ICTS Chalk talk (Jan 23)

I like to study algebraic structures / problems inspried by Physics.

(video by Prof. Subhro Bhattacharjee on 'Phases of quantum matter')

Condensed matter physicysts $\rightsquigarrow$ interested in classification of gapped phases

$\cup$

intrinsic topological order in $(2+1)$ D

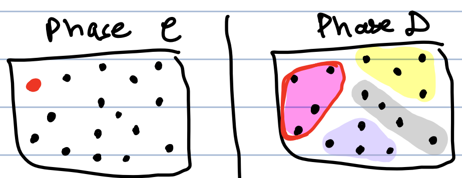
$\cup$

(unitary) Modular tensor categories	$\xleftrightarrow[\text{anyon data}]{\text{long distance}}$	bosonic TO
simple objects		anyons
tensor products		fusion
braiding		exchange statistics
ribbon / twist		topological spin
nondegenerate		no invisible anyons

" this a Physics motivation for studying / classifying MTCs ".

How to get new phases of matter?

$\hookrightarrow$ via phase transition $\rightarrow$ in particular condensation



$\longrightarrow$ $\mathcal{C} \mid \mathcal{D} = \mathcal{C}_A^{loc}$

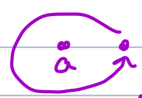
$\mathcal{C}, \mathcal{D}$ are MTC $\quad A = 1 \oplus a \oplus \cdots$ is a 'nice' alg. in $\mathcal{C}$.

take a collection $A = \{1, a, \ldots\}$ bosonic anyons

Declare that they become part of the vacuum.

How are the phases before and after related?

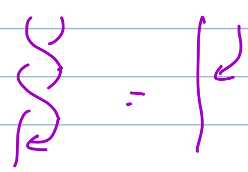
If anyon x has trivial monodromy



then

- x can move in condensate as honest deconfined point particle
- otherwise x is confined.

locality condition



categorically : condensate = comm. alg A

anyons of new phase = A-local modules

domain wall = A-modules

Today's goal: study the local module construction quite generally.

<u>Setting</u>: • $\mathcal{C}$: braided tensor category
- braided monoidal $(\mathbb{1}, \otimes, c_{X,Y})$
- rigid (duals exist)
- $\mathbb{k}$-linear, abelian, locally finite.

no assumption of finiteness OR semisimplicity.

$\mathcal{C}$ is a MTC if it is also
① ribbon
② non-degenerate $\left(c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y} \ \forall Y \Rightarrow X = \mathbb{1} \right)$

• A = commutative algebra in $\mathcal{C}$
- $(A \otimes A \xrightarrow{m} A, \ \mathbb{1} \xrightarrow{u} A)$ s.t. associativity and unit hold
- $m \circ c_{A,A} = m$

<u>Example</u>: ① G = finite group $\rightsquigarrow$ $\mathcal{C} = \mathrm{Rep}(G)$ is a finite semisimple braided tensor category.

f.d. $\mathbb{C}$-reps of G.

Given $H \subset G$, $\mathrm{Fun}(G/H) \in \mathrm{Rep}(G)$ is a commutative algebra in $\mathcal{C}$.

② $\mathcal{C} = \mathrm{Rep}(\widehat{\mathfrak{sl}}(n), k)$ = category of integrable level k reps of affine Lie alg $\widehat{\mathfrak{sl}}(n)$

$\hookrightarrow$ conformal embedding $\widehat{\mathfrak{sl}}(n)_{n+1} \subset \widehat{\mathfrak{sl}}\left(\frac{(n-1)(n+1)}{2}\right)_1$ lead to comm. algs in $\mathrm{Rep}(\widehat{\mathfrak{sl}}(n), k)$

Using $\mathcal{C}$ and A, we can form

• $\mathcal{C}_A$ = cat. of right A-modules = $\{ (M, \triangleleft : M \otimes A \to M) \mid \triangleleft \circ (\triangleleft \otimes \mathrm{id}_A) = \triangleleft \circ (\mathrm{id}_M \otimes m) \}$

this is a monoidal category with

$(M, \triangleleft) \otimes_A (N, \triangleleft) = \mathrm{coeq}\left(\triangleleft \otimes \mathrm{id}_N, \ \mathrm{id}_M \otimes \triangleright \right)$, maps $M \otimes A \otimes N \to M \otimes N$

monoidal unit is A.

