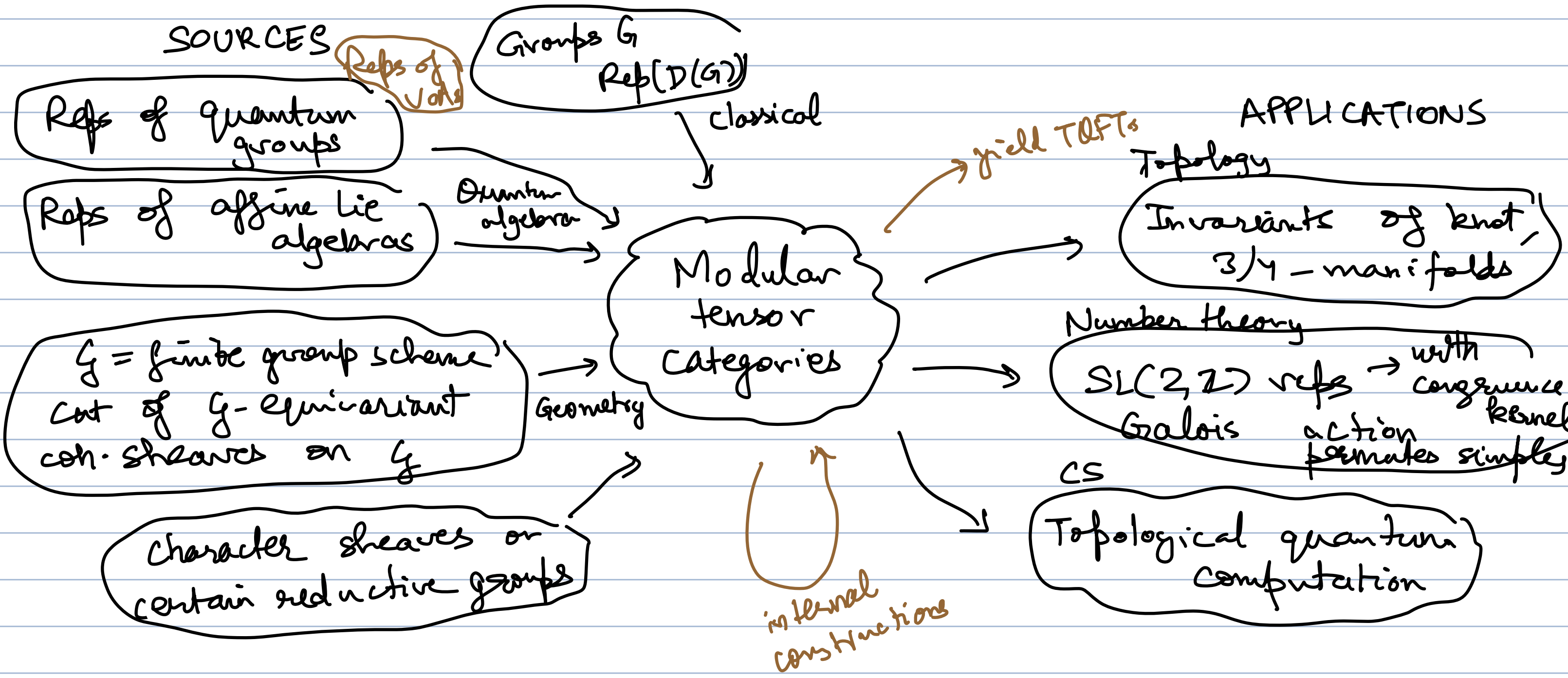
TIFR (Jan 19 2026)

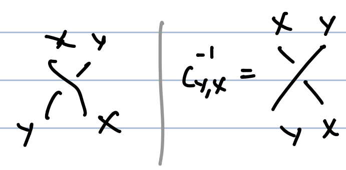
Modular tensor categories (beyond finite & semisimple)



What is a 'modular tensor category'?

It is category $\mathcal{C}$ which

braided tensor category
- $\Bbbk$-Linear, abelian, locally finite
- braided monoidal $(\mathcal{C}, \otimes, \mathbb{1}, c)$
  $c_{X,Y}: X\otimes Y \xrightarrow{\sim} Y\otimes X$



- rigid ($\exists$ dual objects)
- non-degenerate (if X satisfies  $= id_{X\otimes Y}$ $\forall Y \in \mathcal{C}$ then $X = \mathbb{1}$)
- ribbon

We call $\mathcal{C}$
- finite if $\mathcal{C} \simeq Rep^{fd}(A)$ for a f.d. alg. A
- semisimple if every object is direct sum of irreducibles.

Examples:

① $\mathcal{C} = (Vect_{\Bbbk}, \otimes_{\Bbbk}, \Bbbk, c = swap)$

② $\mathcal{C} = Vect_{\mathbb{Z}/2}$ = $\mathbb{Z}/2$-graded vector spaces

two simple $\mathbb{1} = (\Bbbk)_0$ and $X = 0 \oplus \Bbbk = (\Bbbk)_1$
$X \otimes X \simeq \mathbb{1}$

braiding $c_{X,X}: X\otimes X \xrightarrow{-1} X \otimes X$ (other braidings are identity)

③ $\mathcal{C} = Rep(G)$ is braided tensor but not nondegenerate.

MTC from Quantum groups

μ_∞ = roots of unity

	semisimple	non-semisimple
finite	$\overline{\mathrm{Tilt}(U_q(sl_2))}$ certain $q \in \mu_\infty$	$\mathrm{Rep}(u_q(sl_2))$ certain $q \in \mu_\infty$ small quantum group
infinite	$\mathrm{Rep}^1(U_q(sl_2))$ $q \notin \mu_\infty$	$\mathrm{Rep}(u_q^H(sl_2))$ certain $q \in \mu_\infty$ unrolled small quantum group

(*) The above sources and applications of MTCs are only well understood for finite & semisimple MTCs.

