures we observe that B and C vanish while A approaches its zero temperature expression, given by Eq. (4.30),

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \xrightarrow{T\to 0} \begin{pmatrix} A(T=0) & 0 \\ 0 & D(T=0) \end{pmatrix}. \tag{4.99}$$

This corresponds to a decoupling of superfluid and thermal degrees of freedom. This is expected to happen in the zero temperature limit since the few atoms in the thermal cloud have no influence on the dynamics of the condensate. The operator entering the eigenvalue problem is now block diagonal and all zero temperature frequencies are reproduced. They correspond to Stringari's mean field formula given above. (Note that we have neglect the LHY-correction.) It is interesting that D does not vanish. Indeed, the coefficient functions a_1, d_1 and d_2 entering A and D from Eqs. (4.47) and (4.50) satisfy

$$\frac{a_1}{d_1} \xrightarrow{T\to 0} \frac{1}{3}, \qquad d_2 \xrightarrow{T\to 0} -\frac{1}{6}. \tag{4.100}$$

This can be seen by evaluating the integrals for $P(\mu, T)$ and $n_n(\mu, T)$ numerically or applying an expansion for $T \ll \mu$ using only the phonon part of the spectrum, see Sec. 4.4.1. It can be shown that a_1/d_1 corresponds to the ratio of the squared first and second velocities of sound, c_1^2 and c_2^2, respectively. Therefore we find the well-known relation

$$c_2^2 = \frac{c_1^2}{3} \tag{4.101}$$

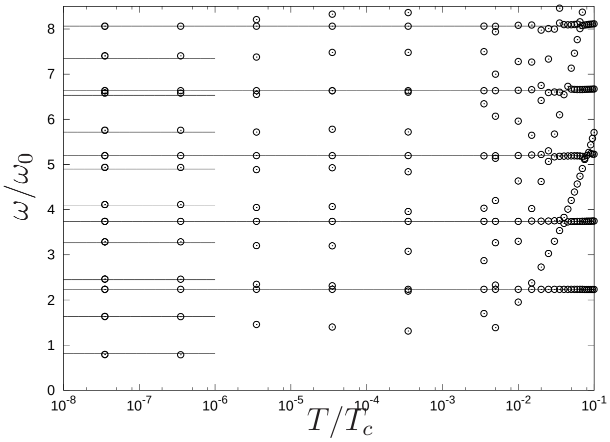

Figure 4.11: Zero temperature limit of the collective oscillations for the same choice of parameters as in Figs. 4.9 and 4.10. In this figure, there is a systematic error for high temperatures, because the thermal cloud is not treated appropriately. The solid lines over the whole range of temperatures correspond to formula (4.71) for superfluid oscillations at $T = 0$. The frequencies given by Eq. (4.102) are indicated by solid lines only for small temperatures, where this formula is expected to be valid.

to be satisfied for very low temperatures.

We have shown in Sec. 4.4.1 that the behavior of the coefficient functions of D as $T \to 0$, Eq. (4.100), can be used to derive the eigenvalues of D by slight modifications from the eigenvalues of A, which are given by Eq. (4.71) with $\alpha = 1$. The corresponding eigenfrequencies are found to be

$$\omega_{nl} = \left(\frac{4n(n+l) - l}{6} \right)^{1/2} \omega_0, \ (n \geq 1). \tag{4.102}$$

The experimental verification of this formula will be difficult, however. Although for low T (and $T \to 0$) these frequencies are solutions to the two-fluid hydrodynamic equations, there are only very few atoms left in the normal phase to oscillate. From Eq. (4.102) it is apparent that for $n = 1, l = 0$ one obtains $\sqrt{2/3}\omega_0$ which is *below* the trapping frequency. This cannot be achieved by formula (4.71) with an effective polytropic index α_{eff}. The existence of such modes below the trapping frequency might be a special feature of superfluid hydrodynamics just as the appearance of a second velocity of sound. Indeed, in their early experiments Stamper-Kurn et al. [7] used a cigar-shaped harmonic potential with trapping frequency $\omega_{0z}/2\pi = 18.04(1)$ Hz in axial direction. They observed an out-of-phase oscillation between condensate and thermal cloud of a

Bose gas at $T = 1\mu K$ in the hydrodynamic regime. The critical temperature was around $1.7\mu K$. The measured frequency of axial motion was found to be $\omega_0/2\pi = 17.26(9)$ Hz, which is below ω_{0z}.

In Fig. 4.11 we show our numerical results for the frequencies in the zero temperature limit using the full Lee–Yang equation of state for the same parameters as in Fig. 4.10. We find formulas (4.71) and (4.102) to be in perfect agreement with our data.

5 Response to an external driving force

In Sec. 4 we have determined the eigenfrequencies of a trapped quantum gas. This finding motivates to assign further observables to the atomic cloud which are typical for oscillating systems. In this section we will show that the response function is the natural generalization of our considerations in Sec. 4.

5.1 The response function

This section provides basic concepts of linear response theory. Furthermore, we use this opportunity to introduce our notation. As before, we work with units such that $\hbar = k_B = m = 1$. For a good introduction into the topic see also [66].

Consider a system described by a Hamilton function H_0 or Hamilton operator $\hat{H}_0$, respectively. To unify our description, we assume all observables to be stochastic variables in the following. Given such an observable x we write $\bar{x}$ for the expectation value of x. For a classical system this includes an averaging over the environment. In the quantum case one also has to account for quantum fluctuations. The state of the system shall be described by a set of observables $\{x_i\}$ such that $\bar{x}_i = 0$. One may think of the following two relevant cases: Suppose a classical system in thermodynamic equilibrium. Then the macroscopic state can be described by density n and temperature T, which are allowed to fluctuate and have expectation values $\bar{n}$ and $\bar{T}$, respectively. Then, $x_1 = n - \bar{n}$ and $x_2 = T - \bar{T}$ have the wished properties. In a quantum system of N particles $x_1, \dots, x_N$ can be associated with the positions of the particles with the expectation value subtracted.

We now apply an external time-dependent force to the system. Then the equilibrium expectation values of $\{x_i\}$ will possibly change from zero to a finite value. If this perturbation is small, the response will also consist in a small effect. We write the corresponding change in the Hamiltonian as

$$\delta H(t) = -x_i f_i(t) \tag{5.1}$$

where the "forces" $f_i(t)$ have to be chosen appropriately and the sum over repeated indices is understood implicitly. We neglect nonlinear terms because we assume the perturbation to be small. The response function $\chi_{ij}(t)$ is defined by

$$\bar{x}_i(t) = \int_0^\infty \mathrm{d}\tau \chi_{ij}(\tau) f_j(t-\tau) = \int_{-\infty}^\infty \mathrm{d}t' \chi_{ij}(t-t')\theta(t-t') f_j(t') \tag{5.2}$$

with Heaviside step-function θ. The conjugates $f_i = -\partial \delta H / \partial x_i$ have to be complete in the sense that an expansion of $\bar{x}_i(t)$ in the above manner is possible. It is important to note that $\chi_{ij}(t)$ does no longer depend on $f_i(t)$ and thus describes a property of the responding system. Causality is ensured by the chosen domain of integration. Expanding

$$\bar{x}_i(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega t} \bar{x}_i(\Omega), \tag{5.3}$$

and analogously for $f(t)$, we arrive at

$$\bar{x}_i(\Omega) = \chi_{ij}(\Omega) f_j(\Omega) \tag{5.4}$$

with

$$\chi_{ij}(\Omega) = \int_{0}^{\infty} \mathrm{d}\tau e^{i\Omega\tau} \chi_{ij}(\tau). \tag{5.5}$$

Note that $\chi_{ij}(\Omega)$ is the Laplace transform of $\chi_{ij}(t)$ and not the Fourier transform. Since $x_i(t)$ and $f_i(t)$ are assumed to be real, we have $\bar{x}_i^*(\Omega) = \bar{x}_i(-\Omega)$ and $f_i^*(\Omega) = f_i(-\Omega)$. This also implies $\chi_{ij}(-\Omega) = \chi_{ij}^*(\Omega)$.

Since $\delta H(t)$ from Eq. (5.1) depends explicitly on time, energy is no longer conserved in the system. Indeed, the energy of the system is given by the expectation value $E(t) = \langle H_0 + \delta H(t) \rangle$ and thus

$$\frac{\mathrm{d}E}{\mathrm{d}t}(t) = \left\langle \frac{\partial \delta H}{\partial t} \right\rangle = -\left\langle x_i \frac{\partial f_i}{\partial t} \right\rangle = -\bar{x}_i(t) \frac{\partial f_i(t)}{\partial t}. \tag{5.6}$$

The only explicitly time-dependent terms appearing in $H_0 + \delta H$ are the $f_i(t)$, and thus the second equality is justified. However, due to the time-dependent force the ground or equilibrium state of the system will eventually also be time-dependent. Therefore, $\bar{x}_i$ on the right-hand side of Eq. (5.6) depends on time.

We illustrate these objects on an important example. Given real forces which are periodic in time with frequency Ω_0,

$$f_i(t) = \mathrm{Re}\left(c_i e^{-i\Omega_0 t}\right) = \frac{1}{2}\left(c_i e^{-i\Omega_0 t} + c_i^* e^{i\Omega_0 t}\right). \tag{5.7}$$

The Fourier transform of $f_i(t)$ is given by

$$f_i(\Omega) = \int_{-\infty}^{\infty} \mathrm{d}t f_i(t) e^{i\Omega t} = \pi c_i \delta(\Omega - \Omega_0) + \pi c_i^* \delta(\Omega + \Omega_0). \tag{5.8}$$

This results in a displacement

$$\begin{aligned} \bar{x}_i(t) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} \bar{x}_i(\Omega) e^{-i\Omega t} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} \chi_{ij}(\Omega) f_j(\Omega) e^{-i\Omega t} \\ &= \frac{1}{2}\left(\chi_{ij}(\Omega_0) c_j e^{-i\Omega_0 t} + \chi_{ij}(-\Omega_0) c_j^* e^{i\Omega_0 t}\right), \end{aligned} \tag{5.9}$$

which is real because of $\chi_{ij}(-\Omega) = \chi^*_{ij}(\Omega)$. The Fourier component of $\bar{x}_i(t)$ reads

$$\bar{x}_i(\Omega) = \pi\chi_{ij}(\Omega)c_j\delta(\Omega - \Omega_0) + \pi\chi_{ij}(-\Omega)c^*_j\delta(\Omega + \Omega_0). \tag{5.10}$$

From $\dot{E} = \mathrm{d}E/\mathrm{d}t$ we can derive the *rate of average dissipated energy* Q by averaging over a small time interval. First we deduce the corresponding formula for a monochromatic force. Then, Eqs. (5.7) and (5.9) for $f_i(t)$ and $\bar{x}_i(t)$ yield

$$\begin{aligned} \dot{E}(t) &= -\bar{x}_i(t)\frac{\mathrm{d}f_i(t)}{\mathrm{d}t} \\ &= -\frac{1}{2}\left(\chi_{ij}(\Omega_0)c_j e^{-i\Omega_0 t} + \chi_{ij}(-\Omega_0)c^*_j e^{i\Omega_0 t}\right)\left(\frac{1}{2}\left(-i\Omega_0 c_i e^{-i\Omega_0 t} + i\Omega_0 c^*_i e^{i\Omega_0 t}\right)\right) \end{aligned} \tag{5.11}$$

Thus we are left with four terms. However, the explicit time-evolution of the system will in most cases not be resolved in experiments but rather a coarse grained, averaged picture emerges. Therefore it is sensible to average over a small interval of time, e.g. the oscillation period $T = 2\pi/\Omega_0$. We then arrive at

$$Q = \frac{1}{T}\int_0^T \mathrm{d}t\dot{E}(t) = \frac{1}{4i}\Omega_0 c^*_i\chi_{ij}(\Omega_0)c_j - \frac{1}{4i}\Omega_0 c^*_j\chi_{ij}(-\Omega_0)c_i = \frac{\Omega_0}{2}\mathrm{Im}\left[c^*_i\chi_{ij}(\Omega_0)c_j\right], \tag{5.12}$$

where we used that the oscillatory contributions from $e^{\pm 2i\Omega_0 t}$ to the integral vanish in the averaging process. We also applied $\chi^*_{ij}(-\Omega_0) = \chi_{ij}(\Omega_0)$. The average dissipation rate Q has dimension of energy over time.

We generalize the notion of average dissipated energy to arbitrary driving forces with Fourier decomposition $f_i(t) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\Omega}{2\pi}e^{-i\Omega t}f_i(\Omega)$. From Eq. (5.6) we find

$$\dot{E}(t) = -\left(\int_{\infty}^{\infty}\frac{\mathrm{d}\Omega}{2\pi}e^{-i\Omega t}\chi_{ij}(\Omega)f_j(\Omega)\right)\left(\int_{-\infty}^{\infty}\frac{\mathrm{d}\Omega'}{2\pi}(-i\Omega')e^{-i\Omega' t}f_i(\Omega')\right). \tag{5.13}$$

Since the forces are not monochromatic there is no characteristic time period T. We therefore integrate $\dot{E}(t)$ over the time of application of the external force. We assume this interval to be finite. However, the domain of integration can be extended to $t \in (-\infty, \infty)$, because of the vanishing contribution from the other regions. We use $\int_{-\infty}^{\infty}\mathrm{d}te^{-i(\Omega+\Omega_0)t} = 2\pi\delta(\Omega + \Omega_0)$ and arrive at

$$\mathcal{E} = \int_{-\infty}^{\infty}\mathrm{d}t\dot{E}(t) = -\int_{-\infty}^{\infty}\frac{\mathrm{d}\Omega}{2\pi}i\Omega f_i(-\Omega)\chi_{ij}(\Omega)f_j(\Omega) \tag{5.14}$$

for the integrated dissipated energy $\mathcal{E}$. Splitting up the integration $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$ we find

$$\mathcal{E} = \int_0^\infty \frac{\mathrm{d}\Omega}{2\pi} 2\Omega \mathrm{Im}\left[f_i(-\Omega)\chi_{ij}(\Omega)f_j(\Omega)\right]. \tag{5.15}$$

This formula generalizes Eq. (5.12). Indeed, inserting $f_i(\Omega)$ from Eq. (5.8) for the monochromatic case ($\Omega_0 > 0$) we obtain

$$\begin{aligned}\mathcal{E} =& \int_0^\infty \frac{\mathrm{d}\Omega}{2\pi} 2\Omega \mathrm{Im}[(\pi c_i \delta(\Omega + \Omega_0) + \pi c_i^* \delta(\Omega - \Omega_0))\chi_{ij}(\Omega) \\ & \times (\pi c_j \delta(\Omega - \Omega_0) + \pi c_j^* \delta(\Omega + \Omega_0))] \\ \overset{\Omega_0 > 0}{=}& \int_0^\infty \frac{\mathrm{d}\Omega}{2\pi} 2\pi\Omega \mathrm{Im}[c_i^* \chi_{ij}(\Omega) c_j \delta(\Omega - \Omega_0)\delta(\Omega - \Omega_0)] \\ =& \Omega_0 \mathrm{Im}[c_i^* \chi_{ij}(\Omega_0) c_j]\delta(\Omega = 0).\end{aligned} \tag{5.16}$$

A comparison of the temporal integrations in Q for the monochromatic force in Eq. (5.12) and $\mathcal{E}$ in Eq. (5.14) shows $\delta(\Omega = 0) =$ length of the time interval $= 2T$, and hence

$$\mathcal{E} = Q \cdot 2T. \tag{5.17}$$

Together with Eq. (5.16) for $\mathcal{E}$, this yields Eq. (5.12) as expected.

5.2 Response function of a classical harmonic oscillator

Consider a classical point particle with Hamiltonian $H = \frac{1}{2m}p^2 + \frac{m}{2}\omega_0^2 x^2$. The equation of motion governing the time evolution of this system is given by

$$\ddot{x}(t) + \omega_0^2 x(t) = 0. \tag{5.18}$$

The solution to this differential equation is found to be $x(t) = \mathrm{Re}(Ae^{-i\omega_0 t})$ with $A \in \mathbb{C}$. If the oscillation is constantly damped by some mechanism we have to introduce a damping constant Γ such that

$$\ddot{x}(t) + \Gamma\dot{x}(t) + \omega_0^2 x(t) = 0. \tag{5.19}$$

The dimension of Γ is inverse time. The solution to Eq. (5.19) is given by $x(t) = \mathrm{Re}(Ae^{-i\Omega t})$ where Ω is a solution of

$$-\Omega^2 - i\Gamma\Omega + \omega_0^2 = 0. \tag{5.20}$$

Different situations arise for different choices of Γ/ω_0. We will be interested in the case where the system oscillates with $\mathrm{Re}(\Omega) \neq 0$ and the decrease in amplitude is only visible over several periods of the oscillation time. For this to be the case, we need to have $\omega_0^2 > \Gamma^2/4$. We then have $\Omega = \Omega_0 - i\Gamma/2$ with $\Omega_0 = (\omega_0^2 - \Gamma^2/4)^{1/2} \in \mathbb{R}$ and the solution to Eq. (5.19) reads $x(t) = \mathrm{Re}(Ae^{-i\Omega_0 t}e^{-\Gamma t/2})$.

In the particular case where $\Gamma^2/4 \ll \omega_0^2$ we find from Eq. (5.20)

$$\Omega \simeq \omega_0 \left(1 - \frac{\Gamma^2}{8\omega_0^2}\right) - \frac{i\Gamma}{2}. \tag{5.21}$$

We observe the frequency of the oscillation to be slightly shifted away from ω_0 and there appears an imaginary term, which corresponds to the damping of the mode.

With regard to the decrease in the oscillation amplitude according to $x(t) = \mathrm{Re}(Ae^{-i\Omega_0 t}e^{-\Gamma t/2})$ we conclude that the kinetic energy of the oscillation decreases in time. This energy is either converted into heat or transferred to the environment, which acts as a sink for energy. Both processes are irreversible and will be summarized by the notion of "dissipation" in the following. Heat of a single particle is here understood as thermal motion of the particle. For a many-body system it has the usual meaning. In the above analysis we introduced a phenomenological damping constant Γ but it should in principle be possible to derive Γ from the microscopic properties of the system. We will come to this later.

Our considerations so far show that after an excitation of the system (e.g. by a transfer of energy), there will be an interval of time where the system oscillates freely with frequency Ω. Due to damping the amplitude of this oscillation shrinks with the length of the time interval depending on Γ/ω_0. After several periods of oscillation, all kinetic energy is dissipated and the system comes to rest, $x(t \to \infty) = 0$. This also corresponds to the equilibrium solution of minimal energy.

A similar but slightly different approach to gain information on the oscillation frequency ω_0 of the point particle is by applying a time-dependent force $f(t)$. If the force is periodic in time with frequency Ω_0, the particle will oscillate with Ω_0 but the amplitude and the amount of dissipated energy will depend on the relation between Ω_0 (force) and ω_0 (particle). A general force yields the equation of motion

$$\ddot{x}(t) + \Gamma\dot{x}(t) + \omega_0^2 x(t) = \frac{1}{m} f(t). \tag{5.22}$$

Applying a Fourier transformation to both $x(t)$ and $f(t)$ as in Eq. (5.3) we obtain

$$(-\Omega^2 - i\Gamma\Omega + \omega_0^2)x(\Omega) = \frac{1}{m} f(\Omega) \tag{5.23}$$

and therefore

$$\chi(\Omega) = x(\Omega)/f(\Omega) = \frac{1}{m}\frac{1}{-\Omega^2 - i\Gamma\Omega + \omega_0^2} \tag{5.24}$$

for the response function. The eigenfrequencies are found as the poles of the response function. We obtain Eq. (5.20) as was expected from our integration of the equation of motion. This proves our statement that information on the oscillation frequencies of a system can also be extracted by response techniques.

