Classical Electrodynamics

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Contents

1 Charges in external fields **3**
1.1 Vectors and vector fields . . . 3
1.2 Lorentz force . . . 7
1.3 Coulomb force and units . . . 10
1.4 Electric potential and voltage . . . 12
1.5 Conductors . . . 17
1.6 Electric currents . . . 18
1.7 Electric and magnetic dipoles . . . 21

1 Charges in external fields

Electrodynamical phenomena have two key components:

(1) The motion of charged particles in electric and magnetic fields. The central equation is the Lorentz force relation.

(2) The dynamics of electric and magnetic fields, i.e. their spatial and temporal behavior, and their mutual interaction. Furthermore, the generation of electric and magnetic fields by charges and currents. The central equations are the Maxwell equations.

Clearly, the "matter" part (charges and currents) and the "field" part (electric and magnetic fields) feature in both (1) and (2), which are thus not independent.

To get started and develop some intuition, we will study in this section the behavior of charges in fixed external electric and magnetic fields (1), while mostly ignoring their origin and dynamics (2).

1.1 Vectors and vector fields

Vectors in three dimensions. We write vectors $\vec{v} \in \mathbb{R}^3$ as column vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \tag{1.1}$$

and denote the components of $\vec{v}$ by v_i with $i = 1, 2, 3 = x, y, z$. The transposed vector is the row

$$\vec{v}^T = (v_1, v_2, v_3). \tag{1.2}$$

We denote the canonical basis of $\mathbb{R}^3$ by the vectors $\vec{e}_i,\ i = 1, 2, 3$, given by

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.3}$$

The vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are also denoted by $\hat{\imath}, \hat{\jmath}, \hat{k}$ in the literature, a practice that I strongly discourage. Every vector can be written

$$\vec{v} = \sum_{i=1}^{3} v_i \vec{e}_i = \sum_{i=1}^{3} \vec{e}_i v_i. \tag{1.4}$$

(Note that the second way of writing is uncommon, but not wrong in principle.) The scalar product between two vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = (a_1, a_2, a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i. \tag{1.5}$$

We write

$$\vec{v}^2 = \vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 \tag{1.6}$$

and define the length or magnitude of $\vec{v}$ as

$$|\vec{v}| = \sqrt{\vec{v}^2} = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{1.7}$$

A vector is normalized or a unit vector if $|\vec{v}| = 1$. If α is the angle between the vectors $\vec{v}$ and $\vec{w}$, then

$$\vec{v} \cdot \vec{w} = |\vec{v}| \; |\vec{w}| \; \cos\alpha. \tag{1.8}$$

Two vectors $\vec{v}, \vec{w}$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$. The vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are an orthonormal basis, meaning

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}, \tag{1.9}$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } j \neq j \end{cases} \tag{1.10}$$

for $i, j = 1, 2, 3$. This is consistent with the scalar product definition

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{3} \sum_{j=1}^{3} v_i w_j \vec{e}_i \cdot \vec{e}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} v_i w_j \delta_{ij} = \sum_{i=1}^{3} v_i w_i. \tag{1.11}$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}, \tag{1.12}$$

which is again a vector. I encourage you to memorize this formula instead of looking it up every single time.

Cartesian coordinates. We denote spatial coordinates in three dimensions by

$$\vec{x} = \vec{r}. \tag{1.13}$$

In Cartesian coordinates, we have

$$\vec{x} = \vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{1.14}$$

We define the length of $\vec{x}$ by

$$r = |\vec{x}| = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}. \tag{1.15}$$

The time coordinates is denoted by t. We call the vector space of pairs $(\vec{x}, t) \in \mathbb{R}^3 \times \mathbb{R}$ "spacetime". (More about this later.) The SI unit of $\vec{x}$ is meters, the SI unit of t is seconds.

Vector and scalar fields. A vector field assigns a vector to each position in space or spacetime according to

$$\vec{E}: \; \mathbb{R}^3 \to \mathbb{R}^3 \tag{1.16}$$

$$\vec{x} \; \mapsto \vec{E}(\vec{x}) \tag{1.17}$$

or

$$\vec{E}: \; \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \tag{1.18}$$

$$(\vec{x}, t) \; \mapsto \vec{E}(\vec{x}, t), \tag{1.19}$$

respectively. That is, for each $\vec{x}$ and t we have

$$\vec{E}(\vec{x}, t) = \begin{pmatrix} E_1(\vec{x}, t) \\ E_2(\vec{x}, t) \\ E_3(\vec{x}, t) \end{pmatrix} = \begin{pmatrix} E_x(\vec{x}, t) \\ E_y(\vec{x}, t) \\ E_z(\vec{x}, t) \end{pmatrix}. \tag{1.20}$$

A scalar field assign a number to each position in space or spacetime according to

$$\phi: \ \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R} \tag{1.21}$$

$$(\vec{x}, t) \ \mapsto \phi(\vec{x}, t). \tag{1.22}$$

Importantly, $\vec{E}$ and ϕ have physical units, which are in general different from those of $\vec{x}$ and t.

Nabla operator. We define

$$\partial_i = \frac{\partial}{\partial x_i} \tag{1.23}$$

for $i = 1, 2, 3 = x, y, z$ and denote the nabla operator ∇ by the column vector

$$\nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}. \tag{1.24}$$

The nabla operator is a vector of differential operators and has the dimension of inverse length, i.e. its SI unit is meters^{-1}. We have

$$\nabla = \sum_{i=1}^{3} \vec{e}_i \partial_i = \vec{e}_1 \partial_1 + \vec{e}_2 \partial_2 + \vec{e}_3 \partial_3. \tag{1.25}$$

We write it like this so that the ∂_is do not act on the $\vec{e}_i$. This will be important later in spherical and cylindrical coordinates. The gradient of a scalar function $\phi(\vec{x})$ is given by

$$\text{grad } \phi(\vec{x}) = \nabla \phi(\vec{x}) = \begin{pmatrix} \partial_x \phi(\vec{x}) \\ \partial_y \phi(\vec{x}) \\ \partial_z \phi(\vec{x}) \end{pmatrix}. \tag{1.26}$$

Note that $\vec{Y}(\vec{x}) = \text{grad } \phi(\vec{x})$ is then a vector field,

$$\vec{Y}: \ \mathbb{R}^3 \to \mathbb{R}^3, \tag{1.27}$$

$$\vec{x} \mapsto \vec{Y}(\vec{x}) = \text{grad } \phi(\vec{x}), \tag{1.28}$$

and we have

$$(\nabla \phi)_i = Y_i = \partial_i \phi. \tag{1.29}$$

We will sometimes also use the notation

$$\nabla \phi(\vec{x}) = \frac{\partial}{\partial \vec{x}} \phi(\vec{x}) = \frac{\partial \phi}{\partial \vec{x}}(\vec{x}), \tag{1.30}$$

which emphasizes the fact that

$$(\nabla \phi)_i = \frac{\partial \phi}{\partial x_i}. \tag{1.31}$$

If needed, we specify which argument ∇ acts on by a subscript, e.g.

$$\nabla_{\vec{x}} f(\vec{x} - \vec{x}') = \frac{\partial}{\partial \vec{x}} f(\vec{x} - \vec{x}'). \tag{1.32}$$

