Filtered Frobenius algebras in monoidal categories

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(joint work with Chelsea Walton)

Motivation

- Filtered vector spaces
$$F_A(0) \subset F_A(1) \subset \cdots \subset F_A(i) \subset \cdots \subset F_A(n) = A$$
- Associated graded vector space $\mathrm{gr}(A)$ with
$$\mathrm{gr}(A)_i = F_A(i)/F_A(i-1)$$

Example

$\mathrm{Cl}(V, B)$ with associated graded $\Lambda(V)$
$\mathcal{U}(\mathfrak{g}, [,])$ with associated graded $\mathrm{Sym}(\mathfrak{g})$

- What properties of the associated graded algebra lift to its filtered deformations?
- Integral domain, Noetherian, prime, etc.

Definition (Frobenius algebra)

It is a tuple $(A, m, u, \varepsilon : A \to \Bbbk)$ such that $(A, m, u) \in \mathrm{Alg}_{\Bbbk}$ and $\ker(\varepsilon)$ does not contain any non-trivial left ideal.

Example

Take a f.d. vector space V with basis $(e_i)_{i=1}^{n}$, then $\Lambda(V)$ is Frobenius with $\varepsilon(a) =$coefficient of $e_1 \wedge \cdots \wedge e_n$ in a.

Connection to TQFTs

$\mathrm{CommFrobAlg}(\mathrm{Vec}_{\Bbbk}) \leftrightarrow 2d\text{-}\mathrm{TQFT}(\mathrm{Vec}_{\Bbbk})$

Theorem (Bongale 1967)

Let A be a finite-dimensional, connected, filtered $\Bbbk$-algebra. If the associated graded algebra of A is Frobenius, then so is A.

Definition (Monoidal category)

It is a tuple $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, $\mathbb{1} \in \mathrm{Obj}(\mathcal{C})$ ('unit' object) and α, l, r are natural isomorphisms satisfying coherence (pentagon and triangle) axioms.

Example

$(\mathrm{Vec}_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk), (\mathrm{Set}, \times, \{a\})$

Why study Frobenius algebras in monoidal categories?

- Conformal Field Theory (work of Fuchs, Runkel, Schweigert, etc.)
- Computer Science (work of Abramsky, Coecke, Vicary, etc.)
- Classification of subfactors (work of Müger, Jones, Snyder etc.)
- TQFTs

GOAL: Generalize Bongale's result to abelian monoidal categories.

Outline

1. Associated graded constructions
2. Frobenius algebras
3. Main Result
4. Further directions

Filtered and Graded categories

Previous works that have prompted this framework include Ardizzoni-Menini 2012, Galatius-Kupers-Randal–Williams 2018, Haugseng-Miller 2016.

- $\mathbb{N}_0$: set of natural numbers including 0.
- $\underline{\mathbb{N}_0}$: category with objects $\mathbb{N}_0$ and with only identity morphisms id_i for all $i \in \mathbb{N}_0$.
- $\underrightarrow{\mathbb{N}_0}$: category with objects $\mathbb{N}_0$ and with morphisms $i \to j$ only for $i, j \in \mathbb{N}_0$ with $i \leq j$.
- From here on $\mathcal{C}$ will always be an abelian category.

Definition ($(\mathbb{N}_0\text{-})\mathrm{Fil}(\mathcal{C})$)

- A *filtration* on $X \in \mathrm{Ob}(\mathcal{C})$ is a functor $F_X \in \mathrm{Fun}(\underrightarrow{\mathbb{N}_0}, \mathcal{C})$ such that $\mathrm{colim}_i(F_X(i)) \cong X$.
- A *filtered object* is a pair (X, F_X) such that $X \in \mathrm{Ob}(\mathcal{C})$ and F_X is a filtration on X.
- A *filtered morphism* $(f, F_f) : (X, F_X)$ and (Y, F_Y) is a tuple (f, F_f) such that $f : X \to Y$ is a morphism in $\mathcal{C}$ and
$$F_f = \{F_f(i) : F_X(i) \to F_Y(i)\}_{i \in \mathbb{N}_0}$$
is a natural transformation such that $\mathrm{colim}_i(F_f(i)) = f$.
- Filtered objects in $\mathcal{C}$ and their morphisms form a category, denoted by Fil(C).