We already noted below Eq. (5.2) that $\chi(\Omega)$ is a property of the system and *does not depend on* $f(t)$. We exemplify this for a monochromatic force $f(t) = \frac{1}{2}(ce^{-\Omega_0 t} + c^* e^{i\Omega_0 t})$. Replacing $x(t)$ in Eq. (5.22) by its Fourier transform we obtain

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} x(\Omega)(-\Omega^2 - i\Gamma\Omega + \omega_0^2)e^{-i\Omega t} = \frac{1}{2m}(ce^{-i\Omega_0 t} + c^* e^{i\Omega_0 t}). \tag{5.25}$$

A projection via $\int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega'}{2\pi} e^{i\Omega' t}(\cdot)$ yields

$$\begin{aligned} x(\Omega)(-\Omega^2 - i\Gamma\Omega + \omega_0^2) &= \frac{1}{2m}(2\pi c\delta(\Omega - \Omega_0) + 2\pi c^*\delta(\Omega + \Omega_0)) \\ &\overset{!}{=} \frac{1}{m}\underbrace{(\pi c\delta(\Omega - \Omega_0) + \pi c^*\delta(\Omega + \Omega_0))}_{f(\Omega)} \end{aligned} \tag{5.26}$$

and again Eq. (5.24) for the response function $\chi(\Omega) = x(\Omega)/f(\Omega)$. We remark, however, that the dissipated energy $\mathcal{E}$ or Q explicitly depend on the particular choice of $f(t)$.

5.3 Response function of a trapped quantum gas

In Sec. 4 we have calculated the eigenfrequencies of an ultracold quantum gas trapped in a harmonic potential. We found that the collective modes without damping show an oscillatory behavior with characteristic frequencies ω_{nl}. The system is therefore formally equivalent to a set of harmonic oscillators. Hence we can apply our above procedure to calculate the response function of the trapped gas. The equations of motion in this particular case are provided by linearized two-fluid hydrodynamics.

5.3.1 Response at zero temperature

We begin our investigation with the zero temperature case. The unperturbed system consists in the equilibrium gas trapped in an external potential $V_{\mathrm{ext}}(\vec{x})$

with density profile $n_0(\vec{x})$, chemical potential $\mu_0(\vec{x})$ etc. We do not use any explicit form of V_{ext} in our derivations. To describe the deviation of the macroscopic state from this equilibrium situation, knowledge of one thermodynamic variable is sufficient at zero temperature. We will later choose the chemical potential for this purpose and denote $\delta\mu = \mu - \mu_0$. In the above notation, $\delta\mu(\vec{x}, t)$ corresponds to $\bar{x}_i(t)$ with (continuous) labeling $i = \vec{x}$.

To bring the system out of equilibrium, we apply a small, explicitly time-dependent variation of the external potential, denoted by $\delta V_{\text{ext}}(\vec{x}, t)$. This will lead to a change

$$\delta H'(t) = \int_{\vec{x}} n(\vec{x}) \delta V_{\text{ext}}(\vec{x}, t) \tag{5.27}$$

in the Hamiltonian of the system. Writing $n = n_0 + \delta n$ we obtain two terms. However, the one containing $n_0 \delta V_{\text{ext}}$ will vanish when we integrate over a small period of time to calculate the energy of the system via $E = \langle H_0 + \delta H(t) \rangle$. We can therefore neglect the contribution from the first term and write

$$\delta H(t) = \int_{\vec{x}} \delta n(\vec{x}) \delta V_{\text{ext}}(\vec{x}, t). \tag{5.28}$$

This is of the form $\delta H(t) = -\bar{x}_i f_i(t)$. Note that the stochastic variable $\delta n(\vec{x})$ is not explicitly time-dependent, but due to the driving force the expectation value of δn will have a nontrivial time evolution and eventually $\delta n = \delta n(\vec{x}, t)$. The integrals $\int_{\vec{x}} = \int \mathrm{d}^d x$ in Eqs. (5.27) and (5.28) extend over the whole space. We define the scalar product

$$\langle f, g \rangle := \int_{\vec{x}} f(\vec{x})^* P_0^{\mu\mu}(\vec{x}) g(\vec{x}), \tag{5.29}$$

for functions $f(\vec{x})$ and $g(\vec{x})$. Here $P_0^{\mu\mu}$ is defined as in Eq. (4.20). Then, for $\delta\mu(\vec{x})$ and $\delta V_{\text{ext}}(\vec{x}, t)$ we get

$$\langle \delta\mu, \delta V_{\text{ext}}(t) \rangle = \int_{\vec{x}} \delta n(\vec{x}) \delta V_{\text{ext}}(\vec{x}, t) = \delta H(t) \tag{5.30}$$

because of $\delta n = P_0^{\mu\mu} \delta\mu$. Moreover, $\langle \delta\mu, \delta\mu \rangle = \int \delta n \delta\mu$ such that the chemical potential is multiplied by its thermodynamic conjugate. Another reason for this particular choice (5.29) of the scalar product is that the operator corresponding to hydrodynamic modes becomes self-adjoint, as we will see below.

We define the response function at zero temperature by

$$\delta\mu(\vec{x}, t) = -\int_0^\infty \mathrm{d}\tau \int_{\vec{y}} \chi(\vec{x}, \vec{y}, \tau) \delta V_{\text{ext}}(\vec{y}, t - \tau). \tag{5.31}$$

Of course, we can also define a response function relating $\delta n(\vec{x},t)$ to $\delta V_{\text{ext}}(\vec{y},t-\tau)$, which reads $P_0^{\mu\mu}(\vec{x})\chi(\vec{x},\vec{y},\tau)$, with $P_0^{\mu\mu} > 0$. Obviously, it is possible to switch between both pictures when the equation of state $P(\mu)$ is known. The minus sign in the definition (5.31) accounts for the fact that a local increase of the trapping potential will lead to a decrease of particle number density. Indeed, the individual atoms seek places of lower potential energy such that $\delta n \propto -\delta V_{\text{ext}}$. Introducing $\chi(\vec{x},\vec{y},\Omega) = \int_0^\infty \mathrm{d}t e^{i\Omega t}\chi(\vec{x},\vec{y},t)$, Eq. (5.31) can be written as

$$\delta\mu(\vec{x},\Omega) = -\int_{\vec{y}} \chi(\vec{x},\vec{y},\Omega)\delta V_{\text{ext}}(\vec{y},\Omega). \tag{5.32}$$

When we include the perturbing potential the linearized equation (2.84) for the superfluid velocity becomes $\partial_t\delta\vec{v}_s + \nabla(\delta\mu + \delta V_{ext}) = 0$. Hence the Stringari wave equation $\partial_t^2\delta n - \text{div}(n_0\nabla\delta\mu) = 0$ is now replaced by

$$(\partial_t^2 + \Gamma\partial_t)P_0^{\mu\mu}\delta\mu - \text{div}(P_0^{\mu}\nabla\delta\mu) = \text{div}(P_0^{\mu}\nabla\delta V_{ext}), \tag{5.33}$$

where we also included damping as in Eq. (5.19). The left-hand side of this equation represents the unperturbed oscillator and the right-hand side corresponds to the external driving force. Applying a Fourier transformation $\delta\mu(\vec{x},t) = \int \frac{\mathrm{d}\Omega}{2\pi}e^{-i\Omega t}\delta\mu(\vec{x},\Omega)$ and analogous for $\delta V_{ext}(\vec{x},t)$ we obtain

$$(-\Omega^2 - i\Gamma\Omega + E)\delta\mu(\vec{x},\Omega) = -E\delta V_{\text{ext}}(\vec{x},\Omega), \tag{5.34}$$

where we defined the operator $E = -(P_0^{\mu\mu})^{-1}\text{div}(P_0^{\mu}\nabla\cdot)$. Note that for isotropic harmonic trapping and $\phi_{nlm}(\vec{x}) = g_{nl}(z)r^l f_{lm}$ we have

$$E\phi_{nlm}(\vec{x}) = r^l f_{lm} A g_{nl}(z) = \omega_{nl}^2\phi_{nlm}(\vec{x}) \tag{5.35}$$

with A from Eq. (4.55) and corresponding eigenfunctions $g_{nl}(z)$ of A. Thus, A and E share the same eigenvalues.

The operator E is self-adjoint with respect to the scalar product $\langle\cdot,\cdot\rangle$ from Eq. (5.29). Indeed, given functions $f(\vec{x})$ and $g(\vec{x})$ we have

$$\langle f, Eg\rangle = -\int_{\vec{x}} f(\vec{x})^*\text{div}(P_0^{\mu}(\vec{x})\nabla g(\vec{x})) = -\int_{\vec{x}}\text{div}(P_0^{\mu}(\vec{x})\nabla f(\vec{x}))^* g(\vec{x}) = \langle Ef, g\rangle\,. \tag{5.36}$$

Thus, the eigenfunctions ϕ_n of E (corresponding to eigenvalues ω_n^2) form a complete and orthogonal set. We assume ϕ_n to be normalized to 1 in the following.

The spatial labeling $i = \vec{x}$ can also be substituted for an expansion in the basis functions ϕ_n. We write

$$\delta\mu(\vec{x},\Omega) = \sum_n \delta\mu_n(\Omega)\phi_n(\vec{x}) \tag{5.37}$$

with $\delta\mu_n(\Omega) = \langle \phi_n, \delta\mu(\Omega)\rangle$. The response function in this representation reads

$$\delta\mu_n(\Omega) = -\sum_{n'} \chi_{n,n'}(\Omega)\delta V_{\mathrm{ext,n'}}(\Omega), \tag{5.38}$$

where $\delta V_{\mathrm{ext,n}}(\Omega) = \langle \phi_n, \delta V_{\mathrm{ext}}(\Omega)\rangle$. Using the equation of motion (5.34) we can project out the response function in this particular basis using the self-adjointness of E. We have

$$\left(-\Omega^2 - i\Gamma\Omega + \omega_n^2\right)\langle \phi_n, \delta\mu(\Omega)\rangle = -\omega_n^2 \langle \phi_n, \delta V_{\mathrm{ext}}(\Omega)\rangle . \tag{5.39}$$

Comparing this with the definition (5.38) we conclude

$$\chi_{n,n'}(\Omega) = \frac{\omega_n^2}{-\Omega^2 - i\Gamma\Omega + \omega_n^2}\delta_{nn'}. \tag{5.40}$$

Eq. (5.40) shows that the mode $\delta\mu \propto \phi_n e^{-i\omega_n t}$ can be excited by applying a periodic perturbation δV_{ext} of exactly this spatial form.

For practical purposes the shape of the oscillatory modes ϕ_n is not known and therefore we cannot apply a perturbation $\delta V_{\mathrm{ext}} \propto \phi_n$. We conclude that in general several modes will be excited. Let δV_{ext} be monochromatic with frequency Ω_0 and shape $H(\vec{x})$, i.e.

$$\delta V_{\mathrm{ext}}(\vec{x}, t) = \frac{1}{2}\left(H(\vec{x})e^{-i\Omega_0 t} + H^*(\vec{x})e^{i\Omega_0 t}\right). \tag{5.41}$$

The perturbation of the external potential leads to a variation of the Hamiltonian given by Eq. (5.28). The energy dissipation $\dot{E} = -\bar{x}_i(t)\dot{f}_i(t)$ is therefore given by

$$\dot{E}(t) = \int_{\vec{x}} \delta n(\vec{x}, t)\partial_t \delta V_{ext}(\vec{x}, t). \tag{5.42}$$

We now average over a small interval of time $T = 2\pi/\Omega_0$ as we did in Eq. (5.12) and obtain the average rate of dissipated energy Q. Using $\delta n = P_0^{\mu\mu}\delta\mu$ this becomes

$$Q = \frac{\Omega_0}{2}\mathrm{Im}\left\{\int_{\vec{x},\vec{y}} H^*(\vec{x})P_0^{\mu\mu}(\vec{x})\chi(\vec{x}, \vec{y}, \Omega_0)H(\vec{y})\right\} = \frac{\Omega_0}{2}\mathrm{Im}\,\langle H, \chi(\Omega_0)H\rangle . \tag{5.43}$$

For an oscillating system without spatial coordinate $\vec{x}$, Eq. (5.43) would yield $Q = (\Omega_0/2)|H_0|^2\mathrm{Im}\chi(\Omega_0)$ for constant $H(\vec{x}) \equiv H_0$, compare Eq. (5.12). If we consider the trapped system as a whole we can use this analogy to define the response of the cloud to the perturbation $H(\vec{x})$. It has a clear physical meaning related to the dissipated energy. Expanding $H(\vec{x}) = \sum_n H_n \phi_n(\vec{x})$ with $H_n = \langle \phi_n, H \rangle$ we arrive at

$$\begin{aligned} Q &= \frac{\Omega_0}{2} \mathrm{Im} \left\{ \sum_{n,n'} \langle H, \phi_n \rangle \langle \phi_n, \chi(\Omega_0) \phi_{n'} \rangle \langle \phi_{n'}, H \rangle \right\} \\ &= \frac{\Omega_0}{2} \mathrm{Im} \left\{ \sum_n |H_n|^2 \frac{\omega_n^2}{-\Omega_0^2 - i\Gamma\Omega_0 + \omega_n^2} \right\}. \end{aligned} \tag{5.44}$$

Therefore, we have

$$\chi(\Omega) = \sum_n \frac{1}{m_n} \frac{1}{-\Omega^2 - i\Gamma\Omega + \omega_n^2} \tag{5.45}$$

with positive response coefficients

$$\frac{1}{m_n} = |H_n|^2 \omega_n^2. \tag{5.46}$$

The response coefficients generalize the concept of mass of a classical oscillating particle to a trapped gas. We emphasize that $\chi(\Omega)$ in Eq. (5.45) no longer depends on $\vec{x}$, but instead the particular shape of δV_{ext} is contained. Compare this to Eq. (5.40) for $\chi_{nn'}$, which is independent of δV_{ext}.

5.3.2 Response at nonvanishing temperature without damping

For nonzero temperature the macroscopic state out of equilibrium has to be described by two independent thermodynamic variables. We choose $\delta\mu = \mu - \mu_0$ and $\delta T = T - T_0$ since the eigenfunctions of collective modes are expressed in these two variables. The response function is defined analogously to the above analysis by

$$\delta n(\vec{x}, t) = -\int_{-\infty}^{\infty} \mathrm{d}t' \int_{\vec{y}} \theta(t - t') \chi^{(n)}(\vec{x}, \vec{y}, t - t') \delta V_{\mathrm{ext}}(\vec{y}, t'). \tag{5.47}$$

The variation of density can be decomposed into $\delta n = P_0^{\mu\mu} \delta\mu + P^{\mu T} \delta T$. Since δn is the quantity which is most easily observed by imaging, it is sensible to work with the density-response $\chi^{(n)}$ given by Eq. (5.47). However, for computational convenience we also introduce the responses of chemical potential and temperature according to

$$\delta\mu(\vec{x}, t) = -\int_{-\infty}^{\infty} \mathrm{d}t' \int_{\vec{y}} \theta(t - t') \chi^{(\mu)}(\vec{x}, \vec{y}, t - t') \delta V_{\mathrm{ext}}(\vec{y}, t'), \tag{5.48}$$

$$\delta T(\vec{x}, t) = -\int_{-\infty}^{\infty} \mathrm{d}t' \int_{\vec{y}} \theta(t - t') \chi^{(T)}(\vec{x}, \vec{y}, t - t') \delta V_{\mathrm{ext}}(\vec{y}, t'). \tag{5.49}$$

We then have $\chi^{(n)}(\vec{x},\vec{y},t) = P_0^{\mu\mu}(\vec{x})\chi^{(\mu)}(\vec{x},\vec{y},t) + P_0^{\mu T}(\vec{x})\chi^{(T)}(\vec{x},\vec{y},t)$. The relative minus sign in our definition (5.47) reflects the fact that an increase in potential energy will lead to a decrease of density.

The equations of motion are given by the linearized ideal two-fluid equations

$$\begin{aligned}
\partial_t \delta n + \mathrm{div}(\delta\vec{g}) &= 0, && (5.50)\\
\partial_t \delta\vec{g} + \nabla\delta P + \delta n \nabla V_{\mathrm{ext}} &= -n_0 \nabla \delta V_{\mathrm{ext}}(t), && (5.51)\\
\partial_t \vec{v}_s + \nabla\delta\mu &= -\nabla\delta V_{\mathrm{ext}}(t), && (5.52)\\
\partial_t \delta s + \mathrm{div}(s_0 \delta\vec{v}_n) &= 0 && (5.53)
\end{aligned}$$

with external potential $V_{\mathrm{ext}} + \delta V_{\mathrm{ext}}(t)$, $|\delta V_{\mathrm{ext}}| \ll |V_{\mathrm{ext}}|$ and $\delta\vec{g} = n_{s,0}\delta\vec{v}_s + n_{n,0}\delta\vec{v}_n$. For further details see Sec. 2.4. Eq. (5.51) can be written as

$$n_{n,0}\partial_t(\vec{v}_n - \vec{v}_s) + s_0 \nabla \delta T = 0. \tag{5.54}$$

The following derivation is similar to Sec. 4.2.2. Eliminating $\vec{v}_s$ and $\vec{g} = n_0\delta\vec{v}_s + n_{n,0}(\delta\vec{v}_n - \vec{v}_s)$ we find

$$\begin{aligned}
0 &= \partial_t^2 \delta n - \mathrm{div}\left(n_0 \nabla\delta\mu + s_0\nabla\delta T + n_0 \nabla\delta V_{\mathrm{ext}}(t)\right), && (5.55)\\
0 &= \partial_t^2 \delta s - \mathrm{div}\left(s_0 \nabla\delta\mu + \tilde{n}_0\nabla\delta T + s_0 \nabla\delta V_{\mathrm{ext}}(t)\right), && (5.56)
\end{aligned}$$

with $\tilde{n} = s^2/n_n$. A Fourier decomposition yields

$$\delta n(\vec{x},t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega t} \left\{ P_0^{\mu\mu}(\vec{x})\delta\mu(\vec{x},\Omega) + P_0^{\mu T}(\vec{x})\delta T(\vec{x},\Omega) \right\}. \tag{5.57}$$

In the same way we introduce $\delta s(\Omega) = P_0^{\mu T}\delta\mu(\Omega) + P_0^{TT}\delta T(\Omega)$ and $\delta V_{\mathrm{ext}}(\Omega)$. We arrive at

$$\begin{aligned}
0 = {} & \Omega^2 (P_0^{\mu\mu}\delta\mu(\vec{x},\Omega) + P^{\mu T}\delta T(\vec{x},\Omega)) \\
& + \mathrm{div}\left(n_0\nabla\delta\mu(\vec{x},\Omega) + s_0\nabla\delta T(\vec{x},\Omega) + n_0 \nabla\delta V_{\mathrm{ext}}(\vec{x},\Omega)\right), && (5.58)\\
0 = {} & \Omega^2 (P_0^{\mu T}\delta\mu(\vec{x},\Omega) + P^{TT}\delta T(\vec{x},\Omega)) \\
& + \mathrm{div}\left(s_0\nabla\delta\mu(\vec{x},\Omega) + \tilde{n}_0\nabla\delta T(\vec{x},\Omega) + s_0 \nabla\delta V_{\mathrm{ext}}(\vec{x},\Omega)\right). && (5.59)
\end{aligned}$$

Rewriting this in a compact matrix notation yields

$$\begin{aligned}
0 = {} & \left[\Omega^2 \begin{pmatrix} P_0^{\mu\mu} & P_0^{\mu T} \\ P_0^{\mu T} & P_0^{TT} \end{pmatrix} + \begin{pmatrix} \mathrm{div}(n_0\nabla\cdot) & \mathrm{div}(s_0\nabla\cdot) \\ \mathrm{div}(s_0\nabla\cdot) & \mathrm{div}(\tilde{n}_0\nabla\cdot) \end{pmatrix} \right] \begin{pmatrix} \delta\mu(\Omega) \\ \delta T(\Omega) \end{pmatrix} \\
& + \begin{pmatrix} \mathrm{div}(n_0\nabla\cdot) & 0 \\ 0 & \mathrm{div}\,(s_0\nabla\cdot) \end{pmatrix} \begin{pmatrix} \delta V_{\mathrm{ext}}(\Omega) \\ \delta V_{\mathrm{ext}}(\Omega) \end{pmatrix}. && (5.60)
\end{aligned}$$