The divergence of a vector field $\vec{E}(\vec{x})$ is

$$\text{div } \vec{E}(\vec{x}) = \nabla \cdot \vec{E}(\vec{x}) = \sum_{i=1}^{3} \partial_i E_i(\vec{x}) = \partial_x E_x(\vec{x}) + \partial_y E_y(\vec{x}) + \partial_z E_z(\vec{x}). \tag{1.33}$$

Note that $\chi(\vec{x}) = \text{div}\vec{E}(\vec{x})$ is a scalar field

$$\chi: \ \mathbb{R}^3 \to \mathbb{R}, \tag{1.34}$$

$$\vec{x} \mapsto \chi(\vec{x}) = \text{div } \vec{E}(\vec{x}). \tag{1.35}$$

The <u>curl</u> of a vector field $\vec{E}(\vec{x})$ is

$$\text{curl } \vec{E}(\vec{x}) = \nabla \times \vec{E}(\vec{x}) = \begin{pmatrix} \partial_2 E_3(\vec{x}) - \partial_3 E_2(\vec{x}) \\ \partial_3 E_1(\vec{x}) - \partial_1 E_3(\vec{x}) \\ \partial_1 E_2(\vec{x}) - \partial_2 E_1(\vec{x}) \end{pmatrix}. \tag{1.36}$$

This is again a vector field.

Four useful identities. The following four identities are easy to derive, but you should just memorize them anyway. We have

$$(1)\ \nabla\ \vec{a}\cdot\vec{r} = \frac{\partial}{\partial\vec{r}}\ \vec{a}\cdot\vec{r} = \vec{a}. \tag{1.37}$$

First note that this makes sense: $\vec{a}\cdot\vec{r}$ is a scalar, and $\nabla(\vec{a}\cdot\vec{r})$ is a vector. To prove it, use

$$\nabla\ \vec{a}\cdot\vec{r} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}\ (a_x x + a_y y + a_z z) \tag{1.38}$$

$$= \begin{pmatrix} \partial_x(a_x x + a_y y + a_z z) \\ \partial_y(a_x x + a_y y + a_z z) \\ \partial_z(a_x x + a_y y + a_z z) \end{pmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \vec{a}.\ \square \tag{1.39}$$

We have

$$(2)\ \text{div}(\vec{r}) = \nabla\cdot\vec{r} = 3. \tag{1.40}$$

Indeed,

$$\nabla\cdot\vec{r} = (\partial_x, \partial_y, \partial_z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \partial_x x + \partial_y y + \partial_z z = 1 + 1 + 1 = 3.\ \square \tag{1.41}$$

Note that the units make sense: ∇ is an inverse length and $\vec{r}$ is a length, hence $\nabla\cdot\vec{r}$ is a dimensionless number. For a function $f(r)$ of $r = |\vec{r}|$ we have

$$(3)\ \nabla f(r) = f'(r)\vec{e}_r, \tag{1.42}$$

where

$$\vec{e}_r = \frac{\vec{r}}{r} \tag{1.43}$$

is the unit vector pointing in r-direction. This identity is extremely useful. Note that the dimensions make sense: ∇ has dimension of inverse length, f has some dimension $[f]$, $f'(r)$ has dimension $[f]$/length, and $\vec{e}_r$

is dimensionless. For the proof, note that

$$\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \begin{pmatrix} \partial_x\sqrt{x^2+y^2+z^2} \\ \partial_y\sqrt{x^2+y^2+z^2} \\ \partial_z\sqrt{x^2+y^2+z^2} \end{pmatrix} \tag{1.44}$$

$$= \begin{pmatrix} \frac{2x}{2(x^2+y^2+z^2)^{1/2}} \\ \frac{2y}{2(x^2+y^2+z^2)^{1/2}} \\ \frac{2z}{2(x^2+y^2+z^2)^{1/2}} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{e}_r, \tag{1.45}$$

which is a special case of (3) for $f(r) = r$ (hence $f'(r) = 1$). Using the chain rule we then have

$$\nabla f(r) = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} f(r) = \begin{pmatrix} \partial_x f(r) \\ \partial_y f(r) \\ \partial_z f(r) \end{pmatrix} \tag{1.46}$$

$$= \begin{pmatrix} f'(r)\partial_x r \\ f'(r)\partial_y r \\ f'(r)\partial_z r \end{pmatrix} = f'(r)\nabla r = f'(r)\vec{e}_r. \ \square \tag{1.47}$$

If you have a <u>constant</u> vector $\vec{B}$ and want to find a vector field $\vec{A}(\vec{r})$ such that curl $\vec{A}(\vec{r}) = \vec{B}$, use

$$(4)\ \mathrm{curl}\Big(\frac{1}{2}\vec{B}\times\vec{r}\Big) = \vec{B}. \tag{1.48}$$

Indeed,

$$\nabla\times\frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}\nabla\times\begin{pmatrix} B_2x_3 - B_3x_2 \\ B_3x_1 - B_1x_3 \\ B_1x_2 - B_2x_1 \end{pmatrix} \tag{1.49}$$

$$= \frac{1}{2}\begin{pmatrix} \partial_2(B_1x_2 - B_2x_1) - \partial_3(B_3x_1 - B_1x_3) \\ \partial_3(B_2x_3 - B_3x_2) - \partial_1(B_1x_2 - B_2x_1) \\ \partial_1(B_3x_1 - B_1x_3) - \partial_2(B_2x_3 - B_3x_2) \end{pmatrix} \tag{1.50}$$

$$= \frac{1}{2}\begin{pmatrix} B_1 + B_1 \\ B_2 + B_2 \\ B_3 + B_3 \end{pmatrix} = \vec{B}. \ \square \tag{1.51}$$

1.2 Lorentz force

A particle of charge q in an external electric and magnetic field experiences the Lorentz force

$$\vec{F} = q\Big(\vec{E} + \vec{v}\times\vec{B}\Big)\ \text{(SI units)}, \tag{1.52}$$

$$\vec{F} = q\Big(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\Big)\ \text{(Gauss units)}. \tag{1.53}$$

More about units in the next section: they distinguish what is meant by $q, \vec{E}, \vec{B}$. In the first part of this lecture, we will use SI units, which are the ones almost exclusively used in experiments and applications. However, the Gauss units have some conceptual advantage and will be employed later.

Experiments find that all charged particles also have a nonzero mass. Consider then a particle of mass m and charge q in an electric field $\vec{E}(\vec{x},t)$ and magnetic field $\vec{B}(\vec{x},t)$. Its trajectory

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{1.54}$$

is given by the solution to the differential equation

$$m\ddot{\vec{x}}(t) = q\Big(\vec{E}(\vec{x}(t),t) + \dot{\vec{x}}(t) \times \vec{B}(\vec{x}(t),t)\Big) \tag{1.55}$$

for $t \geq 0$ with initial conditions

$$\vec{x}(0) = \vec{x}_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix}, \ \dot{\vec{x}}(0) = \vec{v}_0 = \begin{pmatrix} v_{0,1} \\ v_{0,2} \\ v_{0,3} \end{pmatrix}. \tag{1.56}$$

Note how "$\vec{x}(t)$" is inserted into the $\vec{x}$-slot of the vectors fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$.