"For sources, even basic questions like existence of duals become subtle."

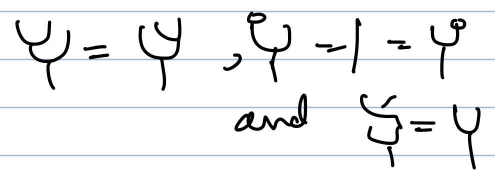
I am interested in studying constructions of MTCs in infinite &/or non semisimple MTC

MAIN BOTTLENECK $\to$ rigidity

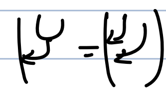
HOW TO OBTAIN NEW MTCs?

$\hookrightarrow$ Local modules construction

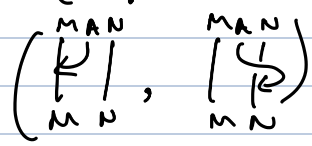
- $\mathcal{C}$ = braided tensor category.
  A = commutative algebra in $\mathcal{C}$ $\quad (A, m, u)$

s.t. 

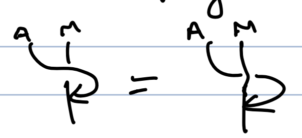
and

using these we can form
$\mathcal{C}_A$ = category of right A-modules $(M, \triangleleft)$ s.t. 

A commutative $\Rightarrow$ $\mathcal{C}_A$ is monoidal with $(M, \triangleleft) \otimes (N, \triangleleft)$

$= \mathrm{Coeq}$ 

but $\mathcal{C}_A$ is not braided in general

consider subcat $\mathcal{C}_A^{loc} = \{ (M, \triangleleft) \mid$  $\}$

then $\mathcal{C}_A^{loc}$ is a braided tensor category.

Eg: $\mathcal{C} = \mathrm{Rep}(G)$ is a Braided tensor category.
For $H \subset G$, $A = \mathrm{Fun}(G/H)$ is a comm. algebra in $\mathcal{C}$.
$\mathcal{C}$ is symmetric.
so $\mathcal{C}_A = \mathcal{C}_A^{loc} \simeq \mathrm{Rep}(H)$

Q When is $\mathcal{C}_A^{loc}$ a MTC?

- For finite semisimple MTC, answer due to Kirillov-Ostrik

↓ we generalize it

Thm: (Shimizu-Y.) Let $\mathcal{C}$ be a finite MTC (char($\mathbb{k}$) is arbitrary) and A a simple comm. symm. Frobenius alg. Then $\mathcal{C}_A^{loc}$ is a finite MTC.

(A comm. alg is called symmetric Frobenius if it admits a map $\chi : A \to \mathbb{1}$ so that $\underset{\chi}{\Upsilon}$ is nondeg. pairing)

↓ going to further general setting of infinite & non-semisimple, we consider infinite algebra $A \in \mathrm{Ind}(\mathcal{C})$.
→ we give conditions ensuring $\mathrm{fg}\text{-}\mathrm{Ind}(\mathcal{C})_A^{loc}$ is rigid, ribbon, etc.

$\mathrm{Ind}(\mathrm{Vec}^{fd}) = \mathrm{Vec}$

$\mathrm{Ind}(\mathrm{corep}(C)) = \mathrm{Corep}(C)$

An A-module M is f.g. if $\exists\, X \in \mathcal{C}$ st. $X \otimes A \twoheadrightarrow M$

"Thm (Shimizu-Y.)" Let $\mathcal{C}$ be a MTC and A be a comm. alg. such that $A = \bigoplus_i X_i \in \mathrm{Ind}(\mathcal{C})$ is an infinite algebra where each X_i is one dimensional.
Then $\mathrm{fg}\text{-}\mathrm{Ind}(\mathcal{C})_A^{loc}$ is rigid and ribbon

Example: $u_q^H(sl_2) = \dfrac{\langle E, F, K^{\pm}, H \rangle}{\left(\begin{array}{c} [H,E] = 2E,\ [H,F] = 2F,\ [H,K] = 0, \\ E^p = 0 = F^p \\ KEK^{-1} = q^2 E \quad , \quad KFK^{-1} = q^{-2} F \\ [E,F] = \dfrac{K - K^{-1}}{q - q^{-1}} \end{array} \right)}$

- $q = e^{\pi i/p}$

this is a qt. Hopf alg.

→ H acts semisimply with f.d. weight spaces.

$\mathcal{C} = \operatorname{Rep}^{wt}(u_q^H sl_2)$ is an infinite non-SS MTC

then $A = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} \cdot 1_{2m} \in \operatorname{Ind}(\mathcal{C})$ is a comm. alg

where E, F, H act on 1_{2m} as $0, 0, 2mp$ resp.

Two main algebraic approaches to 2D-conformal field theory

$$\begin{tikzcd} & \text{2D-CFT} \arrow[dl] \arrow[dr] & \\ \text{MTC} & & \text{VOA} \arrow[ll, dashed, "?"'] \end{tikzcd}$$

- Given VOA, when is $\operatorname{Rep}(V)$ a MTC
- Generally can show $\operatorname{Rep}(V)$ is braided monoidal
  main hurdle is rigidity.
  our work help in overcoming this hurdle.

$\longrightarrow$ Given suitable extension $V \subset W$, then $\operatorname{Rep}(W) = \operatorname{Rep}(V)_A^{loc}$

$\therefore \operatorname{Rep}(V)$ rigid $\Rightarrow \operatorname{Rep}(W)$ rigid