We arrive at the equation

$$\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}\begin{pmatrix} \delta\mu(\vec{x},\Omega) \\ \delta T(\vec{x},\Omega) \end{pmatrix} + \begin{pmatrix} E_5 & E_6 \\ E_7 & E_8 \end{pmatrix}\begin{pmatrix} \delta V_{\text{ext}}(\vec{x},\Omega) \\ \delta V_{\text{ext}}(\vec{x},\Omega) \end{pmatrix} = \Omega^2 \begin{pmatrix} \delta\mu(\vec{x},\Omega) \\ \delta T(\vec{x},\Omega) \end{pmatrix} \tag{5.61}$$

with differential operators

$$\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} = \begin{pmatrix} P_0^{\mu\mu} & P_0^{\mu T} \\ P_0^{\mu T} & P_0^{TT} \end{pmatrix}^{-1} \begin{pmatrix} -\text{div}(n_0\nabla\cdot) & -\text{div}(s_0\nabla\cdot) \\ -\text{div}(s_0\nabla\cdot) & -\text{div}(\tilde{n}_0\nabla\cdot) \end{pmatrix}, \tag{5.62}$$

compare Eq. (4.12), and

$$\begin{pmatrix} E_5 & E_6 \\ E_7 & E_8 \end{pmatrix} = \begin{pmatrix} P_0^{\mu\mu} & P_0^{\mu T} \\ P_0^{\mu T} & P_0^{TT} \end{pmatrix}^{-1} \begin{pmatrix} -\text{div}(n_0\nabla\cdot) & 0 \\ 0 & -\text{div}\,(s_0\nabla\cdot) \end{pmatrix}. \tag{5.63}$$

We specify a scalar product such that the operator $(E_1\ E_2, E_3\ E_4)$ is self-adjoint. Given two-component functions $\Phi(\vec{x})$ and $\Psi(\vec{x})$ we define

$$\langle\Phi,\Psi\rangle = \int_{\vec{x}} (\Phi_1(\vec{x})^*, \Phi_2(\vec{x})^*) \begin{pmatrix} P_0^{\mu\mu}(\vec{x}) & P_0^{\mu T}(\vec{x}) \\ P_0^{\mu T}(\vec{x}) & P_0^{TT}(\vec{x}) \end{pmatrix} \begin{pmatrix} \Psi_1(\vec{x}) \\ \Psi_2(\vec{x}) \end{pmatrix}. \tag{5.64}$$

Thus, the scalar product is defined with the Hessian matrix of the equation of state $P(\mu,T)$. This matrix is positive definite, which can be seen from Eqs. (4.14) and (4.15). As in the zero temperature case we get a well-defined physical object for $\Phi = \Psi = (\delta\mu,\delta T)$. In this particular case we have

$$\begin{aligned} \left\langle \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix}, \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix} \right\rangle &= \int_{\vec{x}} \Big(\delta\mu(P_0^{\mu\mu}\delta\mu + P_0^{\mu T}\delta T) + \delta T(P_0^{\mu T}\delta\mu + P_0^{TT}\delta T)\Big) \\ &= \int_{\vec{x}} (\delta\mu\delta n + \delta T\delta s), \end{aligned} \tag{5.65}$$

where $\delta\mu$ and δT are multiplied with their thermodynamic conjugates δn and δs, respectively. From the representation (5.62) one immediately deduces that $(E_1\ E_2, E_3\ E_4)$ and $(E_5\ E_6, E_7\ E_8)$ are self-adjoint with respect to the scalar product defined in Eq. (5.64) because the Hessian matrix cancels with its inverse and the divergence terms can be "pulled to the left" via partial integration. We conclude that the eigenfunctions

$$\Phi_n(\vec{x}) = \begin{pmatrix} \phi_n(\vec{x}) \\ \psi_n(\vec{x}) \end{pmatrix} \tag{5.66}$$

of $(E_1\ E_2, E_3\ E_4)$ form a complete and orthogonal set.

For an isotropic harmonic trap, the eigenfunctions Φ_{nlm} defined in (5.66) are known from our diagonalization procedure described Sec. 4.2.2. Indeed, one verifies that $(E_1\ E_2, E_3\ E_4)$ in a spherical harmonic trap $V_{\text{ext}}(\vec{x}) = \frac{m}{2}\omega_0^2 r^2 = \frac{1}{2}r^2$ shares the same eigenvalues as $(A\ B, C\ D)$ from Eq. (4.46), i.e.

$$\begin{aligned} \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \begin{pmatrix} g_{nl}(z) r^l f_{lm} \\ h_{nl}(z) r^l f_{lm} \end{pmatrix} &= r^l f_{lm} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} g_{nl}(z) \\ h_{nl}(z) \end{pmatrix} \\ &= \omega_{nl}^2 \begin{pmatrix} g_{nl}(z) r^l f_{lm} \\ g_{nl}(z) r^l f_{lm} \end{pmatrix}. \end{aligned} \tag{5.67}$$

Here we used the exact eigenfunctions $g_{nl}(z)$ and $h_{nl}(z)$ of $(A\ B, C\ D)$. We therefore have

$$\phi_{nlm} = g_{nl}(z) r^l f_{lm}, \tag{5.68}$$

$$\psi_{nlm} = h_{nl}(z) r^l f_{lm} \tag{5.69}$$

for an isotropic harmonic potential. For general external potential the computation of the Φ_n is more involved, but possible. Hence we can use the scalar product $\langle \cdot, \cdot \rangle$ from Eq. (5.64) to project out the Φ_n-components of the equation of motion (5.61). We then arrive at

$$\begin{aligned} 0 =& (\omega_n^2 - \Omega^2) \left\langle \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}, \begin{pmatrix} \delta\mu(\Omega) \\ \delta T(\Omega) \end{pmatrix} \right\rangle \\ &+ \left\langle \begin{pmatrix} E_5 & E_6 \\ E_7 & E_8 \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}, \begin{pmatrix} \delta V_{\text{ext}}(\Omega) \\ \delta V_{\text{ext}}(\Omega) \end{pmatrix} \right\rangle. \end{aligned} \tag{5.70}$$

Contrary to the zero temperature case in Eq. (5.39), the second term on the right-hand side of this equation does not yield $\omega_n^2 \langle \Phi_n, \delta V_{\text{ext}} \rangle$, because the operators $(E_1\ E_2, E_3\ E_4)$ and $(E_5\ E_6, E_7\ E_8)$ do not coincide.

5.3.3 Damping constants from transport coefficients

In Eq. (5.33) we introduced a phenomenological damping constant Γ. To calculate this quantity from first principles is highly attractive. In particular such an analysis reveals that Γ depends on the frequency Ω, i.e. $\Gamma = \Gamma_n$ for the peaks located at ω_n. We expect the damping of collective modes and response to an external driving force to be described by first order dissipative hydrodynamics as it is introduced in Sec. 2.4. In these equations five transport coefficients $\eta, \kappa_T, \zeta_1, \zeta_2$, and ζ_3 appear. In general we have $\eta = \eta(\mu, T)$ etc. We aim at finding the relation between these quantities and the damping function $\Gamma(\Omega)$. Of course, the transport coefficients itself are phenomenological and have to be provided by e.g. non-equilibrium quantum field theory. We already encountered a similar situation with the equation of state $P(\mu, T)$, which has to be computed from the partition function. In Eq. (5.21) we have shown that for the case of weak damping, the location of the poles of the response function is governed by ideal hydrodynamics, and thus by the equation of state. The width instead is given by the damping function and therefore depends on the transport coefficients.

We start from the linearized dissipative two-fluid equations (2.92) - (2.95), i.e.

$$\begin{aligned} \partial_t \delta n + \mathrm{div}(\delta \vec{g}) &= 0, && (5.71)\\ (\partial_t \delta \vec{g} + \nabla \delta P + \delta n \nabla V_{\mathrm{ext}})_i &= \partial_k(\eta \delta \sigma_{ik}) + \partial_i(\zeta_1 \mathrm{div}(\delta \vec{w}) + \zeta_2 \mathrm{div}(\delta \vec{v}_n)) \\ &=: (\vec{\varepsilon}_1)_i + \partial_i \varepsilon_2, && (5.72)\\ \partial_t \delta \vec{v}_s + \nabla \delta \mu &= \nabla(\zeta_3 \mathrm{div}(\delta \vec{w}) + \zeta_1 \mathrm{div}(\delta \vec{v}_n)) =: \nabla \varepsilon_3, && (5.73)\\ \partial_t \delta s + \mathrm{div}(s_0 \delta \vec{v}_n) &= \frac{1}{T} \mathrm{div}(\kappa_T \nabla \delta T) =: \varepsilon_4, && (5.74) \end{aligned}$$

and try to mimic terms involving $\Omega^2 + i\Gamma\Omega$ in the equations of motion. For notational convenience we introduced the short-hands $\vec{\varepsilon}_1, \varepsilon_2$, ε_3, and ε_4. In the ideal case these quantities vanish.

Using $\nabla \delta P + \delta n \nabla V_{\mathrm{ext}} = n_0 \nabla \delta \mu + s_0 \nabla \delta T$ and Eq. (5.73), Eq. (5.72) is seen to be equivalent to

$$n_{n,0} \partial_t (\delta \vec{v}_n - \delta \vec{v}_s) + s_0 \nabla \delta T = \vec{\varepsilon}_1 + \nabla \varepsilon_2 - n_0 \nabla \varepsilon_3. \qquad (5.75)$$

In the ideal case we were able to eliminate the velocity fields by taking a further partial derivative with respect to time of Eqs. (5.71) and (5.74). Including dissipative terms we obtain

$$\begin{aligned} n_{n,0} \partial_t (\delta \vec{v}_n - \delta \vec{v}_s)|_{\mathrm{diss.}} &= -s_0 \nabla \delta T + \vec{\varepsilon}_1 + \nabla \varepsilon_2 - n_0 \nabla \varepsilon_3, && (5.76)\\ \partial_t \delta \vec{v}_s|_{\mathrm{diss.}} &= -\nabla \delta \mu + \nabla \varepsilon_3 && (5.77) \end{aligned}$$

and hence

$$\begin{aligned} 0 &= \partial_t^2 \delta n - \mathrm{div}(n_0 \nabla \delta \mu + s_0 \nabla \delta T) + \mathrm{div}(\vec{\varepsilon}_1) + \Delta \varepsilon_2, && (5.78)\\ 0 &= \partial_t^2 \delta s - \mathrm{div}(s_0 \nabla \delta \mu + \tilde{n}_0 \nabla \delta T) \\ &\quad + \mathrm{div}\left(\frac{s_0}{n_{n,0}} (\vec{\varepsilon}_1 + \nabla \varepsilon_2) + s_0 (1 - \frac{n_0}{n_{n,0}}) \nabla \varepsilon_3 \right) - \partial_t \varepsilon_4. && (5.79) \end{aligned}$$

Obviously, the velocity fields ($\vec{\varepsilon}_1$, ε_2, and ε_3) did not disappear from the equations. We perform a Fourier transformation of all time-dependent quantities, i.e. $\delta\vec{v}_s(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega t}\delta\vec{v}_s(\Omega)$ etc., and arrive at

$$\Omega^2 \begin{pmatrix} \delta\mu(\Omega) \\ \delta T(\Omega) \end{pmatrix} = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \begin{pmatrix} \delta\mu(\Omega) \\ \delta T(\Omega) \end{pmatrix} + \begin{pmatrix} P_0^{\mu\mu} & P_0^{\mu T} \\ P_0^{\mu T} & P_0^{TT} \end{pmatrix}^{-1} \times$$
$$\times \begin{pmatrix} \operatorname{div}(\vec{\varepsilon}_1(\Omega)) + \Delta\varepsilon_2(\Omega) \\ \operatorname{div}\left(\frac{s_0}{n_{n,0}}[\vec{\varepsilon}_1(\Omega) + \nabla\varepsilon_2(\Omega)] + s_0(1 - \frac{n_0}{n_{n,0}})\nabla\varepsilon_3(\Omega)\right) + i\Omega\varepsilon_4(\Omega) \end{pmatrix} \quad (5.80)$$

with operator (E_1 E_2 , E_3 , E_4) from Eq. (4.12). Note that all quantities are still space dependent.

The next step consists in a linear expansion of $\delta\vec{v}_s$ and $\delta\vec{v}_n$, respectively, in $\delta\mu(\Omega)$ and $\delta T(\Omega)$. Considering the ideal equations

$$\partial_t\delta\vec{v}_s(t)|_{\text{id.}} = -\nabla\delta\mu(t), \quad (5.81)$$
$$n_{n,0}\partial_t(\delta\vec{v}_n(t) - \delta\vec{v}_s(t))|_{\text{id.}} = -s_0\nabla\delta T(t), \quad (5.82)$$

we conclude from Eqs. (5.76) and (5.77) that $\delta\vec{v}_{s,n}|_{\text{diss.}} = \delta\vec{v}_{s,n}|_{\text{id.}} + \ldots$ with

$$\delta\vec{v}_s(\Omega)|_{\text{id.}} = \frac{1}{i\Omega}\nabla\delta\mu(\Omega), \quad (5.83)$$
$$\delta\vec{v}_n(\Omega)|_{\text{id.}} = \frac{1}{i\Omega}\left(\nabla\delta\mu(\Omega) + \frac{s_0}{n_{n,0}}\nabla\delta T(\Omega)\right). \quad (5.84)$$

Inserting these expressions into Eqs. (2.88) and (2.89) for $\delta\sigma_{ik}$ and $\delta\vec{w}$ we obtain

$$\delta\sigma_{ik}(\Omega) \simeq \frac{1}{i\Omega}\Bigg[2\partial_i\partial_k\delta\mu(\Omega) - \frac{2}{3}\delta_{ik}\Delta\delta\mu(\Omega)$$
$$+ \partial_i\left(\frac{s_0}{n_{n,0}}\partial_k\delta T(\Omega)\right) + \partial_k\left(\frac{s_0}{n_{n,0}}\partial_i\delta T(\Omega)\right) - \frac{2}{3}\delta_{ik}\operatorname{div}\left(\frac{s_0}{n_{n,0}}\nabla\delta T(\Omega)\right)\Bigg], \quad (5.85)$$
$$\delta\vec{w}(\Omega) \simeq -\frac{1}{i\Omega}\frac{n_{s,0}s_0}{n_{n,0}}\nabla\delta T(\Omega). \quad (5.86)$$

In this approximation the velocity fields are absent in the dissipative hydrodynamic equations. The successive inclusion of higher orders in $\delta\vec{v}_{s,n}|_{\text{diss.}} = \delta\vec{v}_{s,n}|_{\text{id.}} + \ldots$ would yield expressions which are of higher order in the transport coefficients and we assume them to be negligible.

It is now possible to write $\vec{\varepsilon}_1(\Omega)$, $\varepsilon_2(\Omega)$, $\varepsilon_3(\Omega)$ and $\varepsilon_4(\Omega)$ as operators acting on $\delta\mu(\Omega)$ and $\delta T(\Omega)$. We find

$$\begin{aligned}
(\vec{\varepsilon}_1(\Omega))_i &\simeq \frac{1}{i\Omega}\hat{\epsilon}_{1,i}\begin{pmatrix}\delta\mu(\Omega)\\ \delta T(\Omega)\end{pmatrix}, \qquad &(5.87)\\
\varepsilon_2(\Omega) &\simeq \frac{1}{i\Omega}\hat{\epsilon}_{2}\begin{pmatrix}\delta\mu(\Omega)\\ \delta T(\Omega)\end{pmatrix}, &(5.88)\\
\varepsilon_3(\Omega) &\simeq \frac{1}{i\Omega}\hat{\epsilon}_{3}\begin{pmatrix}\delta\mu(\Omega)\\ \delta T(\Omega)\end{pmatrix}, &(5.89)\\
\varepsilon_4(\Omega) &\simeq \hat{\epsilon}_{4}\begin{pmatrix}\delta\mu(\Omega)\\ \delta T(\Omega)\end{pmatrix} &(5.90)
\end{aligned}$$

with (1×2)-operators

$$\begin{aligned}
\hat{\epsilon}_{1,i} =& \Bigg(\partial_k\left[\eta\left\{2\partial_i\partial_k\cdot-\frac{2}{3}\delta_{ik}\Delta\cdot\right\}\right],\\
&\partial_k\left[\eta\left\{\partial_i(\frac{s_0}{n_{n,0}}\partial_k\cdot)+\partial_k(\frac{s_0}{n_{n,0}}\partial_i\cdot)-\frac{2}{3}\delta_{ik}\mathrm{div}\left(\frac{s_0}{n_{n,0}}\nabla\cdot\right)\right\}\right]\Bigg), \qquad &(5.91)\\
\hat{\epsilon}_2 =& \left(\zeta_2\Delta\cdot, \zeta_2\mathrm{div}\left(\frac{s_0}{n_{n,0}}\nabla\cdot\right)-\zeta_1\mathrm{div}\left(\frac{n_{s,0}s_0}{n_{n,0}}\nabla\cdot\right)\right), &(5.92)\\
\hat{\epsilon}_3 =& \left(\zeta_1\Delta\cdot, \zeta_1\mathrm{div}\left(\frac{s_0}{n_{n,0}}\nabla\cdot\right)-\zeta_3\mathrm{div}\left(\frac{n_{s,0}s_0}{n_{n,0}}\nabla\cdot\right)\right), &(5.93)\\
\hat{\epsilon}_4 =& \left(0, \frac{1}{T}\mathrm{div}(\kappa_T\nabla\cdot)\right). &(5.94)
\end{aligned}$$

We write $\hat{\epsilon}_2 = (\hat{\epsilon}_{2\mu}, \hat{\epsilon}_{2T})$ etc. to address the first and second component separately. To this order of approximation the eigenvalue problem (5.80) becomes

$$\Omega^2\begin{pmatrix}\delta\mu(\Omega)\\ \delta T(\Omega)\end{pmatrix} \simeq \begin{pmatrix}E_1 & E_2\\ E_3 & E_4\end{pmatrix}\begin{pmatrix}\delta\mu(\Omega)\\ \delta T(\Omega)\end{pmatrix} - i\Omega\begin{pmatrix}X_1(\Omega) & X_2(\Omega)\\ X_3(\Omega) & X_4(\Omega)\end{pmatrix}\begin{pmatrix}\delta\mu(\Omega)\\ \delta T(\Omega)\end{pmatrix} \tag{5.95}$$

with

$$\begin{pmatrix}X_1(\Omega) & X_2(\Omega)\\ X_3(\Omega) & X_4(\Omega)\end{pmatrix} = \begin{pmatrix}P_0^{\mu\mu} & P_0^{\mu T}\\ P_0^{\mu T} & P_0^{TT}\end{pmatrix}^{-1}\begin{pmatrix}Y_1(\Omega) & Y_2(\Omega)\\ Y_3(\Omega) & Y_4(\Omega)\end{pmatrix} \tag{5.96}$$

and

$$
\begin{aligned}
Y_1(\Omega) &= \frac{1}{\Omega^2}(\partial_i[\hat{\epsilon}_{1,i\mu}\cdot] + \Delta[\hat{\epsilon}_{2\mu}\cdot]), && (5.97)\\
Y_2(\Omega) &= \frac{1}{\Omega^2}(\partial_i[\hat{\epsilon}_{1,iT}\cdot] + \Delta[\hat{\epsilon}_{2T}\cdot]), && (5.98)\\
Y_3(\Omega) &= \frac{1}{\Omega^2}\left\{\partial_i\left(\frac{s_0}{n_{n,0}}[\hat{\epsilon}_{1,i\mu}\cdot]\right) + \mathrm{div}\left(\frac{s_0}{n_{n,0}}\nabla[\hat{\epsilon}_{2\mu}\cdot]\right)\right.\\
&\quad \left. +\mathrm{div}\left(s_0(1-\frac{n_0}{n_{n,0}})\nabla[\hat{\epsilon}_{3\mu}\cdot]\right)\right\}, && (5.99)\\
Y_4(\Omega) &= \frac{1}{\Omega^2}\left\{\partial_i\left(\frac{s_0}{n_{n,0}}[\hat{\epsilon}_{1,iT}\cdot]\right) + \mathrm{div}\left(\frac{s_0}{n_{n,0}}\nabla[\hat{\epsilon}_{2T}\cdot]\right)\right.\\
&\quad \left. +\mathrm{div}\left(s_0(1-\frac{n_0}{n_{n,0}})\nabla[\hat{\epsilon}_{3T}\cdot]\right)\right\} - \hat{\epsilon}_{4T}. && (5.100)
\end{aligned}
$$

We summarize our findings so far in a compact notation. Neglecting higher orders of the transport coefficients we could eliminate the velocity fields from the linearized dissipative hydrodynamics and obtained the equation of motion

$$0 = (-\Omega^2 - i\Omega\hat{X}(\Omega) + \hat{E})f(\Omega) \qquad (5.101)$$

for some function $f(\Omega, \vec{x})$ and differential operators $\hat{E}$ and $\hat{X}(\Omega)$. The similarity to the classical harmonic oscillator is obvious. Ω enters this equation as a parameter, the functional dependence is only on $\vec{x}$. The eigenfunctions $f_n^{(0)}(\vec{x})$ of $\hat{E}$ are assumed to be known. They satisfy $\hat{E}f_n^{(0)} = \omega_n^2 f_n^{(0)}$. We want to find the values of Ω such that there exists a function $f(\Omega, \vec{x})$ which solves Eq. (5.101). The ideal equation ($\hat{X} = 0$) only has a solution for $\Omega \in \{\omega_n\}$.