Consider first the case of constant E- and vanishing B-field. We solve

$$m\ddot{\vec{x}}(t) = q\vec{E} \tag{1.57}$$

to obtain

$$\boxed{\vec{x}(t) = \frac{qt^2}{2}\vec{E} + \vec{v}_0 t + \vec{x}_0.} \tag{1.58}$$

The charge is accelerated by the electric field. A positive charge moves into the direction of the E-field, a negative charge into the opposite direction. This is purely conventional and depends on how we defined $\vec{E}$ instead of $-\vec{E}$ in the equation for $\vec{F}$.

By convention, electric field lines point into the direction of the movement of positive charges.

Next consider the case of constant $\vec{B}$ and vanishing E-field. Without loss of generality assume that

$$\vec{B} = B\vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \tag{1.59}$$

points in the z-direction. We solve

$$m\ddot{\vec{x}}(t) = q\dot{\vec{x}}(t) \times \vec{B} \tag{1.60}$$

with the substitution $\vec{v}(t) = \dot{\vec{x}}(t)$ to obtain

$$\dot{\vec{v}}(t) = \frac{q}{m}\vec{v}(t) \times \vec{B}, \tag{1.61}$$

$$\Leftrightarrow \quad \begin{pmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{v}_3(t) \end{pmatrix} = \frac{q}{m}\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{q}{m}\begin{pmatrix} v_2(t)B \\ -v_1(t)B \\ 0 \end{pmatrix}. \tag{1.62}$$

Note that this is a first-order equation and so has only three (not six) integration constants, related to $\vec{v}_0$. The third equation is solved by

$$v_3(t) = v_{0,3}. \tag{1.63}$$

The first two equations are decoupled by another time-derivative:

$$\ddot{v}_1(t) = \frac{q}{m}\dot{v}_2(t)B = -\Big(\frac{qB}{m}\Big)^2 v_1(t), \tag{1.64}$$

$$\ddot{v}_2(t) = -\frac{q}{m}\dot{v}_1(t)B = -\Big(\frac{qB}{m}\Big)^2 v_2(t). \tag{1.65}$$

These are harmonic oscillator equations and solved by

$$v_1(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t), \tag{1.66}$$
$$v_2(t) = A_2 \cos(\omega t) + B_2 \sin(\omega t) \tag{1.67}$$

with angular frequency

$$\boxed{\omega = \frac{qB}{m}.} \tag{1.68}$$

Only two of the four constants $A_{1,2}$ and $B_{1,2}$ are independent, since we require

$$\dot{v}_1(t) = \frac{qB}{m} v_2(t) \tag{1.69}$$
$$\Leftrightarrow \; -A_1\omega \sin(\omega t) + B_1\omega \cos(\omega t) = \underbrace{\frac{qB}{m}}_{\omega} [A_2 \cos(\omega t) + B_2 \sin(\omega t)] \tag{1.70}$$

for all t. Thus $A_2 = B_1$ and $B_2 = -A_1$ and

$$\vec{v}(t) = A_1 \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \\ 0 \end{pmatrix} + B_1 \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_{3,0} \end{pmatrix}. \tag{1.71}$$

Applying the initial condition $\vec{v}(0) = \vec{v}_0$ we obtain

$$\vec{v}(t) = v_{1,0} \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \\ 0 \end{pmatrix} + v_{2,0} \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_{3,0} \end{pmatrix}. \tag{1.72}$$

Another time-integration yields $\vec{x}(t)$. Assuming for simplicity a particle initially at the origin moving in the x-direction,

$$\vec{x}_0 = 0, \; \vec{v}_0 = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}, \tag{1.73}$$

we find

$$\vec{v}(t) = v \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \\ 0 \end{pmatrix} \tag{1.74}$$

and

$$\vec{x}(t) = \frac{v}{\omega} \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}. \tag{1.75}$$

This describes a particle moving on a circle in the xy-plane with angular frequency $\omega = qB/m$ and radius

$$R = \frac{v}{\omega} = \frac{mv}{qB}. \tag{1.76}$$

The sign of q determined whether the motion is clockwise or counter-clockwise.

We could have obtained this result quicker by realizing that the Lorentz force of magnitude qvB is the centripetal force of that holds the particle on a circular orbit, hence

$$\frac{mv^2}{R} = qvB \;\Rightarrow\; R = \frac{mv}{qB}. \tag{1.77}$$

If the particle has an initial z-component in $\vec{v}_0$, it will continue the uniform motion in z-direction and follow a spiral trajectory.

1.3 Coulomb force and units

Experimentally, the force between two pointlike charges q_1 and q_2 is found to decay with the inverse square law

$$F = k\,\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}, \tag{1.78}$$

where $|\vec{r}_1 - \vec{r}_2|$ is the distance between the charges and $k > 0$ is a positive constant. The force acts along the vector $\hat{\vec{e}}$ connecting the charges, so

$$\vec{F} = k\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}\underbrace{\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}}_{\hat{\vec{e}}} = kq_1q_2\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}. \tag{1.79}$$

The SI unit of charge is the Coulomb (C). Since 2019, it is defined such that the electron charge is fixed as

$$e = 1.602\ 176\ 634 \times 10^{-19}\ \mathrm{C}. \tag{1.80}$$

This determines all other SI units like Ampere A = C/s, Volt V = W/A = J/As, Ohm, Ω = V/A, Farad F = C/V, or Tesla T = Vs/m^2. Since the equation $F = kq_1q_2/|\vec{r}_1 - \vec{r}_2|^2$ basically defines what a charge is, we could give charge any other unit by changing the parameter k. There are two major choices:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}q_1q_2\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}\ \text{(SI units)}, \tag{1.81}$$

$$\vec{F} = q_1q_2\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}\ \text{(Gauss units)}, \tag{1.82}$$

corresponding to $k = (4\pi\varepsilon_0)^{-1}$ and $k = 1$. Here

$$\varepsilon_0 = 8.8854 \times 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}} \tag{1.83}$$

is the vacuum permittivity. Gauss units are one of (several) cgs systems, which use cm, g, and s as units for length, mass, and time. The issue, however, is not that we need to convert cm and g into the SI units m and kg, which would be trivial, but that there are non-unity constants like k involved in the conversion. Since the purely mechanical LHS of

$$m\ddot{\vec{x}} = \vec{F} \tag{1.84}$$

does not know anything about electrodynamics or choice of units, so does the force:

$$F_{\mathrm{G}} = F_{\mathrm{SI}}. \tag{1.85}$$

As a consequence, a charge q_{SI} expressed in SI units corresponds to a charge q_{G} in Gauss units through

$$q_{\mathrm{G}} = \frac{q_{\mathrm{SI}}}{\sqrt{4\pi\varepsilon_0}}. \tag{1.86}$$

Conversion of E- and B-field. We can read the equation for $\vec{F}$ as the force on a test charge $q_1 = q$ at position $\vec{r} = \vec{r}_1$ in the electric field created by the another charge q_2 at position $\vec{r}' = \vec{r}_2$. The corresponding Lorentz force on q is

$$\vec{F} = q\vec{E}(\vec{r}_1), \tag{1.87}$$

where the electric field created by the second charge is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} q_2 \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \text{ (SI units)}, \tag{1.88}$$

$$\vec{E}(\vec{r}) = q_2 \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \text{ (Gauss units)}. \tag{1.89}$$

Consequently,

$$E_{\mathrm{SI}} = \frac{1}{4\pi\varepsilon_0} q_{2,\mathrm{SI}} \frac{1}{r^2} = \frac{1}{\sqrt{4\pi\varepsilon_0}} q_{2,\mathrm{G}} \frac{1}{r^2} = \frac{1}{\sqrt{4\pi\varepsilon_0}} E_{\mathrm{G}}, \tag{1.90}$$

and so

$$E_{\mathrm{G}} = \sqrt{4\pi\varepsilon_0} E_{\mathrm{SI}}. \tag{1.91}$$

Now remember the full the Lorentz force relation

$$\vec{F} = q\Big(\vec{E} + \vec{v} \times \vec{B}\Big) \text{ (SI units)}, \tag{1.92}$$

$$\vec{F} = q\Big(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\Big) \text{ (Gauss units)}. \tag{1.93}$$

Since $\vec{v}/c$ is dimensionless, E- and B-field have the same dimension in Gauss units.