In a similar manner, we can define the category $\mathrm{Gr}(\mathcal{C})$ of graded objects by replacing $\underrightarrow{\mathbb{N}_0}$ by $\underline{\mathbb{N}_0}$ in the definition of $\mathrm{Fil}(\mathcal{C})$.

Monoidal structure

Let $\mathcal{C}$ be a monoidal category with $\otimes$ biexact.

$\otimes$ structure on $\mathsf{Fil}(\mathcal{C})$

For $(X, F_X), (Y, F_Y) \in \mathsf{Ob}(\mathsf{Fil}(\mathcal{C}))$, define

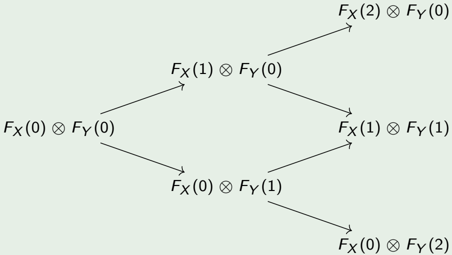
$$(X, F_X) \otimes (Y, F_Y) = (X \otimes Y, F_{X \otimes Y}),$$

where $F_{X\otimes Y}(k) := \mathrm{colim}_{i+j \leq k} F_X(i) \otimes F_Y(j)$.

The unit object is $(\mathbb{1}, F_{\mathbb{1}})$ with the associated filtration $F_{\mathbb{1}} : \underrightarrow{\mathbb{N}_0} \to \mathcal{C}$ given by $F_{\mathbb{1}}(i) = \mathbb{1}$ for all $i \in \mathbb{N}_0$.

Example

$F_{X \otimes Y}(2)$ is the colimit of the following diagram in $\mathcal{C}$.



In a similar manner, we define a monoidal structure on the category $\mathrm{Gr}(\mathcal{C})$.

Associated graded functor gr

Definition (gr)

- For $(X, F_X) \in \mathrm{Fil}(\mathcal{C})$, let $\overline{F_X(i)} = \mathrm{coker}(F_X(i) \to F_X(i-1))$. Define
$$\mathrm{gr}(X, F_X) = \textstyle\coprod_{i \in \mathbb{N}_0} \overline{F_A(i)}.$$
- Given $f : (X, F_X) \to (Y, F_Y)$ in $\mathrm{Fil}(\mathcal{C})$, define
$$\mathrm{gr}(f) : \mathrm{gr}(X, F_X) \to \mathrm{gr}(Y, F_Y)$$
with components coming from the universal property of cokernels:
$$\begin{array}{ccccccc}
F_X(i-1) & \xrightarrow{\iota^X_{i-1}} & F_X(i) & \xrightarrow{\pi^X_i} & \overline{F_X(i)} & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle F_f(i-1)} & & \Big\downarrow{\scriptstyle F_f(i)} & & \vdots\,{\scriptstyle \mathrm{gr}(f)_i} & & \\
F_Y(i-1) & \xrightarrow[\iota^Y_{i-1}]{} & F_Y(i) & \xrightarrow[\pi^Y_i]{} & \overline{F_Y(i)} & \longrightarrow & 0
\end{array}$$

gr is monoidal

Theorem 1 (Walton-Y. 2021)

Let $\mathcal{C}$ be an abelian monoidal category with $\otimes$ biexact, then
$$\mathrm{gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$$
is a monoidal functor.

Thus, we can define associated graded algebras (modules, ideals) in a monoidal category.

Characterizations of Frobenius algebras in $(\mathcal{C}, \otimes)$

Theorem (folklore)

Let $\mathcal{C}$ be a monoidal category and (A, m, u) be an algebra in $\mathcal{C}$. Then the following are equivalent definitions of a Frobenius algebra:

- *There exist morphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{1}$ in $\mathcal{C}$ such that (A, Δ, ε) is a coalgebra in $\mathcal{C}$ and*

$$(id_A \otimes m)(\Delta \otimes id_A) = \Delta\, m = (m \otimes id_A)(id_A \otimes \Delta).$$

- *There exist morphisms $p : A \otimes A \to \mathbb{1}$ and $q : \mathbb{1} \to A \otimes A$ in $\mathcal{C}$ such that*

$$p(m \otimes \mathrm{id}_A) = p(\mathrm{id}_A \otimes m),$$

$$(p \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes q) = \mathrm{id}_A = (\mathrm{id}_A \otimes p)(q \otimes \mathrm{id}_A).$$