For sufficiently small transport coefficients, $\hat{E} - i\Omega\hat{X}(\Omega)$ in Eq. (5.101) is a perturbation of $\hat{E}$. More precisely, $i\Omega\hat{E}^{-1}\hat{X}(\Omega)$ is small in comparison to the identity. Therefore, the eigenfunctions $f_{n,\Omega}(\vec{x})$ of $\hat{E}-i\Omega\hat{X}(\Omega)$ can again be labeled with n and the difference between $f_n^{(0)}$ and $f_{n,\Omega}$ is small. We choose a scalar product such that the operator $\hat{E}$ is self-adjoint. Then $f_{n,\Omega}$ can be expanded in the complete set $\{f_n^{(0)}\}$ and we find

$$f_{n,\Omega}(\vec{x}) = f_n^{(0)}(\vec{x}) + \sum_{m\neq n} c_{nm}(\Omega) f_m^{(0)}(\vec{x}) + \ldots, \qquad (5.102)$$

where the dots correspond to higher order terms and c_{nm} is small compared to 1. Inserting this ansatz into Eq. (5.101), projecting onto $f_k^{(0)}$ for fixed k and using the normalization $\left\langle f_n^{(0)}, f_m^{(0)} \right\rangle = \delta_{nm}$, we arrive at

$$0 = -\Omega^2 - i\Omega \left\langle f_k^{(0)}, \hat{X}(\Omega) f_k^{(0)} \right\rangle + \omega_k^2. \tag{5.103}$$

Here, we neglected the contribution from terms $c_{nm} \left\langle f_k^{(0)}, \hat{X}(\Omega) f_m^{(0)} \right\rangle$ with $n \neq k$. They correspond to higher orders in perturbation theory. The solutions $\{\Omega_k\}$ of Eq. (5.103) correspond to the eigenfrequencies of the damped system and will be denoted by a capital letter as compared to the ideal case. To zeroth order in perturbation theory we have $\Omega_n = \omega_n$ and therefore we can replace Eq. (5.103) to first order by

$$0 = -\Omega^2 - i\Omega \left\langle f_n^{(0)}, \hat{X}(\omega_n) f_n^{(0)} \right\rangle + \omega_n^2. \tag{5.104}$$

Comparing this result with Eq. (5.20) for the classical harmonic oscillator we deduce

$$\Gamma_n \simeq \left\langle f_n^{(0)}, \hat{X}(\omega_n) f_n^{(0)} \right\rangle \tag{5.105}$$

for the damping constant of the (weakly damped) oscillatory mode with frequency

$$\Omega_n \simeq \omega_n \left(1 - \frac{\Gamma_n^2}{8\omega_n^2} \right) - \frac{i\Gamma_n}{2}. \tag{5.106}$$

5.3.4 Response function from dissipative hydrodynamics

In this section we derive a formula for the response function of a trapped quantum gas including dissipative effects. Combining Eqs. (5.61) and (5.95) we obtain the equation of motion

$$\begin{aligned} &\left[-\Omega^2 - i\Omega \begin{pmatrix} X_1(\Omega) & X_2(\Omega) \\ X_3(\Omega) & X_4(\Omega) \end{pmatrix} + \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} \right] \begin{pmatrix} \delta\mu(\vec{x}, \Omega) \\ \delta T(\vec{x}, \Omega) \end{pmatrix} \\ &= - \begin{pmatrix} E_5 & E_6 \\ E_7 & E_8 \end{pmatrix} \begin{pmatrix} \delta V_{\text{ext}}(\vec{x}, \Omega) \\ \delta V_{\text{ext}}(\vec{x}, \Omega) \end{pmatrix} \end{aligned} \tag{5.107}$$

for $\delta\mu(\vec{x}, t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} e^{-i\Omega t} \delta\mu(\vec{x}, \Omega)$ and the corresponding Fourier transforms of $\delta T(\vec{x}, t)$ and $\delta V_{\text{ext}}(\vec{x}, t)$.

From the above considerations we know that the operator $(E_1\ E_2,\ E_3\ E_4)$ is self-adjoint with respect to the scalar product defined in Eq. (5.64). Its eigenfunctions shall be Φ_n (compare Eq. (5.66)) and the corresponding eigenvalues are denoted by lowercase letters ω_n^2. From Eq. (5.63) we see that $(E_5\ E_6,\ E_7\ E_8)$ is also self-adjoint with respect to the chosen scalar product. We write

$$\Gamma_n = \left\langle \Phi_n, \begin{pmatrix} X_1(\omega_n) & X_2(\omega_n) \\ X_3(\omega_n) & X_4(\omega_n) \end{pmatrix} \Phi_n \right\rangle \tag{5.108}$$

for the damping constants of collective oscillations with frequencies $\{\Omega_n\}$. The uppercase letter indicates that these are the eigenfrequencies of the full system including dissipative effects. They are given by the poles of the response function. We have seen in Eq. (5.106) that there is a close relation between Ω_n and ω_n. We project Eq. (5.107) onto Φ_n and approximate the matrix element involving $\hat{X}(\Omega)$ by Eq. (5.108) as we did in Eq. (5.104). This yields

$$\begin{aligned} &(-\Omega^2 - i\Omega\Gamma_n + \omega_n^2)\left\langle \Phi_n, \begin{pmatrix} \delta\mu(\Omega) \\ \delta T(\Omega) \end{pmatrix} \right\rangle \\ &= -\left\langle \begin{pmatrix} E_5 & E_6 \\ E_7 & E_8 \end{pmatrix} \Phi_n, \begin{pmatrix} \delta V_{\text{ext}}(\Omega) \\ \delta V_{\text{ext}}(\Omega) \end{pmatrix} \right\rangle . \end{aligned} \tag{5.109}$$

We can now define the response function analogous to the zero temperature case, Eq. (5.45), by a computation of the average rate of dissipated energy. Assuming a monochromatic perturbation

$$\delta V_{\text{ext}}(\vec{x}, t) = \frac{1}{2}\left(H(\vec{x})e^{-i\Omega_0 t} + H^*(\vec{x})e^{i\Omega_0 t}\right), \tag{5.110}$$

we consider the rate of dissipated energy

$$\dot{E}(t) = \int_{\vec{x}} \delta n(\vec{x}, t)\partial_t \delta V_{\text{ext}}(\vec{x}, t). \tag{5.111}$$

This can be written as

$$\dot{E}(t) = \left\langle \begin{pmatrix} \delta\mu(t) \\ \delta T(t) \end{pmatrix}, \begin{pmatrix} \partial_t \delta V_{\text{ext}}(t) \\ 0 \end{pmatrix} \right\rangle . \tag{5.112}$$

Inserting the corresponding Fourier decompositions, i.e.

$$\delta\mu(\vec{x}, t) = \frac{1}{2}\left(\delta\mu(\vec{x}, \Omega_0)e^{-i\Omega_0 t} + \delta\mu^*(\vec{x}, \Omega_0)e^{i\Omega_0 t}\right), \tag{5.113}$$

for the chemical potential, analogous for $\delta T(\vec{x}, t)$ and $\delta V_{\text{ext}}(\vec{x}, t)$, and averaging over a small period of time, e.g. $2\pi/\Omega_0$, we arrive at the average rate of dissipated energy

$$Q = \langle \dot{E} \rangle = \frac{\Omega_0}{2} \text{Im} \left\langle \begin{pmatrix} \delta\mu(\Omega_0) \\ \delta T(\Omega_0) \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix} \right\rangle . \tag{5.114}$$

Now the equation of motion (5.109) can be applied to find the response function. Indeed, because of $\sum_n |\Phi_n\rangle\langle\Phi_n| = \mathbb{I}$ we have

$$Q = \frac{\Omega_0}{2} \text{Im} \left\{ \sum_n \frac{1}{m_n} \frac{1}{-\Omega_0^2 - i\Omega_0\Gamma_n + \omega_n^2} \right\}, \tag{5.115}$$

with response coefficients

$$\frac{1}{m_n} = \left\langle \begin{pmatrix} H \\ 0 \end{pmatrix}, \Phi_n \right\rangle \left\langle \begin{pmatrix} E_5 & E_6 \\ E_7 & E_8 \end{pmatrix} \Phi_n, \begin{pmatrix} H \\ H \end{pmatrix} \right\rangle . \tag{5.116}$$

The operator ($E_5\ E_6,\ E_7\ E_8$) is given in Eq. (5.63). At $T = 0$ this reduces to Eq. (5.46).

We conclude that the response function of the trapped quantum gas can be defined as

$$\chi(\Omega) = \sum_n \frac{1}{m_n} \frac{1}{-\Omega^2 - i\Omega\Gamma_n + \omega_n^2} \tag{5.117}$$

with eigenfrequencies of the unperturbed system ω_n (see Sec. 4), damping constants Γ_n from Eq. (5.108) and response coefficients m_n from Eq. (5.116). All three quantities are observables which can be computed from the knowledge of the equation of state and the transport coefficients. Note that the response coefficients $m_n = m_n[H]$ explicitly depend on the form of the driving force, i.e. they are functionals of $H(\vec{x})$ from Eq. (5.110).

5.4 Excitation spectrum of a homogeneous quantum gas

In this section we derive the response function for vanishing external potential, $V_{\text{ext}} = 0$. All formulas remain valid in this limiting case because we did not use the explicit form of the trapping potential. For the homogeneous case the equilibrium solution corresponds to spatially constant functions, i.e. $n_0 = n$, etc. The transport coefficients are also constant in space. We expect a continuous spectrum of excitations, labeled by the momentum index $\vec{k}$.

We write equation (5.107) for spatial homogeneous coefficient functions of the operators. We have

$$\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix} = \begin{pmatrix} P^{\mu\mu} & P^{\mu T} \\ P^{\mu T} & P^{TT} \end{pmatrix}^{-1} \begin{pmatrix} -n\Delta\cdot & -s\Delta\cdot \\ -s\Delta\cdot & -\frac{s^2}{n_n}\Delta\cdot \end{pmatrix}, \tag{5.118}$$

and

$$\begin{pmatrix} E_5 & E_6 \\ E_7 & E_8 \end{pmatrix} = \begin{pmatrix} P^{\mu\mu} & P^{\mu T} \\ P^{\mu T} & P^{TT} \end{pmatrix}^{-1} \begin{pmatrix} -n\Delta\cdot & 0 \\ 0 & -s\Delta\cdot \end{pmatrix}. \tag{5.119}$$

The operators $\hat{\epsilon}$ from Eqs. (5.91) - (5.94) become

$$\hat{\epsilon}_{1,i} = \left(\frac{4}{3}\eta\partial_i\Delta\cdot, \frac{4}{3}\eta\frac{s}{n_n}\partial_i\Delta\cdot\right), \tag{5.120}$$

$$\hat{\epsilon}_2 = \left(\zeta_2\Delta\cdot, \zeta_2\frac{s}{n_n}\Delta\cdot - \zeta_1\frac{n_s s}{n_n}\Delta\cdot\right), \tag{5.121}$$

$$\hat{\epsilon}_3 = \left(\zeta_1\Delta\cdot, \zeta_1\frac{s}{n_n}\Delta\cdot - \zeta_3\frac{n_s s}{n_n}\Delta\cdot\right), \tag{5.122}$$

$$\hat{\epsilon}_4 = \left(0, \frac{1}{T}\kappa_T\Delta\cdot\right). \tag{5.123}$$

In order to solve Eq. (5.107) we decompose $\delta\mu(\vec{x},\Omega) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}}\delta\mu(\vec{k},\Omega)$ and analogously for $\delta T(\vec{x},\Omega)$ and $\delta V_{\mathrm{ext}}(\vec{x},\Omega)$. For $(Y_1(\Omega)\ Y_2(\Omega),\ Y_3(\Omega)\ Y_4(\Omega))$ from Eq. (5.96) we then find

$$Y_1(k,\Omega) = \frac{k^4}{\Omega^2}\left(\frac{4}{3}\eta + \zeta_2\right), \tag{5.124}$$

$$Y_2(k,\Omega) = Y_3(k,\Omega) = \frac{k^4}{\Omega^2}\frac{s}{n_n}\left(\frac{4}{3}\eta + \zeta_2 - n_s\zeta_1\right), \tag{5.125}$$

$$Y_4(k,\Omega) = k^2\frac{\kappa_T}{T} + \frac{k^4}{\Omega^2}\frac{s^2}{n_n^2}\left(\frac{4}{3}\eta + \zeta_2 - 2n_s\zeta_1 + n_s^2\zeta_3\right). \tag{5.126}$$

The differential equation (5.107) becomes

$$\begin{pmatrix} a_1(k,\Omega) & a_2(k,\Omega) \\ a_3(k,\Omega) & a_4(k,\Omega) \end{pmatrix} \begin{pmatrix} \delta\mu(\vec{k},\Omega) \\ \delta T(\vec{k},\Omega) \end{pmatrix} = -\begin{pmatrix} nk^2\delta V_{\mathrm{ext}}(\vec{k},\Omega) \\ sk^2\delta V_{\mathrm{ext}}(\vec{k},\Omega) \end{pmatrix} \tag{5.127}$$

with

$$\begin{aligned} a(k,\Omega) = \begin{pmatrix} a_1(k,\Omega) & a_2(k,\Omega) \\ a_3(k,\Omega) & a_4(k,\Omega) \end{pmatrix} &= \begin{pmatrix} -\Omega^2 & 0 \\ 0 & -\Omega^2 \end{pmatrix} \\ &+ \begin{pmatrix} P^{\mu\mu} & P^{\mu T} \\ P^{\mu T} & P^{TT} \end{pmatrix}^{-1} \begin{pmatrix} nk^2 - i\Omega Y_1 & sk^2 - i\Omega Y_2 \\ sk^2 - i\Omega Y_3 & \frac{s^2}{n_n}k^2 - i\Omega Y_4 \end{pmatrix}. \end{aligned} \tag{5.128}$$

We find for the response functions $\chi^{(\mu)}(\vec{k},\Omega)$ and $\chi^{(T)}(\vec{k},\Omega)$ from Eqs. (5.48) and (5.49) the multiplicative representation

$$\delta\mu(\vec{k},\Omega) = -\chi^{(\mu)}(k,\Omega)\delta V_{\mathrm{ext}}(\vec{k},\Omega), \tag{5.129}$$

$$\delta T(\vec{k},\Omega) = -\chi^{(T)}(k,\Omega)\delta V_{\mathrm{ext}}(\vec{k},\Omega). \tag{5.130}$$

Thus we have

$$\begin{pmatrix} \chi^{(\mu)}(k,\Omega) \\ \chi^{(T)}(k,\Omega) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{-1} \begin{pmatrix} nk^2 \\ sk^2 \end{pmatrix}. \tag{5.131}$$

The density response function is given by $\chi^{(n)} = P^{\mu\mu}\chi^{(\mu)} + P^{\mu T}\chi^{(T)}$.

The spectrum of collective excitations of the homogeneous system is given by the poles of the response functions $\chi^{(\mu)}$ and $\chi^{(T)}$. Obviously, the corresponding values of k and $\Omega(k)$ are found from the roots of $\det(a) = a_1a_4 - a_2a_3$. Since the Hessian matrix (P^{ij}) of $P(\mu,T)$ is positive definite, we can equivalently solve $0 = \det(P^{ij})\det(a) = \det(P^{ij}a)$, which simplifies the calculation because of the particular form of a in Eq. (5.128). This yields

$$0 = \det \begin{pmatrix} -P^{\mu\mu}\Omega^2 + nk^2 - i\Omega Y_1 & -P^{\mu T}\Omega^2 + sk^2 - i\Omega Y_2 \\ -P^{\mu T}\Omega^2 + sk^2 - i\Omega Y_2 & -P^{TT}\Omega^2 + \frac{s^2}{n_n}k^2 - i\Omega Y_4 \end{pmatrix}_{\Omega=\Omega(k)}. \tag{5.132}$$

Neglecting the contribution of dissipative terms we find $\Omega(k) = ck$ from Eq. (5.132), where c is a solution of

$$0 = c^4(P^{\mu\mu}P^{TT} - (P^{\mu T})^2) + c^2\left(2P^{\mu T}s - P^{TT}n - \frac{P^{\mu\mu}s^2}{n_n}\right) + s^2\left(1 - \frac{n}{n_n}\right). \tag{5.133}$$

This is Eq. (B.13) and thus yields the velocities of first and second sound, c_1 and c_2, respectively. Including dissipative terms to first order the dominant effects will consist in damped sound propagation and diffusion. The low energy spectrum is therefore expected to consist in branches of the type

$$\Omega(k) = (c - i\alpha)k, \tag{5.134}$$

$$\Omega(k) = -iDk^2 \tag{5.135}$$

with positive real constants c, α, and D. These correspond to either damped sound modes with velocity of sound c or diffusive modes with diffusion constant D. Explicit formulas can be found in Ref. [45].

6 Conclusions and outlook

The aim of this thesis was to identify precision observables for ultracold quantum gases. Their measurement can serve as a stringent test of first principle calculations. For this purpose we considered quantum gases in thermodynamic equilibrium or slightly out of equilibrium and found that the equation of state, collective oscillation frequencies and the response function to an external driving force provide such observables. We summarize here our findings and show perspectives for future research.

A detailed discussion of the equation of state of a Bose gas showed the effects of interaction and thermal fluctuations on thermodynamic quantities. The pressure as a function of chemical potential and temperature receives higher order contributions beyond mean-field already in the perturbative regime of small gas parameter. To observe effects of strong coupling in a Bose gas may be difficult. Indeed, for large values of the gas parameter one expects the system to become unstable due to three-body losses. Therefore an experimental realization will be restricted to short time-scales and thus thermal equilibrium is not achieved. However, this issue is not yet fully settled. In Ref. [67] Salomon et al. report on measurements of the strongly interacting Bose gas. More promising than elementary bosons are ultracold fermions. Using a two-component Fermi gas, the interaction can be tuned through all values of the scattering length via a Feshbach resonance in the BEC-BCS crossover. In particular, in the unitary limit of infinite scattering length the system becomes strongly coupled and non-perturbative effects are dominant. Moreover, in the imbalanced case, a difference in the chemical potentials of the two fermion species is accompanied by interesting effects of phase separation [3]. Since in a trapping potential many values of the chemical potential are tested, the measurement of density profiles always reveals information on the phase structure of the system. Thus, predictions on (quantum) phase diagrams can immediately be tested. The location of the phase boundaries and the behavior of thermodynamic quantities at the critical values has to be provided from sophisticated techniques beyond mean-field. We conclude that measurements of the equation of state open a rich field for connecting theory and experiment.