This is one of the main feats of Gauss units, because E- and B-fields are essentially the same because of special relativity. In the SI system, E- and B-field have different units: The SI unit of E is

$$[E] = \frac{\mathrm{V}}{\mathrm{m}}, \tag{1.94}$$

whereas the SI unit of B is the Tesla

$$[B] = T = \frac{\mathrm{Vs}}{\mathrm{m}^2} = \frac{[E]}{\mathrm{m/s}}. \tag{1.95}$$

Remember that cB has the same dimension as E in SI units.

We have

$$q_{\mathrm{SI}} v B_{\mathrm{SI}} = F_{\mathrm{SI}} = F_{\mathrm{G}} = q_{\mathrm{G}} \frac{v}{c} B_{\mathrm{G}} = \frac{q_{\mathrm{SI}}}{\sqrt{4\pi\varepsilon_0}} \frac{v}{c} B_{\mathrm{G}}, \tag{1.96}$$

and thus

$$B_{\mathrm{G}} = \sqrt{4\pi\varepsilon_0}\; c\; B_{\mathrm{SI}}. \tag{1.97}$$

We note here that the exact relation

$$\varepsilon_0 \mu_0 = \frac{1}{c^2} \tag{1.98}$$

with vacuum permeability

$$\mu_0 = 1.257 \times 10^{-6} \frac{\text{V s}}{\text{A m}}, \tag{1.99}$$

which we will derive later, implies

$$B_{\text{G}} = \sqrt{4\pi\varepsilon_0 c^2} B_{\text{SI}} = \sqrt{\frac{4\pi}{\mu_0}} B_{\text{SI}}. \tag{1.100}$$

The situation outlined here becomes slightly more fiddly when studying electrodynamics in matter, where several additional changes are made to the equations.

My recommendation. Do not worry too much about all of this. Most results in electrodynamics follow (at least schematically) in a few lines from the Maxwell equations and the Lorentz force relation. If you know these central equations in both SI and Gauss units, then you can always re-derive formulas from scratch in the unit system that you need. When applying formulas that you find in the scientific literature, always make sure that you know whether they are in SI or Gauss units. Typically, if an ε_0 or μ_0 appears in the equation, it is in SI units. If E and B have the same dimension, it is probably in Gauss units (unless $c = 1$).

In the following, we will continue with SI units. Only much later in the lecture, when we discuss special relativity and the covariant formalism, will we move to Gauss units.

1.4 Electric potential and voltage

The electric potential has many application and appears in our daily live through the notion of voltage. When you hear the word potential, think potential energy. The two quantities are almost identical: The potential energy of a test charge q at position $\vec{r}$ in an electric potential $\phi(\vec{r})$ is

$$\boxed{E_{\text{pot}}(\vec{r}) = q\phi(\vec{r}).} \tag{1.101}$$

The potential $\phi(\vec{r})$ is independent of the test charge.

Gravitational analogy. An analogy to the gravitational potential (which also has an inverse-square force law) might help to build some intuition. Consider a test mass m in the gravitational field of the Earth, M, with Earth being placed at the origin. The force on m is

$$F(\vec{r}) = -mGM\frac{\vec{r}}{r^3}. \tag{1.102}$$

The potential energy of m at position $\vec{r}$ is

$$E_{\text{pot}}(\vec{r}) = m\phi(\vec{r}), \tag{1.103}$$

where the gravitational potential created by the Earth is

$$\phi(\vec{r}) = -\frac{MG}{r} + \phi_0. \tag{1.104}$$

Here ϕ_0 is an irrelevant additive constant (resulting in a constant shift of energy). Indeed, if an agent wants to lift the test mass m from a radial distance r_1 to a larger distance $r_2 > r_1$, the agent needs to supply the work

$$\begin{aligned} W_{1\to 2} &= -\int_{r_1}^{r_2} \text{d}\vec{r} \cdot \vec{F}(\vec{r}) = -\int_{r_1}^{r_2} \text{d}r\ F(\vec{r}) &(1.105)\\ &= mGM \int_{r_1}^{r_2} \frac{\text{d}r}{r^2} = mGM \frac{-1}{r}\Big|_{r=r_1}^{r_2} &(1.106)\\ &= mGM \underbrace{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}_{>0} = E_{\text{pot}}(r_2) - E_{\text{pot}}(r_1) > 0. &(1.107) \end{aligned}$$

(Minus sign in $-F$ because gravity pulls the mass down, and the agent has to act against this force.) This number is positive, because the test mass has more potential energy at r_2 than at r_1. Indeed, for $r = R + z$ with $z \ll R$ the height of the test mass above Earth's surface, and inserting the values $M = 6 \times 10^{24}$kg, $R = 6.4 \times 10^6$m, and $G = 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ for Earth, we get the familiar

$$E_{\mathrm{pot}}(z) = \frac{-mGM}{R+z} = \frac{-mGM}{R(1+z/R)} \simeq \frac{-mGM}{R}\Big(1 - \frac{z}{R}\Big) = mgz + E_0 \tag{1.108}$$

with

$$g = \frac{MG}{R^2} = 9.8\ \frac{\mathrm{m}}{\mathrm{s}^2}. \tag{1.109}$$

Conservative forces. The electric potential $\phi(\vec{r})$ exists because the Coulomb force is a conservative force. This means that the work supplied by an agent in moving a charge from $\vec{r}_1$ to $\vec{r}_2$,

$$W_{1\to2} = -\int_{\vec{r}_1}^{\vec{r}_2} \mathrm{d}\vec{r}\cdot\vec{F}(\vec{r}), \tag{1.110}$$

is independent of the path taken to connect $\vec{r}_1$ with $\vec{r}_2$ (more on line integrals later). This is equivalent to

$$\mathrm{curl}\ \vec{F}(\vec{r}) = 0 \tag{1.111}$$

for all $\vec{r}$. More precisely, if Ω is a simply connected region of space that contains $\vec{r}_1$ and $\vec{r}_2$, then we require curl $\vec{F}(\vec{r}) = 0$ for all $\vec{r} \in \Omega$. We then loosely say that $\vec{F}(\vec{r})$ is eddy-free. Any eddy-free vector field can be written as