– however it is often not braided.

- $\mathcal{C}_A^{loc}$ = cat. of local A-modules $= \{(M,\mu) \in \mathcal{C}_A \mid c^{\prime}_{\mu} = \mu\}$

  ↳ $\mathcal{C}_A^{loc} \subset \mathcal{C}_A$ is a braided subcategory.

$\underline{Q}$ When is $\mathcal{C}_A^{loc}$ rigid, ribbon, non-degenerate ?

- [Kirillov - Ostrik] answered under finite, semisimple assumption
- [Shimizu - Y.] finite, non-semisimple setting

Generalize further --- to consider infinite algebra.

$\mathcal{C}$ = braided tensor cat $\rightsquigarrow$ $\mathrm{Ind}(\mathcal{C})$ well behaved enlargement of $\mathcal{C}$ to allow ∞ objects.

(eg. $\mathcal{C} = \mathrm{Vec}^{fd}$, $\mathrm{Ind}(\mathcal{C}) = \mathrm{Vec}$)

take A = comm alg in $\mathrm{Ind}(\mathcal{C})$.

→ form $\mathrm{Ind}(\mathcal{C})_A^{loc}$

– too big

– very rarely rigid

but we can certain finite part

$\mathcal{D} = fg\text{-}\mathrm{Ind}(\mathcal{C})_A^{loc}$ = finitely generated local A-modules in $\mathrm{Ind}(\mathcal{C})$.

$\underline{Q}$ When is $fg\text{-}\mathrm{Ind}(\mathcal{C})_A^{loc}$ locally finite, rigid, ribbon ?

Study A $\left.\begin{array}{c}\text{Artinian} \\ \text{exact} \\ \text{Frobenius}\end{array}\right\}$ alg $\Rightarrow$ $\left\{\begin{array}{c}\text{locally finite} \\ \text{rigid} \\ \text{ribbon}\end{array}\right\}$

$\underline{\text{Thm}}$: If $A = \oplus X_i$ where each X_i is a simple invertible object, then $fg\text{-}\mathrm{Ind}(\mathcal{C})_A^{loc}$ is a braided tensor cat.

If $\mathcal{C}$ is ribbon, so is $fg\text{-}\mathrm{Ind}(\mathcal{C})_A^{loc}$

$\boxed{\text{Example}}$ $\mathcal{C} = \mathrm{Rep}^{wt}(u_q^H(sl_2))$

↳ unrolled small quantum group

$u_q^H(sl_2) = \dfrac{\Bbbk\langle E, F, K^{\pm}, H\rangle}{}$

$$q = e^{2\pi i/2p} \quad \left(\begin{array}{c} KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2}E, \quad EF - FE = K - K^{-1}/q - q^{-1} \\ [H,E] = 2E, \quad [H,F] = 2F, \quad [H,K] = 0 \end{array} \right)$$

$u_q^H(sl_2)$ is an ∞-diml Hopf algebra.

and $\mathcal{C}$ = cat. of wt. modules

is a MTC that is non-finite & non-semisimple.

Let $1_m = \mathbb{C}$ with $u_q^H(sl_2)$ action

E, F, H act by $0, 0, 2mp$ (K acts a q^H)

Then $A := \bigoplus_{m \in \mathbb{Z}} 1_m$ is a comm. alg in $\mathrm{Ind}(\mathcal{C})$

Then $fg\text{-}\mathrm{Ind}(\mathcal{C})_A^{loc}$ is a new MTC !!

$= \mathrm{Rep}\,(\underbrace{u_q^{\phi}}_{} sl_2)$

quasi version of the small quantum group.

Strategy of rigidity

- classify simple objects
- show they direct summands of free modules
- free modules are rigid

$\Rightarrow$ all simples are rigid

$\Rightarrow$ the category is rigid.

$$\mathcal{C} \longrightarrow fg\text{-}\mathrm{Ind}(\mathcal{C})_A$$
$$X \longmapsto X \otimes A$$