From the equation of state one can compute the frequencies of collective oscillations of a trapped quantum gas in the hydrodynamic regime. We have seen that these frequencies are found as the eigenvalues of a self-adjoint partial differential operator, which depends solely on the equation of state and the form of the trapping potential. For simplicity we considered a spherical symmetric harmonic trap in our numerics, which relies on the discretization of the differential operator. It is possible to generalize this implementation for an arbitrary external potential. The study of the oscillations at zero temperature is particularly simple because the normal fluid fraction vanishes and the system can be described by the superfluid degrees of freedom only. Moreover, due to the absence of thermal fluctuations, the frequencies can be computed from the oscillations of the chemical potential. By a comparison with exact results for a polytropic equation of state, i.e. of the form $P(\mu) \propto \mu^{\alpha+1}$, we tested the precision of our numerics. The agreement of the numerical values to the exact eigenfrequencies was found to be valid for at least five digits and the (experimental more interesting) lower lying frequencies agreed even better. Using a finer grid in the discretization scheme the precision can be improved and we therefore expect the errors in the frequencies to be solely systematic. Indeed, the conditions for applying the hydrodynamic description might not be satisfied or the precision of the equation of state is insufficient.

The successive improvement of the equation of state of a three-dimensional dilute Bose gas at zero temperature using perturbation theory revealed an expected shift of the frequencies which is proportional to the square root of the gas parameter. With an equation of state obtained from non-perturbative Functional Renormalization Group (FRG) we could reproduce this shift. This proves the accuracy of the microscopic method in this case. For larger values of the gas parameter we found a deviation from the simple scaling behavior of the perturbative regime. In this region of parameters the result becomes non-universal because the dependence on the ultraviolet cutoff has to be taken into account. Indeed, the non-relativistic Bose gas is described by an effective non-renormalizable quantum field theory and therefore has a natural microscopic cutoff scale. For distances below this scale (or equivalently too high energies) the effective model becomes invalid. This cutoff scale also becomes important in the analysis of the two-dimensional dilute Bose gas, because the coupling constant g is dimensionless in this case and therefore the dimensionless gas parameter built out of density or chemical potential necessarily depends on the cutoff. With an equation of state provided by FRG we computed the frequencies of the Bose gas in two dimensions at zero temperature. We found small, gas parameter-independent deviations from the mean-field result at small coupling, but significant differences for larger values of the coupling constant in the interval $1 \lesssim g \lesssim 1.6$. Moreover, the high coupling frequencies also show a non-trivial dependence on the gas parameter, i.e. on the chemical potential in the center of the trap and the cutoff. In experiments the cutoff of the two-dimensional Bose gas can be associated with the length scale of tight confinement of the effectively two-dimensional trap.

A solution of the eigenvalue problem of collective oscillations of a Bose gas in three dimensions at nonvanishing temperature has been given with the equation of state from Lee–Yang theory. The latter describes hard-core bosons, which constitute a good model for cold weakly interacting Bose atoms. Indeed, since interactions in this regime can be described solely by the s-wave scattering length, a universality argument proves this correspondence. Lee–Yang theory depends on a saddle-point approximation of the quantum partition function and therefore has to be classified as a mean-field approach. However, it serves as a solid testing ground for our considerations. We do not aim at a proof of the validity of Lee–Yang theory but on identifying characteristics of oscillations at nonzero temperature. We explicitly showed how the zero temperature limit is reached and also found an additional branch in the zero temperature spectrum of collective modes, which corresponds to oscillations of the thermal component. The latter cannot be found within a description of purely superfluid motion. However, the experimental verification of this result might be difficult because at such low temperatures only a few atoms are left in the normal phase. The oscillation frequencies as a function of temperature have been computed and we found that the frequencies of the oscillations of the condensate vanish from the spectrum at the critical temperature. The two-fluid model is still valid above the critical temperature where it describes the gas in its symmetric phase. This formal property is reflected in our result on the collective modes of the normal component, which continue smoothly across the phase transition. The ratio T/T_c can be changed without heating up the system, because $T_c = T_c(N, a, \omega_0)$ depends on particle number, interaction strength and trap parameters like the trapping frequency ω_0 in a harmonic trap. Varying these parameters adiabatically, T/T_c can be manipulated in a systematic way.

The non-trivial dependence of the frequencies of collective modes on the system parameters may be used for precision measurements of these parameters. In particular, in well understood regimes the precise determination of oscillation frequencies could provide information on temperature, chemical potential, particle number or interaction strength. For example, measuring a few frequencies the comparison of the result with a theoretical temperature-dependent spectrum allows for thermometry. If the system possesses a small expansion parameter the frequencies will show a simple scaling behavior on this parameter in a certain interval. We showed this for the three-dimensional dilute Bose gas, where the frequency shift was proportional to the square root of the gas parameter. However, the scaling statement is quite general and holds for any system. Thus, collecting data in a double-logarithmic plot one can read off the scaling exponent and calibrate the measurement of frequencies for measurements of the small parameter. We conclude that the precise measurement of collective frequencies can be used for stringent tests of many-body theory and also for improvements of the determination and understanding of system parameters.

In order to measure several frequencies individually without destroying the system, we suggest to use in situ response techniques. The response function of a trapped quantum gas thus arises naturally as an observable in our analysis. It contains information on the frequencies and damping constants of collective excitations and also the amplitude of response of the modes to a perturbation of the external potential. Of course, the frequencies and damping constants of the lowest modes can also be measured via an excitation-and-release experiment and the fit of an exponentially decaying sinus function to the density profiles as a function of time. The computation of the response function in the nearly ideal case can be achieved with the same techniques that were used for the eigenvalue problem of collective modes. Indeed, the eigenfunctions of collective modes are found from the discretization of the ideal two-fluid hydrodynamic equations. With this result one can calculate the damping constants and response coefficients as expectation values of certain operators. For the damping constants knowledge of five transport coefficients as a function of chemical potential and temperature is required. We have derived the corresponding formulas under the condition of small transport coefficients. This is a valid assumption for cold gases where already ideal hydrodynamics provides good explanations of experimental data.

The calculation of the response function of a two-component Fermi gas will be an important step towards precision tests of microscopic methods. As explained above, this system is promising both from an experimental and theoretical point of view. In particular the situation simplifies in the unitary limit. There the measured equation of state has a simple structure (though difficult to calculate) and two of the five transport coefficients vanish due to conformal symmetry. Using simple models or Kubo formulas for the transport coefficients the damping constants are accessible. It is an open question whether there are systems in nature with a ratio of shear viscosity over entropy density being smaller than $\hbar/4\pi k_B$, see Ref. [46] for details.

To summarize, we have shown that it is possible to relate first principle calculations to experiments through the knowledge of the equation of state and transport coefficients. The corresponding formulas hold for a large class of systems. Particular emphasis was on the dilute Bose gas which served as a generic example. Extensions of this work seem most interesting in lower dimensions or fermionic systems.

A Natural units

Physical systems can be described in arbitrary (consistent) units. However, one choice or the other will be more favorable. In order to interpret and express experimental results standardized units (SI) provide an appropriate base. For theoretical calculations natural units are more convenient which do not depend on external scales but only on intrinsic parameters. For example when describing a system of Helium atoms it is useful to express masses in terms of the Helium mass m_0. The mass $M = \alpha m_0$ of any object will then be fully characterized by the numerical prefactor α. Therefore, one can set $m_0 = 1$ in all formulas with the additional information provided that masses are measured in units of the Helium mass. What are sensible natural units? For high energy physics it is convenient to set $\hbar = k_B = c = 1$. For non-relativistic systems, the velocity of light c does not appear in any relations and it is therefore more appropriate to set $m = 1$ or $m_0 = 2m = 1$. The former choice is made in this thesis because it simplifies the hydrodynamic description through the equality of mass density and particle number density. The latter system of units is convenient for first principle calculations of thermodynamic quantities because there $2m$ appears in the Lagrangian or Hamiltonian.

We take a set of physical quantities $\{A\} = \{n, \mu, a, \dots\}$ and express them in terms of dimensionless numbers $\hat{A}$ which are denoted by a hat and some powers of $\hbar$, k_B, m_0 and ℓ_0. No further dimensionless quantities than the four latter are required. While $\hbar$ and k_B have their usual meaning, the choice of m_0 and ℓ_0 is arbitrary at this moment. If we take a gas of cold atoms with atom mass m and scattering length a we get

$$m = \hat{m} \cdot m_0, \tag{A.1}$$

$$a = \hat{a} \cdot \ell_0. \tag{A.2}$$

One particular choice of m_0 and ℓ_0 could be $m_0 = m$ and $\ell_0 = a$. We than trivially arrive at

$$\hat{m} = 1, \tag{A.3}$$

$$\hat{a} = 1. \tag{A.4}$$

In order to express the equation of state in dimensionless numbers, we give explicit formulas for time t, energy E, chemical potential μ, temperature T, density n, pressure P, entropy density s, $P^{\mu\mu} = \partial n/\partial\mu$, $P^{TT} = \partial s/\partial T$, $P^{\mu T} = \partial n/\partial T$, specific heat $C = T\partial S/\partial T$ and compressibility $\kappa = -V^{-1}\partial V/\partial P$. Note that it is not of importance for the present discussion, whether we are considering C_P or

C_V, respectively. The same holds for κ_T and κ_S. We find

$$t = \hat{t} \cdot \frac{\ell_0^2 m_0}{\hbar}, \qquad E = \hat{E} \cdot \hbar t_0^{-1} = \hat{E} \cdot \frac{\hbar^2}{\ell_0^2 m_0}, \tag{A.5}$$

$$\mu = \hat{\mu} \cdot \frac{\hbar^2}{\ell_0^2 m_0}, \qquad T = \hat{T} \cdot \frac{\hbar^2}{\ell_0^2 m_0 k_B}, \tag{A.6}$$

$$n = \hat{n} \cdot \frac{1}{\ell_0^d}, \qquad P = \hat{P} \cdot \frac{\hbar^2}{\ell_0^{d+2} m_0}, \tag{A.7}$$

$$s = \hat{s} \cdot \frac{k_B}{\ell_0^d}, \qquad P^{\mu\mu} = \hat{P^{\mu\mu}} \cdot \frac{m_0}{\hbar^2 \ell_0^{d-2}}, \tag{A.8}$$

$$P^{TT} = \hat{P^{TT}} \cdot \frac{k_B^2 m_0}{\hbar^2 \ell_0^{d-2}}, \qquad P^{\mu T} = \hat{P^{\mu T}} \cdot \frac{k_B m_0}{\hbar^2 \ell_0^{d-2}}, \tag{A.9}$$

$$C = \hat{C} \cdot k_B, \qquad \kappa = \hat{\kappa} \cdot \frac{\ell_0^{d+2} m_0}{\hbar^2}. \tag{A.10}$$

We want to compare two particular systems of units. The first one will be denoted by (1) and corresponds to $\hbar = k_B = 1$, $m_0 = m_{(1)}$ and $\ell_0 = \ell_{(1)}$, while the second one, denoted by (2), also assumes $\hbar = k_B = 1$ but in general $m_0 = m_{(2)}$ and $\ell_0 = \ell_{(2)}$ will be different. We ask how to convert these systems into each other. More precisely, given a set of data $\{\hat{A}_{(1)}\}$ we want to have a closed expression for $\{\hat{A}_{(2)}\}$. The translation rules can be derived by expressing a physical quantity A in both systems under consideration, $A = \hat{A}_{(1)} A_{0,(1)}$ and $A = \hat{A}_{(2)} A_{0,(2)}$, and then equating both expressions. For example we get for the chemical potential

$$\frac{1}{m_{(1)} \ell_{(1)}^2} \cdot \hat{\mu}_{(1)} = \mu = \frac{1}{m_{(2)} \ell_{(2)}^2} \cdot \hat{\mu}_{(2)}, \tag{A.11}$$

which implies

$$\hat{\mu}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right)^{-1} \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{-2} \cdot \hat{\mu}_{(1)}. \tag{A.12}$$

In a similar fashion we arrive at

$$\hat{a}_{(2)} = \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right) \cdot \hat{a}_{(1)}, \qquad \hat{t}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right) \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{2} \cdot \hat{t}_{(1)}, \tag{A.13}$$

$$\hat{E}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right)^{-1} \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{-2} \cdot \hat{E}_{(1)}, \qquad \hat{T}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right)^{-1} \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{-2} \cdot \hat{T}_{(1)}, \tag{A.14}$$

$$\hat{n}_{(2)} = \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{-d} \cdot \hat{n}_{(1)}, \qquad \hat{P}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right)^{-1} \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{-(d+2)} \cdot \hat{P}_{(1)}, \tag{A.15}$$

$$\hat{s}_{(2)} = \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{-d} \cdot \hat{s}_{(1)}, \qquad P^{\widehat{\mu\mu}}{}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right) \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{2-d} \cdot P^{\widehat{\mu\mu}}{}_{(1)}, \tag{A.16}$$

$$P^{\widehat{TT}}{}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right) \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{2-d} \cdot P^{\widehat{TT}}{}_{(1)}, \quad P^{\widehat{\mu T}}{}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right) \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{2-d} \cdot P^{\widehat{\mu T}}{}_{(1)}, \tag{A.17}$$

$$\hat{C}_{(2)} = \hat{C}_{(1)}, \qquad \hat{\kappa}_{(2)} = \left(\frac{m_{(1)}}{m_{(2)}}\right) \left(\frac{\ell_{(1)}}{\ell_{(2)}}\right)^{d+2} \cdot \hat{\kappa}_{(1)}. \tag{A.18}$$

It has to be emphasized that m_0 (or correspondingly $m_{(i)}$) is the quantity which is set to 1 in the end. Thus, if the mass of atoms is m, then typical calculations in many-body theory, where $2m$ is set to one, correspond to $m_{(2)} = 2m = 1$ in the above formulas.

B Thermodynamic relations

This appendix provides some formulas related to the thermodynamic properties of superfluid systems. We also derive expressions for the first and second velocities of sound which emerge as solutions of the two-fluid hydrodynamic equations in homogeneous space.

We define specific heat per particle at constant volume (C_V/N) or pressure

(C_P/N) and isothermal (κ_T) and adiabatic compressibility (κ_S) by

$$\frac{C_V}{N} = \frac{T}{N}\left(\frac{\partial S}{\partial T}\right)_{V,N} = \frac{T}{N/V}\left(\frac{\partial (S/V)}{\partial T}\right)_n = \frac{T}{n}\left(\frac{\partial s}{\partial T}\right)_n, \tag{B.1}$$

$$\frac{C_P}{N} = \frac{T}{N}\left(\frac{\partial S}{\partial T}\right)_{P,N} = T\left(\frac{\partial (S/N)}{\partial T}\right)_P = T\left(\frac{\partial (s/n)}{\partial T}\right)_P, \tag{B.2}$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = -\frac{N}{V}\left(\frac{\partial (1/n)}{\partial P}\right)_T = \frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_T, \tag{B.3}$$

$$\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N} = -\frac{N}{V}\left(\frac{\partial (1/n)}{\partial P}\right)_{S/N} = \frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_{s/n}. \tag{B.4}$$

Our aim is to relate these quantities to $P(\mu, T)$. The Jacobian

$$\frac{\partial (a,b)}{\partial (c,d)} = \left(\frac{\partial a}{\partial c}\right)_d \left(\frac{\partial b}{\partial d}\right)_c - \left(\frac{\partial a}{\partial d}\right)_c \left(\frac{\partial b}{\partial c}\right)_d \tag{B.5}$$

will play an important role.

We start with the heat capacities. At constant volume we obtain

$$\frac{C_V}{N} = \frac{T}{n}\left(\frac{\partial s}{\partial T}\right)_n = \frac{T}{n}\frac{\partial (s,n)}{\partial (T,\mu)}\frac{\partial (T,\mu)}{\partial (T,n)} = \frac{T}{P^\mu P^{\mu\mu}}\det P. \tag{B.6}$$

At constant pressure instead we find

$$\begin{aligned}\frac{C_P}{N} &= T\left(\frac{\partial (s/n)}{\partial T}\right)_P = T\left\{\frac{1}{n}\left(\frac{\partial s}{\partial T}\right)_T - \frac{s}{n^2}\left(\frac{\partial n}{\partial T}\right)_P\right\} \\ &= \frac{T}{(P^\mu)^3}(P^{TT}(P^\mu)^2 - 2P^\mu P^T P^{\mu T} + P^{\mu\mu}(P^T)^2).\end{aligned} \tag{B.7}$$

We have for the compressibility at constant temperature

$$\kappa_T = \frac{1}{n}\left(\frac{\partial n}{\partial P}\right)_T = \frac{1}{n}\frac{\partial (n,T)}{\partial (\mu,T)}\frac{\partial (\mu,T)}{\partial (P,T)} = \frac{P^{\mu\mu}}{(P^\mu)^2}. \tag{B.8}$$

The adiabatic compressibility can now be found with the help of the thermodynamic relation $C_V/C_P = \kappa_S/\kappa_T$ and is given by

$$\kappa_S = \frac{\det P}{P^{TT}(P^\mu)^2 - 2P^\mu P^T P^{\mu T} + P^{\mu\mu}(P^T)^2}. \tag{B.9}$$

The thermodynamic quantities derived so far do not depend on the normal fluid density ρ_n. Observables which possess this feature are the first and second velocities of sound. We assume a homogeneous situation. Hence we do not need to apply local density approximation in Eqs. (4.38) and (4.39). We arrive at

$$\omega^2 \delta n + P^\mu \Delta\delta\mu + P^T \Delta\delta T = 0, \tag{B.10}$$

$$\omega^2 \delta s + P^T \Delta\delta\mu + \frac{(P^T)^2}{n_n} \Delta\delta T = 0. \tag{B.11}$$

Asking for wave-like solutions $\delta\mu(\vec{x})$, $\delta T(\vec{x})$, $\delta n(\vec{x})$, $\delta s(\vec{x}) \propto \exp(i\vec{n}\cdot\vec{x}\omega/c)$ with $\vec{n}^2 = 1$ we find

$$\begin{pmatrix} P^{\mu\mu} - P^T/c^2 & P^{\mu T} - P^T/c^2 \\ P^{\mu T} - P^T/c^2 & P^{TT} - \frac{(P^T)^2}{c^2 n_n} \end{pmatrix} \begin{pmatrix} \delta\mu \\ \delta T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{B.12}$$

implying the necessary condition

$$\begin{aligned} &\left(\frac{P^\mu (P^T)^2}{n_n} - (P^T)^2\right) + c^4 \left(P^{\mu\mu}P^{TT} - (P^{\mu T})^2\right) \\ &+ c^2 \left(2P^{\mu T}P^T - P^{TT}P^\mu - \frac{P^{\mu\mu}(P^T)^2}{n_n}\right) = 0 \end{aligned} \tag{B.13}$$

on possible values of c. Our aim is to show that Eq. (B.13) is equivalent to the more familiar condition

$$\begin{aligned} 0 = &\left(\left(\frac{\partial(s/n)}{\partial T}\right)_P \left(\frac{\partial n}{\partial P}\right)_T - \left(\frac{\partial(s/n)}{\partial P}\right)_T \left(\frac{\partial n}{\partial T}\right)_P\right) c^4 \\ &- \left(\left(\frac{\partial(s/n)}{\partial T}\right)_P + \frac{n_s (s/n)^2}{n_n} \left(\frac{\partial n}{\partial P}\right)_T\right) c^2 + \frac{n_s (s/n)^2}{n_n} \end{aligned} \tag{B.14}$$

given in Ref. [43]. We note that

$$\left(\frac{\partial(s/n)}{\partial T}\right)_P = \frac{1}{n}\left(\frac{\partial s}{\partial T}\right)_P - \frac{s}{n^2}\left(\frac{\partial n}{\partial T}\right)_P, \tag{B.15}$$

$$\left(\frac{\partial(s/n)}{\partial P}\right)_T = \frac{1}{n}\left(\frac{\partial s}{\partial P}\right)_T - \frac{s}{n^2}\left(\frac{\partial n}{\partial P}\right)_T. \tag{B.16}$$