$$\vec{F}(\vec{r}) = \nabla h(\vec{r}) \tag{1.112}$$

with some scalar function $h(\vec{r})$. Indeed, this immediately implies

$$\mathrm{curl}\ \vec{F} = \nabla\times\nabla h = \begin{pmatrix}\partial_1\\ \partial_2\\ \partial_2\end{pmatrix} \times \begin{pmatrix}\partial_1 h\\ \partial_2 h\\ \partial_3 h\end{pmatrix} = \begin{pmatrix}\partial_2\partial_3 h - \partial_3\partial_2 h\\ \partial_3\partial_1 h - \partial_1\partial_3 h\\ \partial_1\partial_2 h - \partial_2\partial_1 h\end{pmatrix} = 0, \tag{1.113}$$

because second derivatives commute. The function $h(\vec{r})$ is given by the line integral

$$h(\vec{r}) = \int_{\vec{r}_0}^{\vec{r}} \mathrm{d}\vec{s}\cdot\vec{F}(\vec{s}), \tag{1.114}$$

where the path connects an arbitrary start point $\vec{r}_0$ to $\vec{r}$. Choosing a different start point $\vec{r}_0{}'$ only gives a constant shift,

$$h(\vec{r}) = \int_{\vec{r}_0'}^{\vec{r}} \mathrm{d}\vec{s}\cdot\vec{F}(\vec{s}) = \underbrace{\int_{\vec{r}_0'}^{\vec{r}_0} \mathrm{d}\vec{s}\cdot\vec{F}(\vec{s})}_{h_0} + \int_{\vec{r}_0}^{\vec{r}} \mathrm{d}\vec{s}\cdot\vec{F}(\vec{s}), \tag{1.115}$$

which does not affect the relation

$$\vec{F}(\vec{r}) = \nabla h(\vec{r}). \tag{1.116}$$

Electric potential. Consider a charge Q located at the origin, and a test charge q. The force on the test charge at position $\vec{r}$ is

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r}) = q\frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}. \tag{1.117}$$

In the spirit of potential vs. potential energy, we remove the test charge q from the discussion and consider the electric field created by Q,

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}. \tag{1.118}$$

The E-field is, of course, also conservative. Hence there is and electric potential $\phi(\vec{r})$ such that

$$\boxed{\vec{E}(\vec{r}) = -\nabla\phi(\vec{r}).} \tag{1.119}$$

The minus sign is convention. We have

$$\boxed{\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} + \phi_0.} \tag{1.120}$$

Indeed,

$$-\nabla\phi(\vec{r}) = -\frac{Q}{4\pi\varepsilon_0}\nabla\frac{1}{r} \tag{1.121}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\Big(\frac{-1}{r^2}\Big)\vec{e}_r \tag{1.122}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}. \tag{1.123}$$

The potential energy of the test charge q at position $\vec{r}$ is

$$E_{\rm pot}(\vec{r}) = q\phi(\vec{r}) = \frac{qQ}{4\pi\varepsilon_0}\frac{1}{r} + E_0. \tag{1.124}$$

To convince yourself that the sign is right: Assume q and Q have the same sign, so that they repel each other. Having them close together (small r) creates a lot of potential energy for q. Indeed, $E_{\rm pot}(\vec{r})$ is a positive function that decreases with r. The electric potential is often defined as

$$\boxed{\phi(\vec{r}) = -\int_\infty^{\vec{r}} \mathrm{d}\vec{s}\cdot\vec{E}(\vec{s}),} \tag{1.125}$$

corresponding to $\vec{r}_0 = \infty$, with the normalization $\phi(\infty) = 0$. This means that

$$q\phi(\vec{r}) = -W_{\infty\to\vec{r}} = W_{\vec{r}\to\infty}, \tag{1.126}$$

and so $q\phi(\vec{r})$ is the energy an agent needs to supply (if $W > 0$) or gains (if $W < 0$) to bring the charge from $\vec{r}$ to ∞.

Equipotential surfaces. A surface where $\phi(\vec{r}) = \phi$ is constant for all points $\vec{r}$ on the surface is called an equipotential surface. Electric field lines

$$\vec{E}(\vec{r}) = -\nabla(\vec{r}) \tag{1.127}$$

are perpendicular to equipotential surfaces. Moving a charge along an equipotential surface does not require work.

We can now complete our analogy to the gravitational case:

	Gravity	Electrostatics
Force	$\vec{F} = m\vec{\mathcal{G}}$	$\vec{F} = q\vec{E}$
Field	$\vec{\mathcal{G}} = -MG\frac{\vec{r}}{r^3}$	$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$
Conservative Field	$\vec{\mathcal{G}} = -\nabla\phi_{\mathrm{g}}$	$\vec{E} = -\nabla\phi$
Potential	$\phi_{\mathrm{g}} = -MG\frac{1}{r} + \phi_0$	$\phi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} + \phi_0$
Potential energy	$E_{\mathrm{pot}} = m\phi_{\mathrm{g}}$	$E_{\mathrm{pot}} = q\phi$

Note that by choosing $Q < 0$ and $q > 0$ (attractive case), the analogy becomes perfect.

Application. Earth's electric field.

Voltage. The electric potential at point $\vec{r}$ is

$$\phi(\vec{r}) = -\int_{\infty}^{\vec{r}} \mathrm{d}\vec{s} \cdot \vec{E}(\vec{s}). \tag{1.128}$$

Consider two points $\vec{r}_1$ and $\vec{r}_2$. The voltage between the two points is defined as

$$\boxed{U = \int_{\vec{r}_1}^{\vec{r}_2} \mathrm{d}\vec{s} \cdot \vec{E}(\vec{s}).} \tag{1.129}$$

Hence

$$\boxed{U = \phi(\vec{r}_1) - \phi(\vec{r}_2),} \tag{1.130}$$

which is the *opposite* of the potential difference. Indeed,

$$\phi(\vec{r}_1) - \phi(\vec{r}_2) = -\int_{\infty}^{\vec{r}_1} \mathrm{d}\vec{s} \cdot \vec{E}(\vec{s}) + \int_{\infty}^{\vec{r}_2} \mathrm{d}\vec{s} \cdot \vec{E}(\vec{s}) \tag{1.131}$$

$$= \int_{\vec{r}_1}^{\infty} \mathrm{d}\vec{s} \cdot \vec{E}(\vec{s}) + \int_{\infty}^{\vec{r}_2} \mathrm{d}\vec{s} \cdot \vec{E}(\vec{s}) \tag{1.132}$$

$$= \int_{\vec{r}_1}^{\vec{r}_2} \mathrm{d}\vec{s} \cdot \vec{E}(\vec{s}) = U. \tag{1.133}$$

We have

$$\Delta E_{\mathrm{pot}} = -qU. \tag{1.134}$$

Consider again the potential of a positive charge $Q > 0$ at the origin,

$$\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} \tag{1.135}$$

and compute the voltage between two radial distances $r_1 < r_2$ as

$$U = \phi(r_1) - \phi(r_2) = \frac{Q}{4\pi\varepsilon_0}\underbrace{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}_{>0} > 0. \tag{1.136}$$

Hence U is positive in the radial direction, which is the direction where a positive test charge $q > 0$ would move:

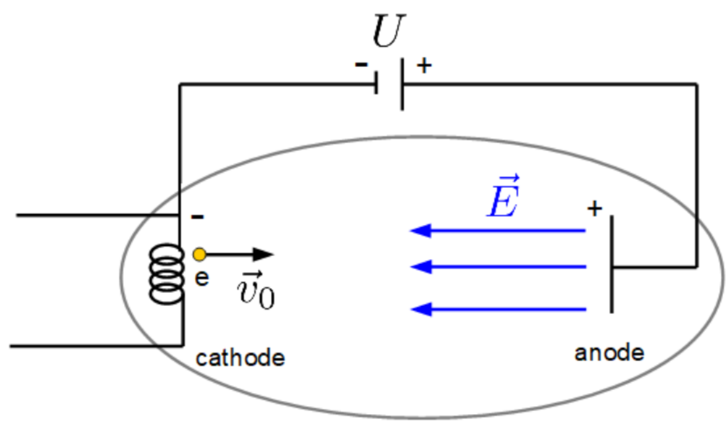
The voltage U is positive in the direction where a positive charge would move.