Another characterization

Theorem (Fuchs-Stigner 2008)

Consider an algebra (A, m, u) in a rigid monoidal category $\mathcal{C}$. The following are equivalent:

- *A is a Frobenius algebra.*
- *There exists an isomorphism $\Phi_l : A \to {}^*A$ of left A-modules in $\mathcal{C}$, with left A-action maps $\lambda_A = m$ and*

$$\lambda_{^*A} = (\mathrm{id}_{^*A} \otimes \mathrm{ev}'_A)(\mathrm{id}_{^*A} \otimes m \otimes \mathrm{id}_{^*A})(\mathrm{coev}'_A \otimes \mathrm{id}_{A \otimes {}^*A}).$$

- *There exists an isomorphism $\Phi_r : A \to A^*$ of right A-modules in $\mathcal{C}$, with right A-action maps $\rho_A = m$ and*

$$\rho_{A^*} = (\mathrm{ev}_A \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^*} \otimes m \otimes \mathrm{id}_{A^*})(\mathrm{id}_{A^* \otimes A} \otimes \mathrm{coev}_A).$$

A new equivalent characterization

Definition

Consider an algebra (A, m, u) in monoidal category $\mathcal{C}$. A weak left ideal is a tuple $(I, \phi_I : I \to A, \lambda_I : A \otimes I \to I)$ satisfying:

$$\lambda_I\,(m \otimes \mathrm{id}_I) = \lambda\,(\mathrm{id}_A \otimes \lambda) \quad , \quad \lambda_I(u \otimes \mathrm{id}_I) = \mathrm{id}_I,$$

$$\phi_I\,\lambda_I = m\,(\mathrm{id}_A \otimes \phi_I).$$

Theorem 2 (Walton-Y. 2021)

Let $\mathcal{C}$ be an abelian, rigid, monoidal category. An algebra (A, m, u) in $\mathcal{C}$ is Frobenius if and only if there exists a morphism $\nu : A \to \mathbb{1}$ in $\mathcal{C}$ so that, if a left or a right weak ideal (I, ϕ_I) of A factors through $\ker(\nu)$, then ϕ_I is a zero morphism in $\mathcal{C}$.

Main result

Theorem 3 (Walton-Y. 2021)

Let $\mathcal{C}$ be an abelian rigid monoidal category, and let A be a connected filtered algebra in $\mathcal{C}$ with finite monic filtration. If the associated graded algebra $\mathrm{gr}(A)$ is a Frobenius algebra in $\mathcal{C}$, then so is A.

Proof sketch: Since A has finite filtration $\exists n \in \mathbb{N}$ such $F_A(n) = F_A(n+k)$ for all $k > 0$. Then $A = F_A(n)$.

- A connected $\Rightarrow$ $\mathrm{gr}(A)$ connected $\Rightarrow$ $\mathrm{gr}(A)_n = \overline{F_A(n)} = \mathbb{1}$.
- Consider $\nu : A \to \overline{F_A(n)} = \mathbb{1}$. By Theorem 2, it suffices to show that no ideal of A factors through $\ker(\nu) = F_A(n-1)$.
- Suppose not, get an I. By Theorem 1, $\mathrm{gr}(I)$ is a weak left ideal of $\mathrm{gr}(A)$ that factors through the kernel of a Frobenius form on it.
- Obtain contradiction by Theorem 2.

Theorem (Bongale 1967)

Let A be a finite-dimensional, connected, filtered $\Bbbk$-algebra. If the associated graded algebra of A is Frobenius, then so is A.

Theorem 3 (Walton-Y. 2021)

Let $\mathcal{C}$ be an abelian, rigid monoidal category , and let A be a connected filtered algebra in $\mathcal{C}$ with finite monic filtration. If the associated graded algebra $\mathrm{gr}(A)$ is a Frobenius algebra in $\mathcal{C}$, then so is A.

Further directions

1. Can Theorem 3 be obtained via the means of a Frobenius monoidal functor $\mathsf{Gr}(\mathcal{C}) \to \mathsf{Fil}(\mathcal{C})$?
2. Can we get rid of the connectedness assumption?
3. Let H be a filtered bialgebra in a braided tensor category $\mathcal{C}$ such that its associated graded algebra is a Hopf algebras. When is then H a Hopf algebra?

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