The derivatives with respect to P can be evaluated and yield

$$\left(\frac{\partial s}{\partial P}\right)_T = \frac{\partial(s,T)}{\partial(P,T)} = \frac{\partial(s,T)}{\partial(\mu,T)}\frac{\partial(\mu,T)}{\partial(P,T)} = \frac{P^{\mu T}}{P^\mu}, \tag{B.17}$$

$$\left(\frac{\partial n}{\partial P}\right)_T = \frac{\partial(n,T)}{\partial(\mu,T)}\frac{\partial(\mu,T)}{\partial(P,T)} = \frac{P^{\mu\mu}}{P^\mu}. \tag{B.18}$$

Since $s(T,P)=s(T,\mu(P,T))$ and $n(T,P)=n(T,\mu(T,P))$ we have

$$\begin{aligned}
\left(\frac{\partial s}{\partial T}\right)_P &= \left(\frac{\partial s}{\partial T}\right)_\mu + \left(\frac{\partial s}{\partial \mu}\right)_T \left(\frac{\partial \mu}{\partial T}\right)_P \\
&= \left(\frac{\partial s}{\partial T}\right)_\mu + \left(\frac{\partial s}{\partial \mu}\right)_T \frac{\partial(\mu,P)}{\partial(\mu,T)}\frac{\partial(\mu,T)}{\partial(T,P)} \\
&= P^{TT} - \frac{P^{\mu T}P^T}{P^\mu}, \qquad \text{(B.19)} \\
\left(\frac{\partial n}{\partial T}\right)_P &= \left(\frac{\partial n}{\partial T}\right)_\mu + \left(\frac{\partial n}{\partial \mu}\right)_T \left(\frac{\partial \mu}{\partial T}\right)_P \\
&= P^{\mu T} - \frac{P^{\mu\mu}P^T}{P^\mu}. \qquad \text{(B.20)}
\end{aligned}$$

Inserting these expressions into the coefficients of Eq. (B.14) we get

$$\begin{aligned}
&\left(\frac{\partial(s/n)}{\partial T}\right)_P \left(\frac{\partial n}{\partial P}\right)_T - \left(\frac{\partial(s/n)}{\partial P}\right)_T \left(\frac{\partial n}{\partial T}\right)_P \\
&= \frac{P^{TT}P^{\mu\mu} - (P^{\mu T})^2}{(P^\mu)^2}, \qquad \text{(B.21)} \\
&\left(\frac{\partial(s/n)}{\partial T}\right)_P + \frac{n_s(s/n)^2}{n_n}\left(\frac{\partial n}{\partial P}\right)_T \\
&= \left(\frac{\partial(s/n)}{\partial T}\right)_P + \left(\frac{n}{n_n}-1\right)\frac{s^2}{n^2}\left(\frac{\partial n}{\partial P}\right)_T \\
&= \frac{1}{(P^\mu)^2}\left(-2P^TP^{\mu T} + P^{TT}P^\mu + \frac{(P^T)^2P^{\mu\mu}}{n_n}\right) \qquad \text{(B.22)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{n_s(s/n)^2}{n_n} &= (n/n_n - 1)\frac{s^2}{n^2} \\
&= \frac{1}{(P^\mu)^2}\left(\frac{P^\mu(P^T)^2}{n_n} - (P^T)^2\right). \qquad \text{(B.23)}
\end{aligned}$$

This proves the equivalence of Eqs. (B.13) and (B.14).

The velocities of first and second sound are now found as the roots of Eq. (B.13)). In the normal phase ($T \geq T_c$) we have $n_n = P^\mu$ and Eq. (B.13) reduces to

$$c^2(c^2 - v_S^2) = 0, \qquad \text{(B.24)}$$

where $v_S^2 = (\partial P/\partial n)_{s/n} = (n\kappa_S)^{-1}$ is the adiabatic velocity of sound and κ_S is given by Eq. (B.9). This solution is expected for a one-component fluid. Below the critical temperature we have

$$\begin{aligned} 0 &= c^4 - \left(\frac{1}{n\kappa_S} + \frac{P^{\mu\mu}(P^T)^2}{P^\mu \mathrm{det}P}\left(\frac{P^\mu}{n_n} - 1\right)\right)c^2 + \frac{(P^T)^2}{\mathrm{det}P}\left(\frac{P^\mu}{n_n} - 1\right) \\ &= c^4 - \left(v_S^2 + \frac{Ts^2n_s}{n^2n_n(C_V/N)}\right)c^2 + \frac{Ts^2n_s}{n^2n_n(C_V/N)}v_T^2, \end{aligned} \tag{B.25}$$

where we used Eq. (B.6) for C_V/N and defined the isothermal velocity of sound $v_T^2 = (\partial P/\partial n)_T = P^\mu/P^{\mu\mu}$. The general expression for first and second velocity of sound can be obtained from this equation. A particularly simple solution is provided in the case of very *low temperatures* where $v_S^2 \simeq v_T^2$ can be assumed. We then find

$$\begin{aligned} c_{1,2}^2(T\to 0) &= \frac{v_T^2}{2}\left(1 \pm \sqrt{1 - \frac{4Ts^2n_s}{n^2n_n(C_V/N)v_T^2}}\right) \\ &\simeq \begin{cases} v_T^2, \\ (Ts^2n_s)/(n_nn^2(C_V/N)). \end{cases} \end{aligned} \tag{B.26}$$

We now define an analogue to κ_S in the symmetry broken phase by

$$\tilde{\kappa}_S := \frac{\mathrm{det}P}{P^{TT}(P^\mu)^2 - 2P^TP^\mu P^{\mu T} + P^{\mu\mu}(P^T)^2 \cdot \frac{P^\mu}{n_n}}. \tag{B.27}$$

Obviously, $\tilde{\kappa}_S \to \kappa_S$ for $T \to T_c$. We observe that the combination $1/(n\tilde{\kappa}_S)$ enters Eq. (B.25) as the prefactor of c^2 and thus has to be non-negative. Therefore we can define a quantity analogue to the squared adiabatic velocity of sound, $\tilde{v}_S^2 = 1/(n\tilde{\kappa}_S)$. From Eq. (B.27) we obtain

$$\tilde{v}_S^2 := \frac{1}{n\tilde{\kappa}_S} = \frac{P^{TT}P^\mu - 2P^TP^{\mu T} + P^{\mu\mu}\tilde{n}}{\mathrm{det}P} = v_S^2 + \frac{Ts^2n_s}{n^2n_n(C_V/N)}, \tag{B.28}$$

with $\tilde{n} = s^2/n_n$. Eq. (B.25) can now be written as

$$0 = c^4 - \tilde{v}_S^2c^2 + (\tilde{v}_S^2 - v_S^2)v_T^2. \tag{B.29}$$

With the relations derived so far, we can draw conclusions on the thermodynamic meaning of some of the coefficient functions of our eigenvalue problem form Sec. 4 in Eqs. (4.51) - (4.53). From Eq. (B.20) we find for c_1 from Eq. (4.52)

$$c_1 = \left[\frac{n}{\mathrm{det}P}\left(\frac{\partial n}{\partial T}\right)_P\right]_0. \tag{B.30}$$

The thermodynamic derivative on the right hand side of this equation is always non-positive, $(\partial n/\partial T)_P \leq 0$. This can be understood as follows. If we increase the temperature there will be more collisions between the particles. But if the pressure shall be kept constant we have to dilute the system and thus decrease the density. Analogously, a_1 is related to Eq. (B.19) via

$$a_1 = \left[\frac{n}{\det P}\left(\frac{\partial s}{\partial T}\right)_P\right]_0 . \tag{B.31}$$

The sum of a_1 and d_1 from Eqs. (4.51) and (4.53), respectively, is given by

$$a_1 + d_1 = -\tilde{v}_{S,0}^2, \tag{B.32}$$

with $\tilde{v}_S^2$ from Eq. (B.28).

C Visualization of Lee–Yang theory for Bose gas

The following figures illustrate the Lee–Yang equation of state of a weakly interacting Bose gas. Besides the pressure $P(\mu, T)$ and its derivatives with respect to μ and T, we also show the superfluid density. The phase diagram can be obtained from the bird's eye view on the pictures. Furthermore, we show thermodynamic quantities, which are composed of the derivatives of $P(\mu, T)$. They are the heat capacity per particle number at fixed pressure or volume, C_P/N and C_V/N, and the isothermal and adiabatic compressibility, κ_T and κ_S, respectively.

We show dimensionless thermodynamic functions such that all quantities are given in units of the scattering length a, which sets a scale for the homogeneous system, see App. A. Temperature and chemical potential are then found to be $\hat{T} = Ta^2mk_B/\hbar^2$ and $\hat{\mu} = \mu a^2 m/\hbar^2$, respectively. Note that for a trapped gas the characteristic length of the external potential, e.g. the oscillator length, enters as an additional scale. Furthermore, beyond perturbation theory, the inverse momentum cutoff Λ^{-1} provides a further unit of length. Such a cutoff is necessary in the treatment of the non-renormalizable three-dimensional quantum field theory of a Bose gas with contact interactions. This issue will not be respected in the following.

For reasons of better visibility, some figures are rotated in the $(\hat{T}, \hat{\mu})$-plane.

When interpreting the results one should keep in mind that larger, positive values of the chemical potential correspond to the superfluid phase, while smaller and in particular negative values of μ describe the normal phase. The critical values of chemical potential and temperature are related by Eq. (3.50),

$$\hat{T}_c = \frac{T_c a^2 m k_B}{\hbar^2} = 2\pi \left(\frac{\mu a^2 m/\hbar^2}{4\pi\zeta(3/2)} \right)^{3/2} = 2\pi \left(\frac{\hat{\mu}}{4\pi\zeta(3/2)} \right)^{3/2} . \tag{C.1}$$

The phase boundary appears to be scraggly in some of the three-dimensional plots. This is due to the interpolation from two-dimensional slices which precedes the visualizations and not a short-coming of the calculation of the equation of state.

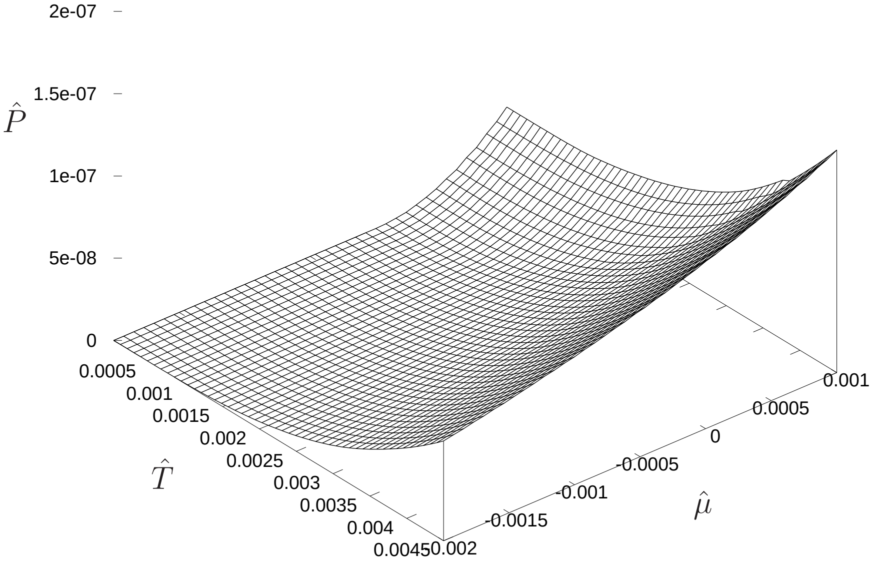

Figure C.1: Pressure $\hat{P} = Pa^5m/\hbar^2$ as a function of $\hat{T}$ and $\hat{\mu}$

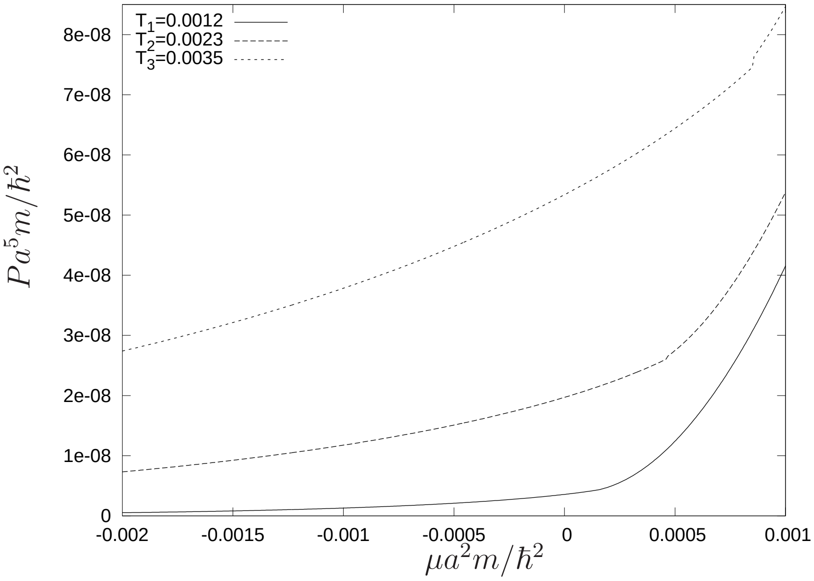

Figure C.2: $\hat{P}(\hat{\mu})$ for three distinct values of $\hat{T}$

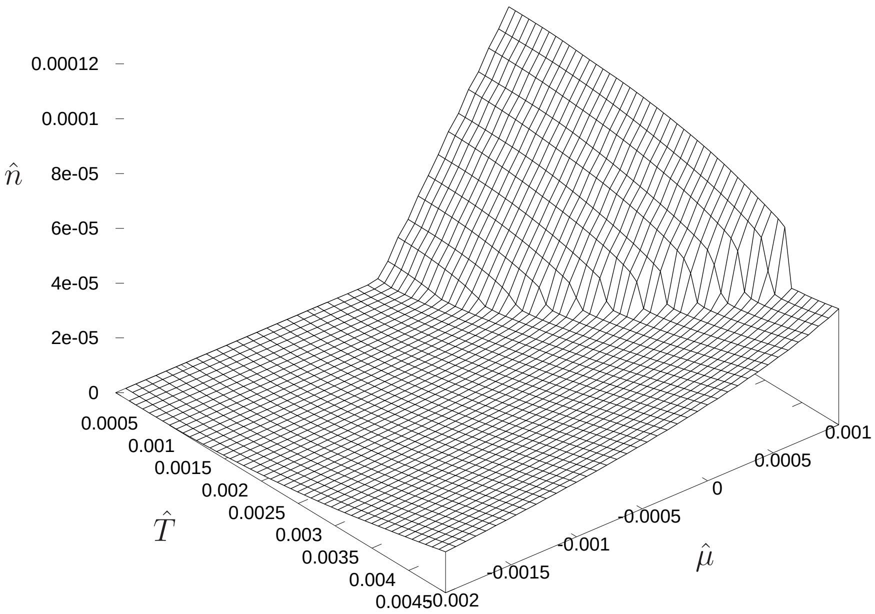

Figure C.3: Density $\hat{n} = na^3$ as a function of $\hat{T}$ and $\hat{\mu}$

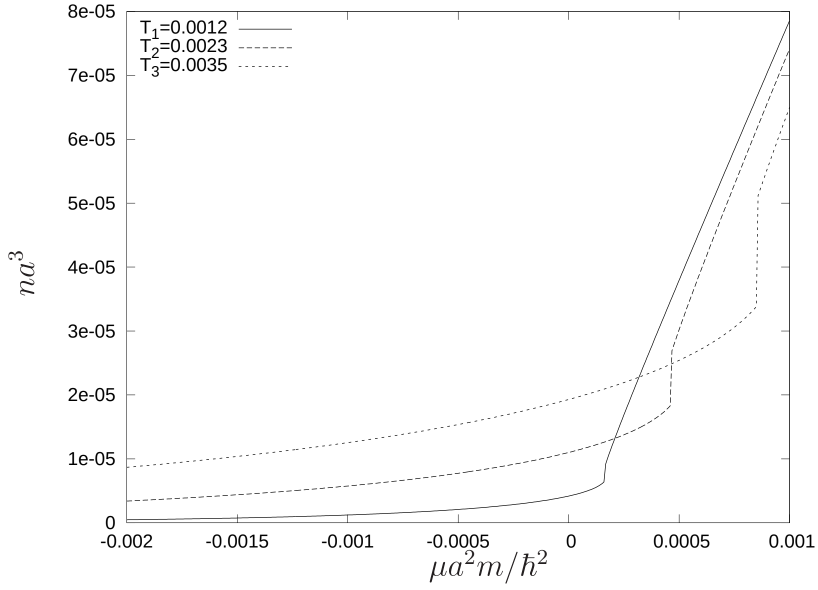

Figure C.4: $\hat{n}(\hat{\mu})$ for three distinct values of $\hat{T}$

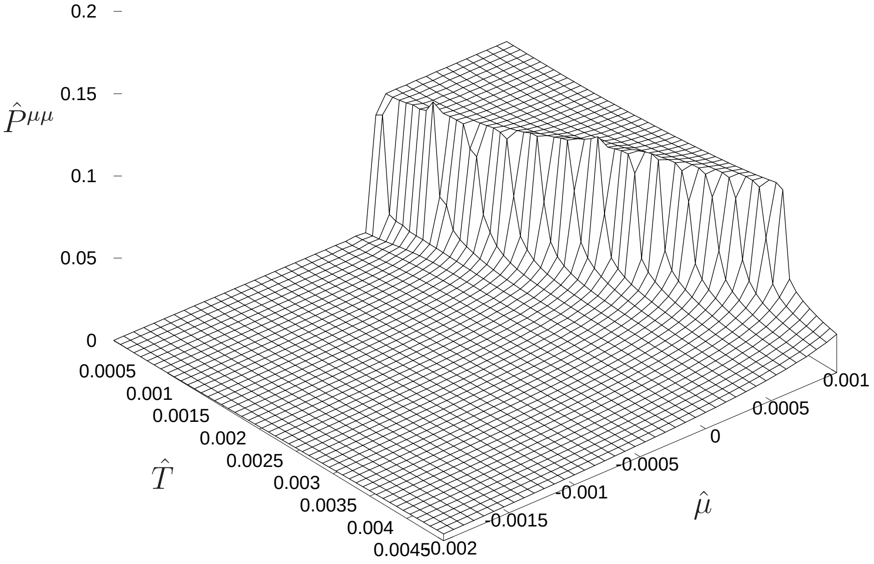

Figure C.5: $\hat{P}^{\mu\mu} = P^{\mu\mu} a\hbar^2/m$ as a function of $\hat{T}$ and $\hat{\mu}$

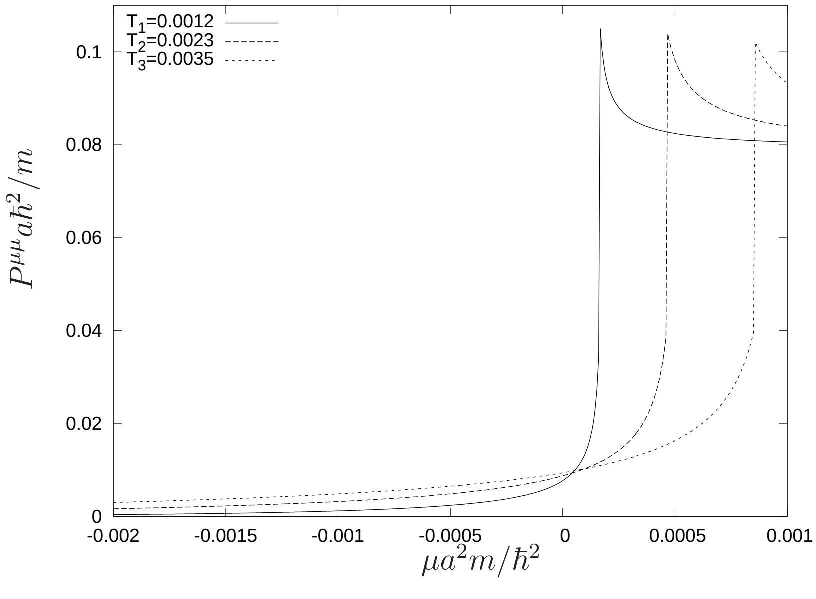

Figure C.6: $\hat{P}^{\mu\mu}(\hat{\mu})$ for three distinct values of $\hat{T}$

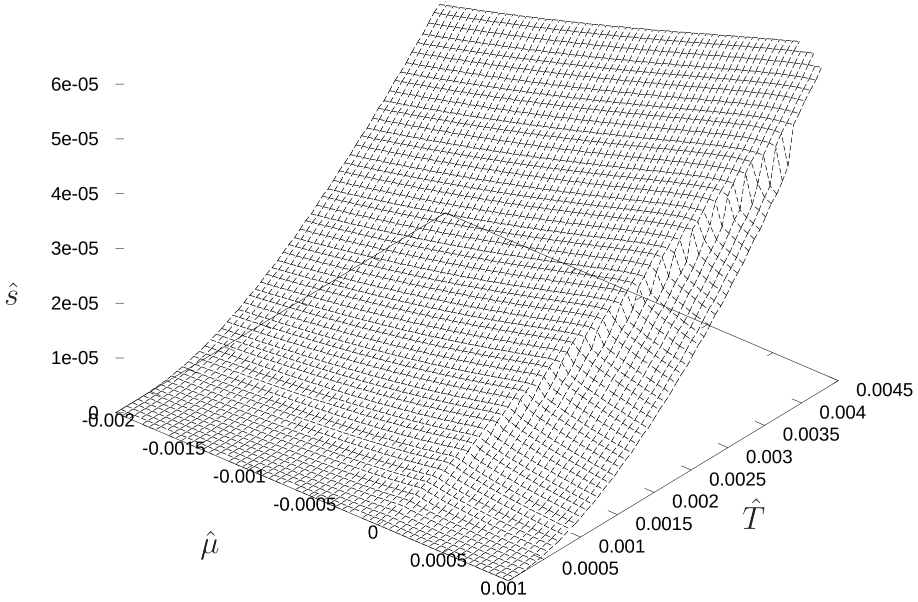

Figure C.7: Entropy density $\hat{s} = sa^3/k_B$. The $(\hat{\mu}, \hat{T})$-plane is rotated.