Of course, U as a scalar has no direction, so we clarify:

The voltage U between two points $\vec{r}_1$ and $\vec{r}_2$ s positive if positive charges would move from $\vec{r}_1$ to $\vec{r}_2$.

The SI unit of voltage is the volt, V. Recall that the SI unit of the E-field is V/m.

Example. Electrons are emitted from a heated cathode in an evacuated tube. The electrons leave the cathode with an average velocity v_0. A large voltage U is applied between the anode and the cathode, which accelerates the emitted electrons. What is the velocity v of the the electrons when they hit the anode?



The initial energy of the electrons ($q = -e < 0$) emitted from the cathode is

$$E_{\mathrm{i}} = E_{\mathrm{kin,i}} + E_{\mathrm{pot,i}} = \frac{m}{2}v_0^2 + q\phi_{\mathrm{c}}, \tag{1.137}$$

whereas the final energy of the electrons at the anode is

$$E_{\mathrm{f}} = E_{\mathrm{kin,f}}(2) + E_{\mathrm{pot,f}} = \frac{m}{2}v^2 + q\phi_{\mathrm{a}}. \tag{1.138}$$

The voltage between cathode and anode is *negative*, because positive charges would go from anode to cathode. Hence

$$U = \phi_{\mathrm{c}} - \phi_{\mathrm{a}} = -|U| < 0. \tag{1.139}$$

We have $E_{\mathrm{i}} = E_{\mathrm{f}}$, and thus

$$\frac{m}{2}v^2 = \frac{m}{2}v_0^2 + q(\phi_{\mathrm{c}} - \phi_{\mathrm{a}}) = \frac{m}{2}v_0^2 + qU = \frac{m}{2}v^2 + e|U|. \tag{1.140}$$

Hence

$$v = \sqrt{v_0^2 + \frac{2e|U|}{m}} \approx \sqrt{\frac{2e|U|}{m}}, \tag{1.141}$$

where we assumed $v_0^2 \ll 2e|U|/m$. For instance, with

$$U = 50\mathrm{V}, \tag{1.142}$$

$$m = 9.1 \times 10^{-31}\ \mathrm{kg}, \tag{1.143}$$

$$e = 1.602 \times 10^{-19}\ \mathrm{C}, \tag{1.144}$$

we get

$$v = 4 \times 10^6 \ \frac{\text{m}}{\text{s}}, \tag{1.145}$$

which is 1.4% of the speed of light.

1.5 Conductors

Conductors play an important role in ED applications. They are materials with one predominant mobile charge carrier. For simplicity, we assume in the following that conductors are metals. There, the negatively charged conduction-band electrons are mobile, while the remaining positively charged ions stay at rest. Note that, as a whole, the conductor is electrically neutral.

In the following, we derive 5 important properties of conductors.

Conductors in electric fields. Assume a conductor in the form of a cube is brought into an external electric field in the x-direction (i.e. E points to the right),

$$\vec{E}_{\text{ext}} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}. \tag{1.146}$$

The electrons with charge $q = -e$ experience the force

$$\vec{F} = -e\vec{E}_{\text{ext}} \tag{1.147}$$

and move to the left, whereas the ions stay at rest. The electrons cannot leave the material and will accumulate on the left boundary of the cube. The boundary side of the cube then has an access of electrons, hence negative surface charge $-Q < 0$, whereas the right boundary has a shortage of electrons, hence a positive surface charge $+Q > 0$. The surface charges create an induced electric field pointing in the negative x-direction that precisely cancels $\vec{E}$:

$$\vec{E}_{\text{ind}} = -\vec{E}_{\text{ext}}. \tag{1.148}$$

The total field inside the conductor vanishes,

$$\vec{E} = \vec{E}_{\text{ext}} + E_{\text{ind}} = 0. \tag{1.149}$$

This is because the surface charges $\pm Q$ will be precisely such that the movement of the negative electrons stops (i.e. Q is neither too small, nor too large). This result is generally valid:

(1) The interior of conductors is free of electric field.

Now assume the conductor is not a cube, but has some funny shape like a sphere or any other 3D geometry. Again, in an electric field $\vec{E}_{\text{ext}}$, the electrons will move to the surfaces such that $\vec{E} = 0$ inside. The charges will arrange such that:

(2) Electric field lines outside the conductor meet the conductor surface perpendicular.

If the E-field had a small component tangential to the surface, it would result in movement of charges on the surface, leading to their re-arrangement such that the assumed tangential component is precisely cancelled. An important corollary to this is:

(3) The surface of a conductor is an equipotential surface, i.e. all points $\vec{R}$ on the surface have the same electric potential $\phi(\vec{R})$.

Excess charges on conductors. A conductor is neutral. If you bring excess charges onto a conductor, the movement of electrons will again be such that the interior is field free, and the net excess charge is on the surface

(4) Excess charges brought onto a conductor are gathered on the conductor surface.

Importantly, the interior is *still field-free* and the E-field lines *still meet the boundary orthogonally*, i.e. (1)-(3) remain true even for charged conducting bodies.

Application: Faraday cage. Consider a conducting body with surface $\mathcal{S} \subset \mathbb{R}^3$. The interior is free of electric field because of the arrangement of induced surface charges. Even if excess charges are added to the conductor, they are restricted to the surface $\mathcal{S}$. Now remove the interior of the conductor, and consider instead only the thin, closed conducting surface $\mathcal{S}$. All the arguments of the arrangement of charges remains valid:

(5) The volume surrounded by a conducting surface $\mathcal{S}$ is free of electric field.

This is the principle of a Faraday cage: to protect any object from external electric fields, just surround it by a conducting surface.

1.6 Electric currents

Stationary currents. Electric currents $I(t)$ are related to charges via

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{1.150}$$

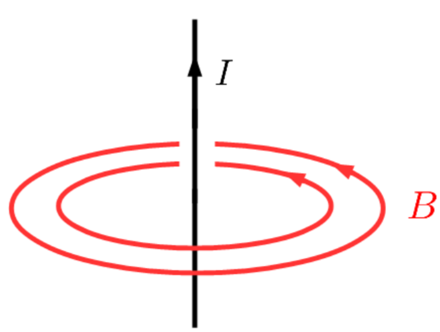
A current is called stationary if $I(t) = I_0$. Direct currents (DC) are stationary, whereas alternating currents (AC) with

$$I(t) = I_0 \cos(\omega t) \tag{1.151}$$

are non-stationary. Moving charges constitute currents. The electric current in electrical circuits is typically made from the mobile electron in conducting wires.

By convention, the direction of electric currents points towards the flow of positive charges. This is opposite to the actual flow of electrons.