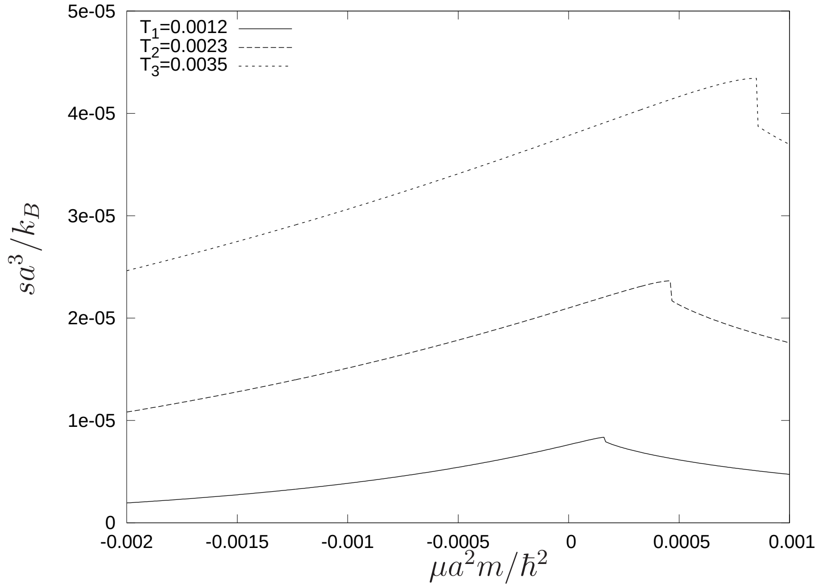

Figure C.8: $\hat{s}(\hat{\mu})$ for three distinct values of $\hat{T}$

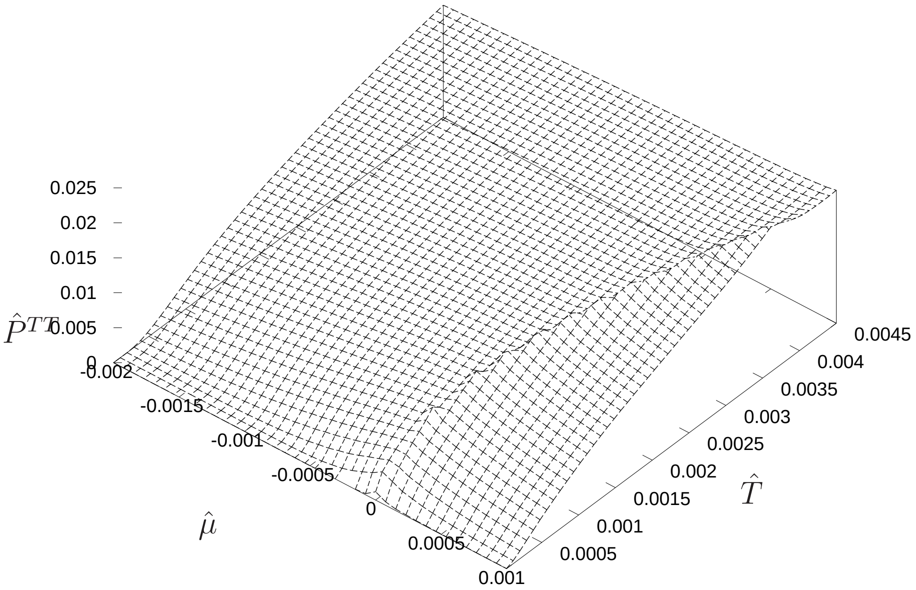

Figure C.9: $\hat{P}^{TT} = P^{TT} a\hbar^2/k_B^2 m$. The $(\hat{\mu}, \hat{T})$-plane is rotated.

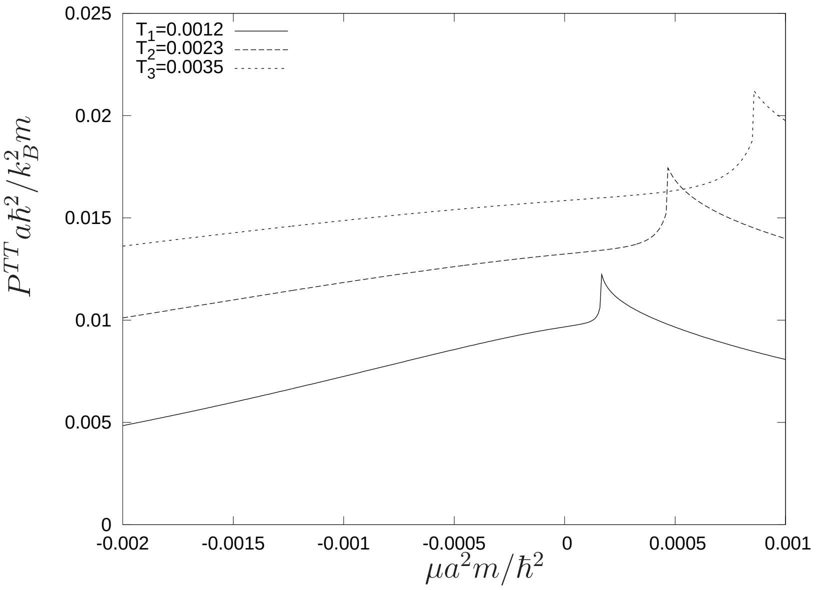

Figure C.10: $\hat{P}^{TT}(\hat{\mu})$ for three distinct values of $\hat{T}$

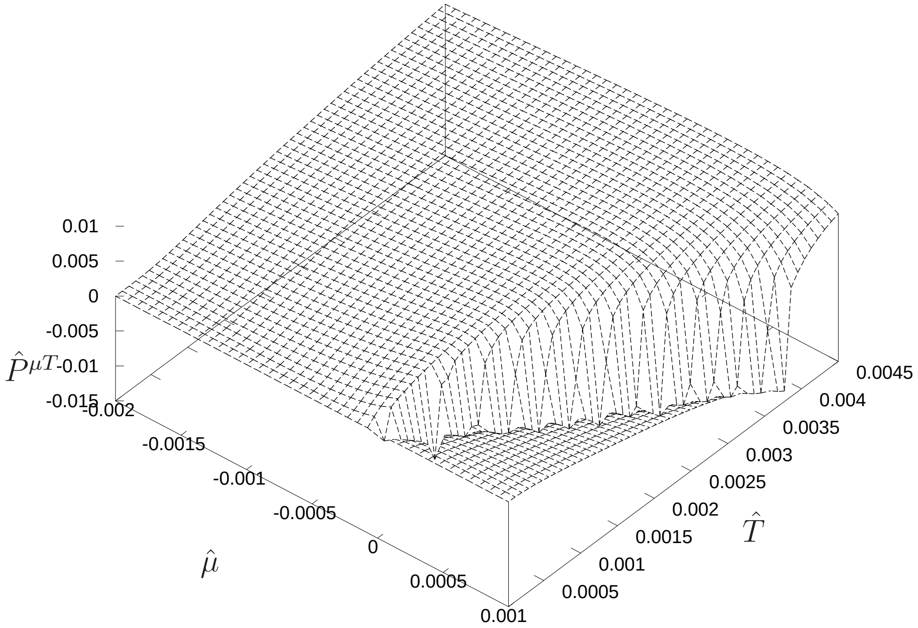

Figure C.11: $\hat{P}^{\mu T} = P^{\mu T} a\hbar^2/k_B m$. The $(\hat{\mu}, \hat{T})$-plane is rotated.

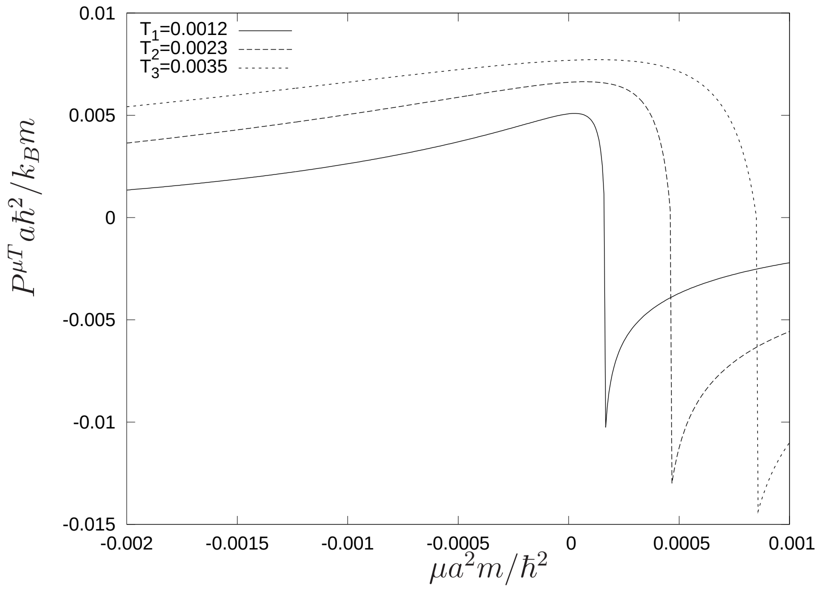

Figure C.12: $\hat{P}^{\mu T}(\hat{\mu})$ for three distinct values of $\hat{T}$

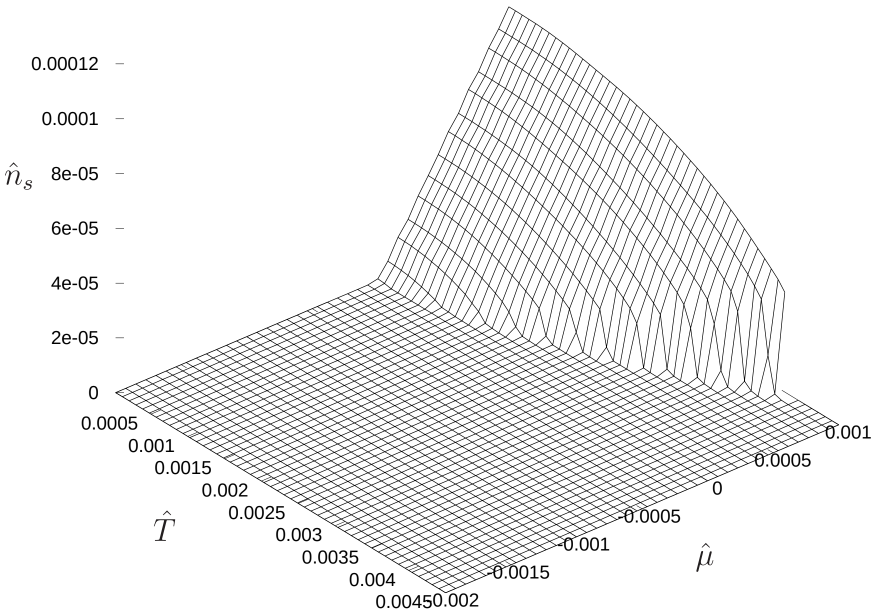

Figure C.13: Superfluid density $\hat{n}_s = n_s a^3$ as a function of $\hat{T}$ and $\hat{\mu}$

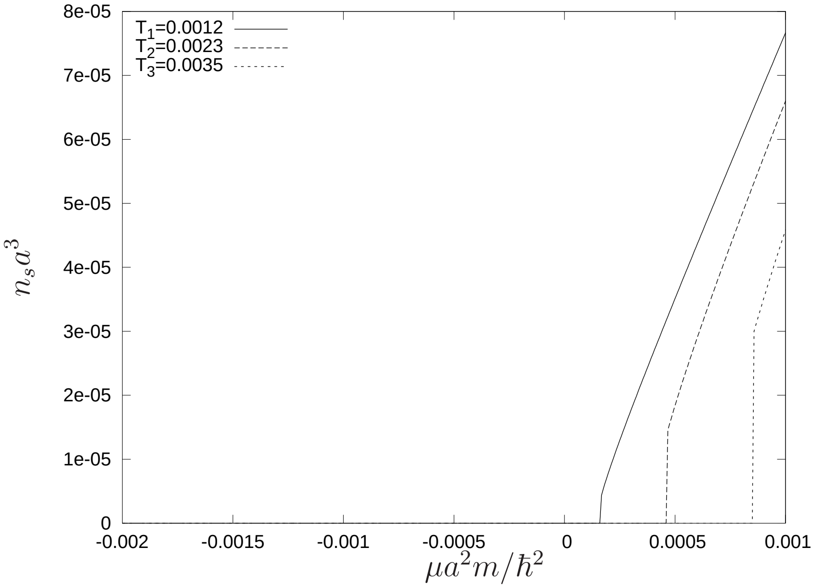

Figure C.14: $\hat{n}_s(\hat{\mu})$ for three distinct values of $\hat{T}$

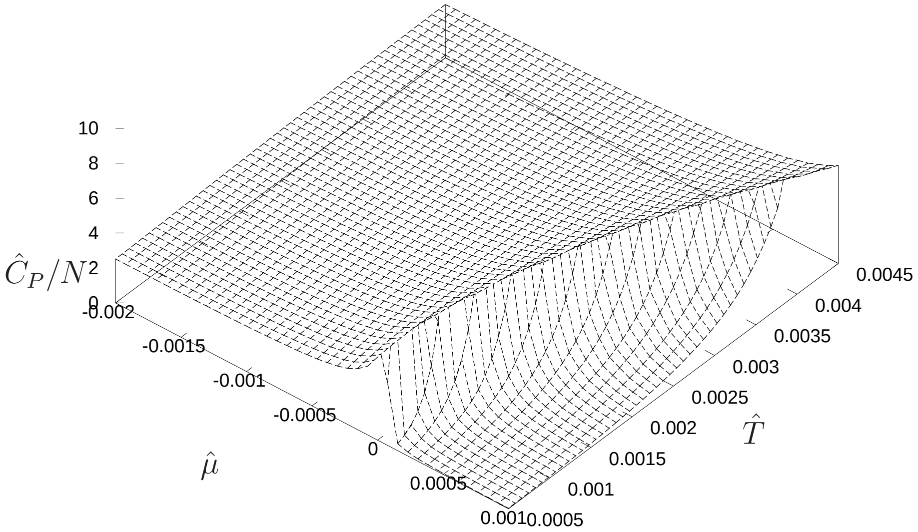

Figure C.15: $\hat{C}_P/N = C_P/Nk_B$. The $(\hat{\mu}, \hat{T})$-plane is rotated.

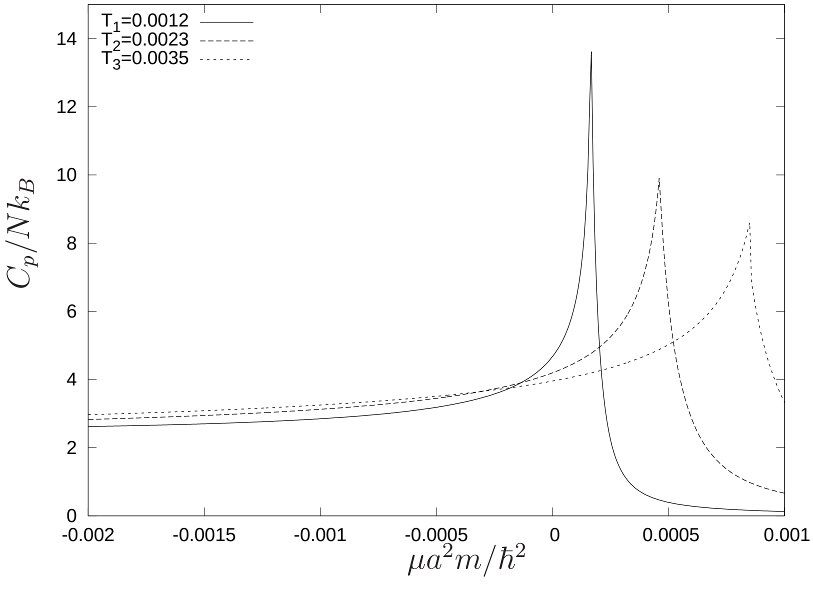

Figure C.16: $\hat{C}_P(\hat{\mu})/N$ for three distinct values of $\hat{T}$

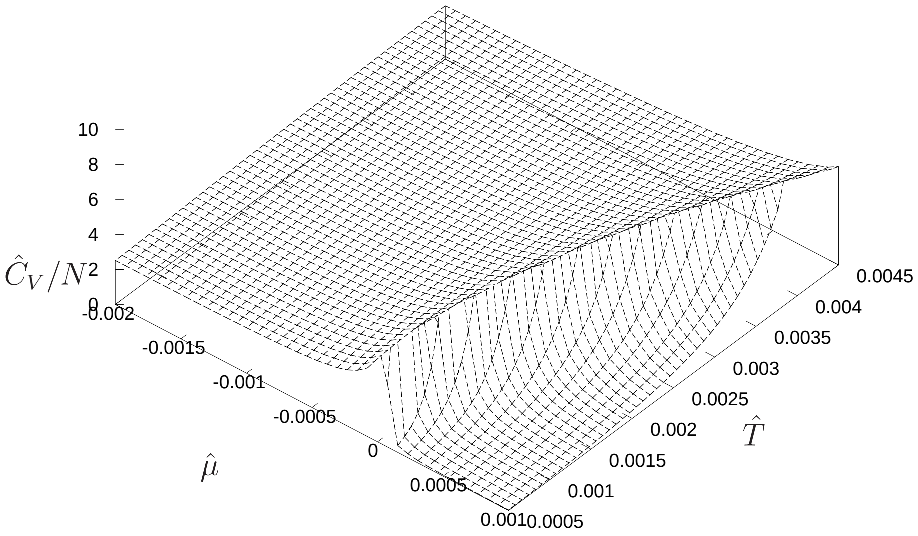

Figure C.17: $\hat{C}_V/N = C_V/Nk_B$. The $(\hat{\mu}, \hat{T})$-plane is rotated.