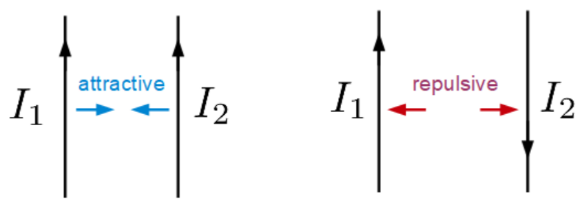
A magnetic field is created around a wire that carries a stationary current I:



The B-field points into the azimuthal direction and forms a "right-hand-system" with I, where the thumb points in the direction of I and the fingers curl into the direction of B. Consider two wires carrying the same currents I at a distance R apart. One at $x = y = 0$ along the z-direction, one at $x = R, y = 0$ along the z-direction. A single electron in the second wire, with $q = -e$, $\vec{v} = -v\vec{e}_z$, experiences the Lorentz force with $\vec{B} = B\vec{e}_y$ as

$$\vec{F} = q\vec{v} \times \vec{B} = B\vec{e}_z \times \vec{e}_y = -evB\vec{e}_x, \tag{1.152}$$

pointing towards the first wire. Hence they attract each other:

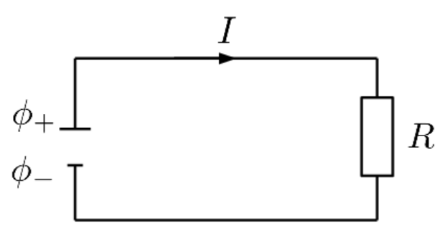


The force per length between two wires at a distance R is

$$\frac{F}{L} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{R}. \tag{1.153}$$

(We later derive this from the Biot–Savart law.) Historically, the SI unit of Ampere (A) was defined as the current $I_1 = I_2 = I = 1$ A such that the attractive force per length at a distance of $R = 1$ m is $F/L = 2 \times 10^{-7}$N/m. The new SI definition was discussed earlier.

Electric circuits. Here is an example of an electric circuit:



The current I flows in the direction of the voltage $U > 0$ with

$$U = \phi_+ - \phi, \tag{1.154}$$

i.e. from ϕ_+ to ϕ_-. We may write

$$\phi_+ = U + \phi_0, \tag{1.155}$$
$$\phi_- = \phi_0. \tag{1.156}$$

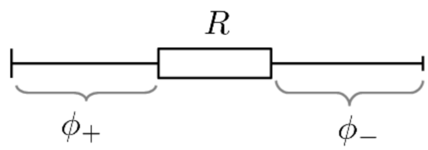
Typically we set the constant $\phi_0 = 0$. Assume that $U = 5$ V, then

$$\phi_+ = 5 \text{ V}, \tag{1.157}$$
$$\phi_- = 0. \tag{1.158}$$

The wires are lines of conductors. Now remember that conductors are equipotential surface.

The electric potential along each conducting wire is constant.

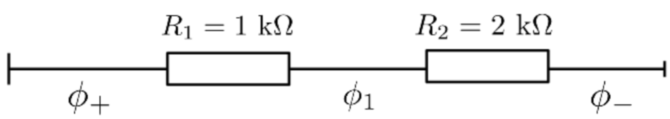
The above circuit is equivalent to:



Hence the potential must change from ϕ_+ on the left connection ("lead") of the resistor R to ϕ_- on the right connection point by the voltage drop

$$U = R\,I. \tag{1.159}$$

For $R = 1$ kΩ, the corresponding current is $I = \frac{5V}{1000\Omega} = 5$ mA. Another example:



First, the voltage has to drop from ϕ_+ to ϕ_1 with $U_1 = \phi_+ - \phi_1 = R_1 I$, and then drop from ϕ_1 to ϕ_- with $U_2 = \phi_1 - \phi_- = R_2 I$. Since

$$U = \phi_+ - \phi_- = (\phi_+ - \phi_1) - (\phi_1 - \phi_-) = U_1 + U_2, \tag{1.160}$$

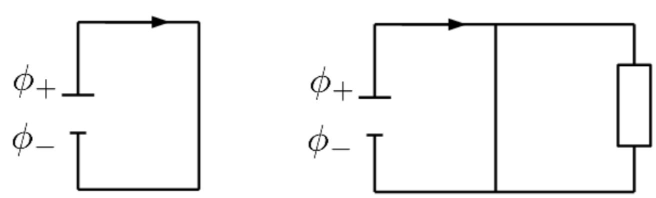
we have

$$U = R_{\text{tot}} I,\ R_{\text{tot}} = R_1 + R_2 \tag{1.161}$$

(resistors in parallel). Plugging in numbers, we have $I = \frac{U}{R_{\text{tot}}} = 1.67$ mA and

$$U_1 = R_1 I = 1.67 \text{ V}, \tag{1.162}$$
$$U_2 = R_2 I = 3.33 \text{ V}. \tag{1.163}$$

Indeed, $U_1 + U_2 = U = 5$V. At last, these are examples of short circuits:



They contain a wire that connects two points of supposedly different potential $\phi_+ \neq \phi_-$. This leads to the flow of an extremely large current that heats up the wire (Joule heating) and typically destroys the circuit.

1.7 Electric and magnetic dipoles

Electric dipoles. An electric dipole consists of two opposite charges, $-Q$ and $+Q$, that are separated by a distance d. We define its electric dipole moment as

$$\vec{p} = Q\vec{d}. \tag{1.164}$$

By convention, the direction $\vec{d}$ of an electric dipole points from the negative to the positive charge.

Choose the position of the charges to be

$$\vec{r}_{\pm} = \pm\frac{\vec{d}}{2} \tag{1.165}$$

so that the center is at the origin. The potential created by the dipole is

$$\phi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\Big(\frac{1}{|\vec{r}-\vec{r}_+|} - \frac{1}{|\vec{r}-\vec{r}_-|}\Big) \tag{1.166}$$

$$= \frac{Q}{4\pi\varepsilon_0}\Big(\frac{1}{|\vec{r}-\vec{d}/2|} - \frac{1}{|\vec{r}+\vec{d}/2|}\Big). \tag{1.167}$$

Using

$$\nabla\frac{1}{|\vec{r}-\vec{r}\,'|} = \nabla\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \tag{1.168}$$

$$= \frac{-\frac{1}{2}}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}\begin{pmatrix}2(x-x')\\2(y-y')\\2(z-z')\end{pmatrix} \tag{1.169}$$

$$= -\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}, \tag{1.170}$$

the electric field is

$$\vec{E}(\vec{r}) = -\nabla\phi(\vec{r}) \tag{1.171}$$

$$= \frac{Q}{4\pi\varepsilon_0}\Big(\frac{\vec{r}-\vec{d}/2}{|\vec{r}-\vec{d}/2|^3} - \frac{\vec{r}+\vec{d}/2}{|\vec{r}+\vec{d}/2|^3}\Big). \tag{1.172}$$