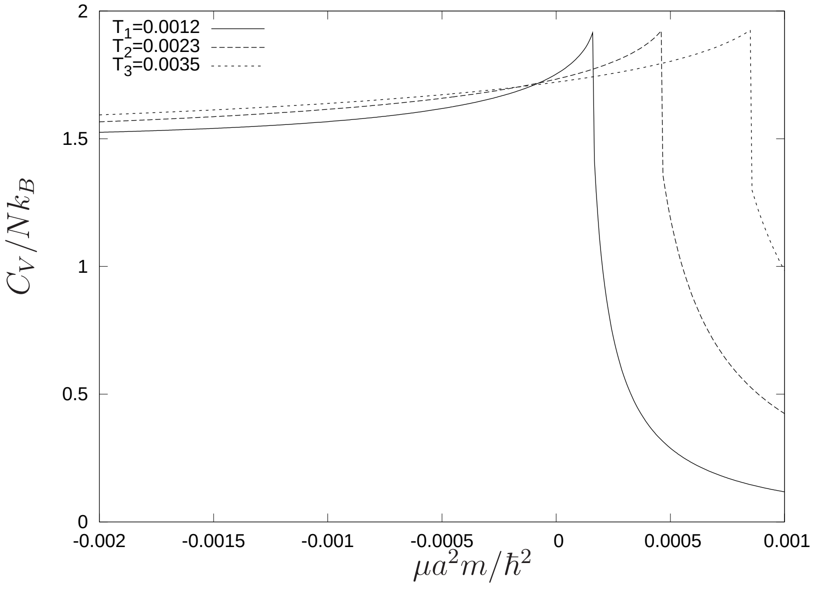

Figure C.18: $\hat{C}_V(\hat{\mu})/N$ for three distinct values of $\hat{T}$

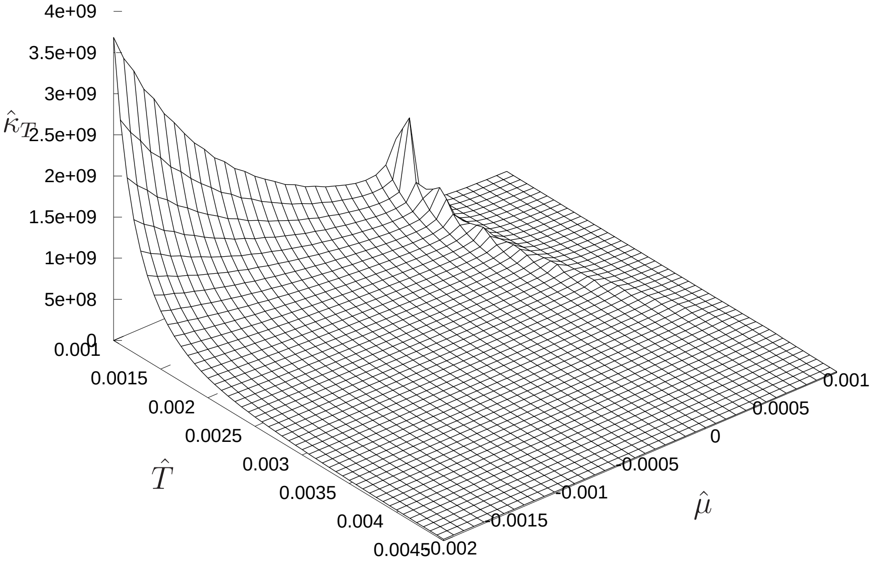

Figure C.19: Isothermal compressibility $\hat{\kappa}_T = \kappa_T \hbar^2/a^5 m$ versus $\hat{T}$ and $\hat{\mu}$

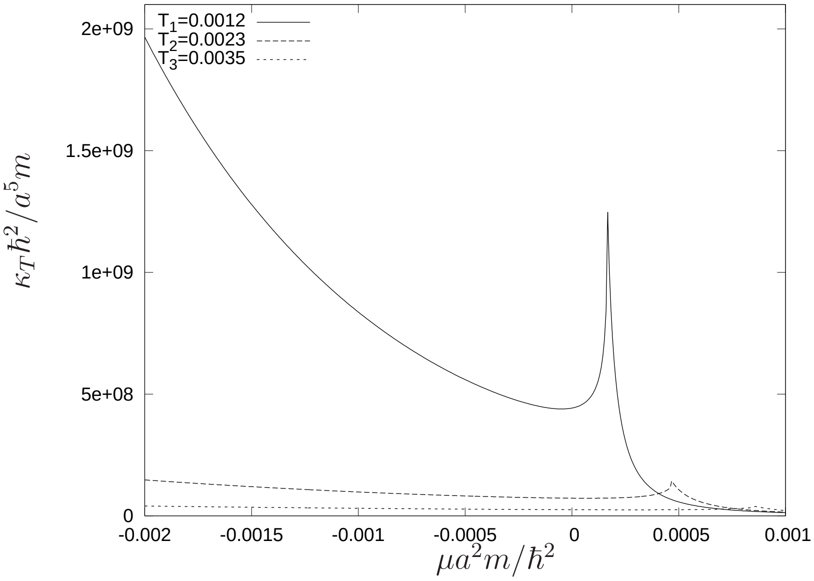

Figure C.20: $\hat{\kappa}_T(\hat{\mu})/N$ for three distinct values of $\hat{T}$

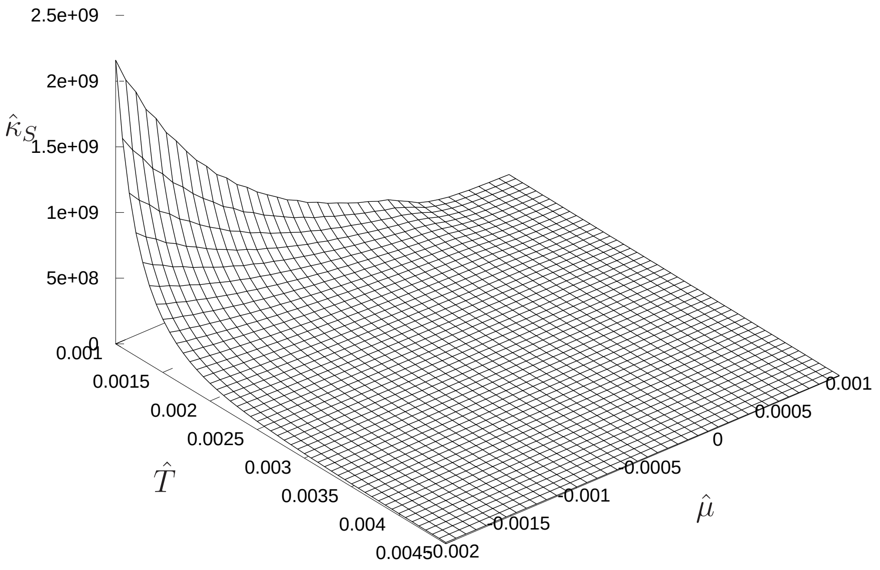

Figure C.21: Adiabatic compressibility $\hat{\kappa}_S = \kappa_S \hbar^2/a^5 m$ versus $\hat{T}$ and $\hat{\mu}$

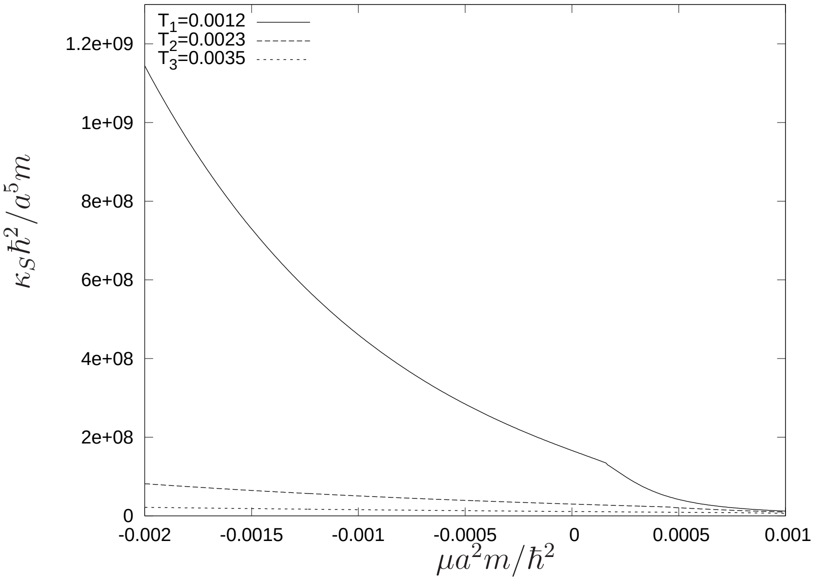

Figure C.22: $\hat{\kappa}_S(\hat{\mu})/N$ for three distinct values of $\hat{T}$

D Spherical harmonics in one and two dimensions

We define spherical harmonics in d dimensions for $l = 0, 1, 2, ...$ as complex valued functions f_{lm} on the unit sphere $\mathbb{S}^{d-1} = \{\vec{x} \in \mathbb{R}^d | r^2 = |\vec{x}|^2 = 1\}$ through the relation $\Delta(r^l f_{lm}) = 0$, where Δ is the Laplacian, $\Delta = (\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_d)^2$, and demand the orthonormality condition

$$\int_{\mathbb{S}^{d-1}} f^*_{lm} f_{l'm'} = \delta_{ll'} \delta_{mm'}. \tag{D.1}$$

m is a set of further indices. For $d = 3$ we have $m \in \{-l, ..., l\}$ and $f_{lm} = Y_{lm}$ in traditional notation. In $d = 2$ dimensions, writing $x_1 = r \cos\phi, x_2 = r \sin\phi$, we need to solve

$$0 = \Delta(r^l f_{lm}(\phi)) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \right) r^l f_{lm}(\phi), \tag{D.2}$$

which leads to

$$f_{l,\pm 1} = \frac{1}{\sqrt{2\pi}} e^{\pm il\phi} \tag{D.3}$$

for $l = 0, 1, 2, ...$ We see that all values of l are allowed for $d = 2$. This is no longer true in the one-dimensional case as can already be guessed by physical reasoning. For $d = 1$ there are only monopole and dipole modes. Indeed, $\mathbb{S}^0 = \{-1, 1\}$ is a discrete point set and one-dimensional spherical coordinates read $x = \sigma r$ with $r = |x|$ and $\sigma = x/r$. The angular variable thus tells us whether we are considering the positive or the negative half line of the real axis. The expression $g(x) = g(r) f_l(\sigma)$ has to be understood as

$$g(x) = \begin{cases} g(r) f_l(-1), & x < 0 \\ g(0), & x = 0 \\ g(r) f_l(1), & x > 0. \end{cases} \tag{D.4}$$

Since the harmonic functions in $d = 1$, i.e. the ones satisfying $g''(x) = 0$, are exactly the affine linear functions, the relation $\Delta(r^l f_l(\sigma)) = 0$ can only be true for $l = 0, 1$. The normalization condition

$$\begin{aligned} \int_{\mathbb{S}^0} f^*_{lm} f_{l'm'} &= f^*_{lm}(-1) f_{l'm'}(-1) + f^*_{lm}(1) f_{l'm'}(1) \\ &= \delta_{ll'} \delta_{mm'} \end{aligned} \tag{D.5}$$

is valid for

$$\begin{aligned} f_0(\sigma) &= \begin{cases} 1/\sqrt{2}, & \sigma = -1 \\ 1/\sqrt{2}, & \sigma = 1 \end{cases} , \\ f_1(\sigma) &= \begin{cases} -1/\sqrt{2}, & \sigma = -1 \\ 1/\sqrt{2}, & \sigma = 1 \end{cases} . \end{aligned} \tag{D.6}$$

f_0 and f_1 defined in this way have the correct parity $(-1)^l$. We do not give explicit constructions of spherical harmonics for higher dimensions than three.

References

[1] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008)

[2] T.-L. Ho and Q. Zhou, Nature Phys. **6**, 131 (2010)

[3] N. Navon, S. Nascimbène, F. Chevy, and C. Salomon, Science **328**, 729 (2010)

[4] D. S. Jin, J.R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **77**, 420 (1996)

[5] M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. G. Townsend, and W. Ketterle, Phys. Rev. Lett. **77**, 988 (1996)

[6] D. S. Jin, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **78**, 764 (1997)

[7] D. M. Stamper-Kurn, H.-J. Miesner, S. Inouye, M. R. Andrews, and W. Ketterle, Phys. Rev. Lett. **81**, 500 (1998)

[8] F. Chevy, V. Bretin, P. Rosenbusch, K. W. Madison, and J. Dalibard, Phys. Rev. Lett. **88**, 250402 (2002)

[9] H. Moritz, T. Stöferle, M. Köhl, and T. Esslinger, Phys. Rev. Lett. **91**, 250402 (2003)

[10] A. Altmeyer, S. Riedl, C. Kohstall, M. J. Wright, R. Geursen, M. Bartenstein, C. Chin, J. Hecker Denschlag, and R. Grimm, Phys. Rev. Lett. **98**, 040401 (2007)

[11] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, H. Hecker Denschlag, and R. Grimm, Phys. Rev. Lett **92**, 203201 (2004)

[12] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004)

[13] J. Kinast, A. Turlapov, and J.E. Thomas, Phys. Rev. A. **70**, 051401(R) (2004)

[14] J. Kinast, A. Turlapov, and J.E. Thomas, Phys. Rev. Lett. **94**, 170404 (2005)

[15] M. Kottke, T. Schulte, L. Cacciapuoti, D. Hellweg, S. Drenkelforth, W. Ertmer, and J. J. Arlt, Phys. Rev. A. **72**, 053631 (2005)

[16] S. Riedl, E. R. Sánchez Guajardo, C. Kohstall, A. Altmeyer, M. J. Wright, J. Hecker Denschlag, R. Grimm, G. M. Bruun, and H. Smith, Phys. Rev. A **78**, 053609 (2008)

[17] S. E. Pollack, D. Dries, R. G. Hulet, K. M. F. Magalhães, E. A. L. Henn, E. R. F. Ramos, M. A. Caracanhas, and V. S. Bagnato, Phys Rev. A. **81**, 053627 (2010)

[18] M. Edwards, P. A. Ruprecht, K. Burnett, R. J. Dodd, and C. W. Clark, Phys. Rev. Lett. **77**, 1671 (1996)

[19] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996)

[20] V. M. Pérez-García, H. Michinel, J. I. Cirac, M Lewenstein, and P. Zoller, Phys. Rev. Lett. **77**, 5320 (1996)

[21] A. Griffin, W.-C. Wu, and S. Stringari, Phys. Rev. Lett **78**, 1838 (1997)

[22] V. B. Shenoy and T.-L. Ho, Phys. Rev. Lett. **80**, 3895 (1998)

[23] L. P. Pitaevskii and S. Stringari, Phys. Rev. Lett. **83**, 4237 (1998)

[24] E. Braaten and J. Pearson, Phys. Rev. Lett. **82**, 255 (1999)

[25] M. Amoruso, I. Meccoli, A. Minguzzi, and M. P. Tosi, Eur. Phys. J. D. **7**, 441 (1999)

[26] B. Jackson and E. Zaremba, Phys. Rev. Lett. **88**, 180402 (2002)

[27] C. Menotti and S. Stringari, Phys. Rev. A **66**, 043610 (2002)

[28] J. N. Fuchs, X. Leyronas, and R. Combescot, Phys. Rev. A **68**, 043610 (2003)

[29] H. Heiselberg, Phys. Rev. Lett **93**, 040402 (2004)

[30] S. Stringari, Europhys. Lett. **65**, 749 (2004)

[31] H. Hu, A. Minguzzi, X.-J. Liu, and M. P. Tosi, Phys. Rev. Lett. **93**, 190403 (2004)

[32] T. Nikuni and A. Griffin, Phys. Rev. A **69**, 023604 (2004)

[33] Y. E. Kim and A. L. Zubarev, Phys. Rev. A. **70**, 033612 (2004)

[34] G. E. Astrakharchik, R. Combescot, X. Leyronas, and S. Stringari, Phys. Rev. Lett. **95**, 030404 (2005)

[35] A. Bulgac and G. F. Bertsch, Phys. Rev. Lett. **94**, 070401 (2005)

[36] E. Taylor and A. Griffin, Phys. Rev. A **72**, 053630 (2005)

[37] K. Huang, *Statistical Mechanics* (Wiley, New York, 1998), 2nd edition

[38] L.D. Landau and E.M. Lifschitz, *Hydrodynamik* (Harri Deutsch, Frankfurt am Main, 2007) 5. Auflage

[39] P. Kapitza, Nature **141**, 74 (1938)

[40] J. F. Allen and A. D. Misener, Nature **141**, 75 (1938)

[41] L. D. Landau, J. Phys. Lett (Moscow) **5**, 71 (1941) This paper can be found as a reprint in [42].

[42] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, New York, 1965)

[43] L. P. Pitaevskii and S. Stringari, *Bose–Einstein Condensation* (Oxford University Press, Oxford, 2003)

[44] L.D. Landau and E.M. Lifschitz, *Klassische Mechanik* (Harri Deutsch, Frankfurt am Main, 2007) 14., korrigierte Auflage

[45] S. J. Putterman, *Superfluid Hydrodynamics*, (North Holland, Amsterdam, 1974)

[46] T. Schäfer and D. Teaney, Rep. Prog. Phys. **72**, 126001 (2009)

[47] J. O. Andersen, Rev. Mod. Phys. **76**, 599 (2004)

[48] N. N. Bogoliubov, J. Phys USSR **11**, 23 (1947)

[49] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119-1120 (1957)

[50] T. D. Lee, K. Huang and C. N. Yang, Phys. Rev. **106**, 1135-1145 (1957)

[51] T. T. Wu, Phys. Rev. **115**, 1390-1404 (1959)

[52] T. D. Lee and C. N. Yang, Phys. Rev. **112**, 1419-1429 (1958)

[53] T. D. Lee and C. N. Yang, Phys. Rev. **113**, 1406-1413 (1959)

[54] A. Pelissetto and E. Vicari, Phys. Rep. **368**, 549 (2002)

[55] I. Boettcher, S. Floerchinger, and C. Wetterich, arXiv:1103.5342 (2011)

[56] C. J. Pethik and H. Smith, *Bose–Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, 2002)

[57] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (U.S. Government Printing Office, Washington, DC, 2964)

[58] C. Wetterich, Phys. Lett. B **301**, 90 (1993)

[59] S. Floerchinger and C. Wetterich, Phys. Rev. A **77**, 053603 (2008)

[60] S. Floerchinger and C. Wetterich, Phys. Rev. A **79**, 013601 (2009)

[61] S. Floerchinger and C. Wetterich, Phys. Rev. A **79**, 063602 (2009)

[62] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966)

[63] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967)

[64] P. Pedri, S. De Palo, E. Orignac, R. Citro and M. L. Chiofalo, Phys. Rev. A **77**, 015601 (2008)

[65] R. Citro, E. Orignac, S. De Palo and M. L. Chiofalo, Phys. Rev. A **75**, 051602(R) (2007)

[66] L. D. Landau and E. M. Lifschitz, *Statistische Physik, Teil I*, (Harri Deutsch, Frankfurt am Main, 2008), 8., korrigierte Auflage

[67] N. Navon, S. Piatecki, K. J. Günter, B. Rem, T. C. Nguyen, F. Chevy, W. Krauth, C. Salomon, arXiv:1103.4449 (2011)

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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