Far away from the dipole, for $r \gg d$, we can use $(1+\epsilon)^\alpha \simeq 1+\alpha\epsilon$ to write

$$\frac{1}{|\vec{r}\pm\vec{d}/2|^3} = \frac{1}{[r^2 \pm 2\vec{r}\cdot\vec{d}+d^2]^{3/2}} \tag{1.173}$$

$$\simeq \frac{1}{r^3}\left(1 \pm \frac{2\vec{r}\cdot\vec{d}}{r^2}\right)^{-3/2} \tag{1.174}$$

$$\simeq \frac{1}{r^3}\left(1 \pm \Big(\frac{-3}{2}\Big)\frac{2\vec{r}\cdot\vec{d}}{r^2}\right) \tag{1.175}$$

$$\simeq \frac{1}{r^3}\left(1 \mp \frac{3\vec{r}\cdot\vec{d}}{r^2}\right). \tag{1.176}$$

We arrive at

$$\vec{E}(\vec{r}) \simeq \frac{Q}{4\pi\varepsilon_0 r^3}\left[\left(\vec{r}-\frac{\vec{d}}{2}\right)\left(1+\frac{3\vec{r}\cdot\vec{d}}{r^2}\right)-\left(\vec{r}+\frac{\vec{d}}{2}\right)\left(1-\frac{3\vec{r}\cdot\vec{d}}{r^2}\right)\right] \tag{1.177}$$

$$= \frac{Q}{4\pi\varepsilon_0 r^3}\left[-2\frac{\vec{d}}{2}+2\vec{r}\frac{3(\vec{r}\cdot\vec{d})}{r^2}+\mathcal{O}\left(\frac{d^2}{r^2}\right)\right] \tag{1.178}$$

$$= \frac{Q}{4\pi\varepsilon_0}\frac{3(\vec{d}\cdot\vec{r})\vec{r}-r^2\vec{d}}{r^5}. \tag{1.179}$$

Write

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{3(\vec{d}\cdot\vec{e}_r)\vec{e}_r-\vec{d}}{r^3} \tag{1.180}$$

to highlight that $E \sim r^{-3}$ at large distance (in contrast to $E \sim r^{-2}$ for point charges).

Force, torque, and potential energy. In an external electric field $\vec{E}(\vec{r})$, the force on a dipole located at the origin is

$$\vec{F} = \vec{F}_+ + \vec{F}_- = Q\Big[\vec{E}(\vec{r}_+) - \vec{E}(\vec{r}_-)\Big]. \tag{1.181}$$

In a homogeneous E-field, where $\vec{E}(\vec{r}) = \vec{E}_0$, there is no force:

$$\boxed{\vec{F}_{\text{hom}} = 0.} \tag{1.182}$$

However, there is a torque,

$$\vec{\tau} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- \tag{1.183}$$

$$= \vec{r}_+ \times Q\vec{E}(\vec{r}_+) - \vec{r}_- \times Q\vec{E}(\vec{r}_-) \tag{1.184}$$

$$= Q(\vec{r}_+ - \vec{r}_-) \times \vec{E}_0 \tag{1.185}$$

$$= Q\vec{d} \times \vec{E}_0, \tag{1.186}$$

thus

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}_0.} \tag{1.187}$$

Since $\vec{E}_0 = -\nabla\phi_{\text{hom}}$ with

$$\phi_{\text{hom}}(\vec{r}) = -\vec{E}_0 \cdot \vec{r} \tag{1.188}$$

for a homogeneous field, the dipole has the potential energy

$$E_{\text{pot}} = Q\phi_{\text{hom}}(\vec{r}_+) - Q\phi_{\text{hom}}(\vec{R}_-) \tag{1.189}$$

$$= -Q\vec{E}_0 \cdot (\vec{r}_+ - \vec{r}_-), \tag{1.190}$$

or

$$\boxed{E_{\text{pot}} = -\vec{p} \cdot \vec{E}_0.} \tag{1.191}$$

The potential energy is minimal when the electric dipole is anti-parallel to the electric field. The nonzero torque will rotate it into this direction.

In an inhomogeneous field $\vec{E}(\vec{r}) \neq \vec{E}_0$, there is a nonzero force. We have

$$F_i = Q\Big[E_i(\vec{r}_+) - E_i(\vec{r}_-)\Big] \tag{1.192}$$

$$= Q\Big[E_i\Big(\frac{\vec{d}}{2}\Big) - E_i\Big(-\frac{\vec{d}}{2}\Big)\Big] \tag{1.193}$$

$$= Q\Big[\Big(\frac{\vec{d}}{2}\cdot\nabla\Big)E_i(\vec{r}) - \Big(-\frac{\vec{d}}{2}\cdot\nabla\Big)E_i(\vec{r})\Big]_{\vec{r}=0} + \dots \tag{1.194}$$

$$= Q(\vec{d}\cdot\nabla)E_i(\vec{r})\Big|_{\vec{r}=0} + \dots, \tag{1.195}$$

hence

$$\boxed{\vec{F} = (\vec{p}\cdot\nabla)\vec{E}(\vec{r})\Big|_{\vec{r}=0}.} \tag{1.196}$$

Magnetic dipoles. Permanent magnets such as compass needles are solids that have a North (N) and a South (S) pole. Experimentally, N-N and S-S repel each other by a force, whereas N-S attract each other. We say the magnet has a magnetic dipole moment $\vec{m}$, which is the analogue to the charge Q in the Coulomb force.

The B-field lines outside a permanent magnet look strikingly similar to the E-field lines outside an electric dipole. But there are two *crucial differences*:

(1) If we cut an electric dipole in the middle, we obtain two separates charges or monopoles $+Q$ and $-Q$. If you break a magnetic dipole, we obtain two new magnetic dipoles instead.

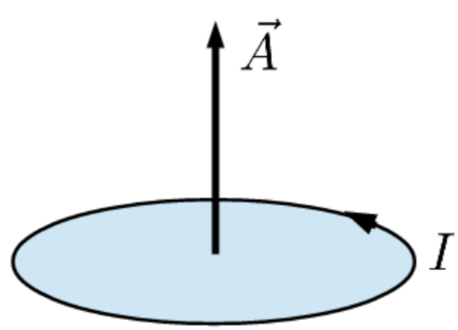
(2) The E-field lines in an electric dipole *always* point from the positive to the negative charge. They start and end at the charges, which constitute monopoles. The B-field lines of magnetic dipoles continue inside the dipole and are *always closed lines.* There are no sources and sinks of magnetic field, i.e. there are *no magnetic monopoles.*

To visualize the continuous magnetic field lines inside the magnet, we can put the magnet under a plastic glass surface and scatter little magnetic particles or compass needles onto the surface. They will align the with B-field lines.

A good way to think about magnetic dipoles are electromagnets. The magnetic moment of a conductor loop of area A carrying a current I is the product

$$m = I\ A. \tag{1.197}$$

Define the surface normal vector $\vec{n}$ (with $\vec{n}^2 = 1$) such that the current I flows as in this sketch:



Then $\vec{A} = A\vec{n}$ denotes the area vector and

$$\boxed{\vec{m} = I \ \vec{A}.} \tag{1.198}$$

The magnetic moment of a conducting coil with N windings is accordingly

$$\vec{m} = N \ I \ \vec{A} \tag{1.199}$$

In a homogeneous external magnetic field $\vec{B}(\vec{r}) = \vec{B}_0$, a magnetic dipole experiences the torque

$$\boxed{\vec{\tau} = \vec{m} \times \vec{B}_0} \tag{1.200}$$

and has potential energy

$$\boxed{E_{\text{pot}} = -\vec{m} \cdot \vec{B}_0,} \tag{1.201}$$

fully analogous to electric dipoles.

Example. Earth's magnetic field.