# From Finite Groups to Vertex Operator Algebras: A Gentle Tour of Tensor Categories


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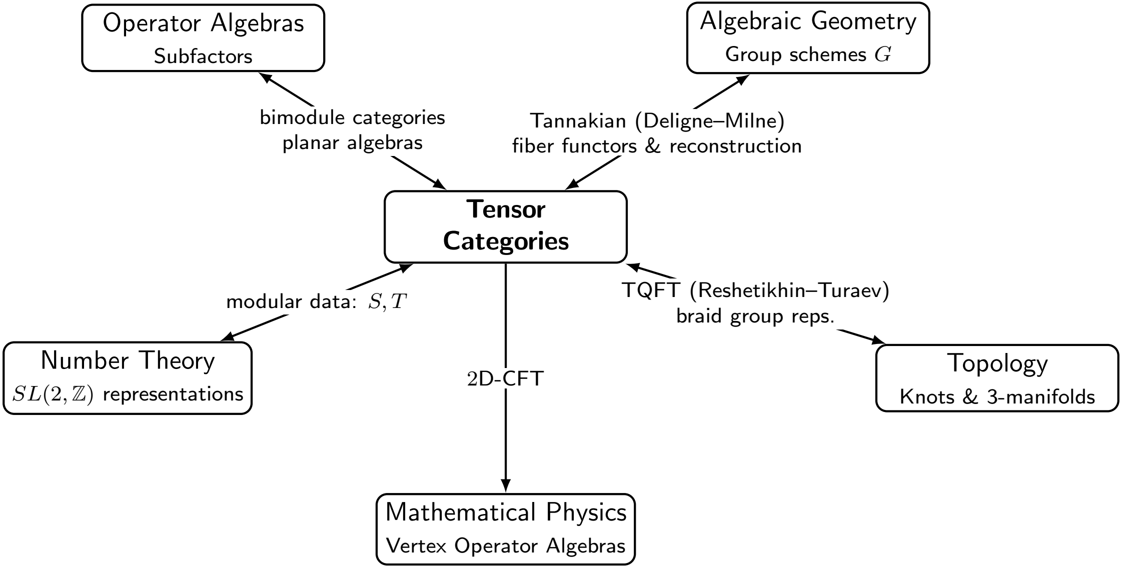
# Part 1: Tensor Categories Across Mathematics



# $\mathrm{Vec}_{\Bbbk}$: the baseline

Let $\mathrm{Vec}_{\Bbbk}$ be the category of finite dimensional $\Bbbk$-vector spaces.

- Objects: finite-dimensional $\Bbbk$-vector spaces; morphisms: $\Bbbk$-linear maps.
- Can tensor objects: $(V, W) \mapsto V \otimes_{\Bbbk} W$.
- tensor product is associative and unital up to isomorphism:

$$(U \otimes V) \otimes W \xrightarrow[a_{U,V,W}]{\sim} U \otimes (V \otimes W), \quad V \otimes \Bbbk \xrightarrow[l_V]{\sim} V \xleftarrow[r_V]{\sim} \Bbbk \otimes V$$

- Rigid: dual $V^* = \mathrm{Hom}_{\Bbbk}(V, k)$ is also an object in our category.

A *monoidal category* is a tuple $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ where $\otimes$ is a way of taking tensor product of objects and $\mathbb{1}$ is the unit object which. Moreover, we have associativity isomorphism $a$ and unit isomorphisms $l, r$ as above, satisfying certain conditions.

A *tensor category* is a monoidal category that is rigid, $\Bbbk$-linear abelian, locally finite with $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$.

# Rep$(G)$: a familiar tensor category

Let $G$ be a finite group. The category Rep$(G)$ of finite-dimensional representations of $G$ over $\mathbb{k}$ is a tensor category.

- Objects: finite-dimensional $\mathbb{k}$-modules of $G$.
- Monoidal: $(V, W) \mapsto V \otimes W$ with diagonal $G$-action:

$$g \cdot (v \otimes_{\mathbb{k}} w) := g \cdot v \otimes_{\mathbb{k}} g \cdot w.$$

- $\mathbb{1} := \mathbb{k}$ is a $G$-module with trivial action $g \cdot x := x$.
- Rigid: given a $G$-module $V$, dual $V^*$ with $(g \cdot f)(v) := f(g^{-1} \cdot v)$ is also a $G$-module.

A *fusion category* is a semisimple tensor category with finitely many simple objects (up to isomorphism). The *rank* of a fusion category is the number of isomorphism classes of simple objects.

- The tensor category Rep$(G)$ is a fusion category $\iff$ char$(\mathbb{k}) \nmid |G|$.

i) For $\mathrm{GL}_2$ with standard $V$: $V \otimes V \cong S^2V \oplus \wedge^2 V$ (i.e. $4 = 3 + 1$).
ii) The rank of $\mathcal{C} =$ Rep$(S_n)$ is the number of partitions of $n$.

# Braided and symmetric tensor categories

Notice that in $\mathsf{Vec}_{\Bbbk}$ and $\mathsf{Rep}(G)$, we have isomorphisms $c_{X,Y}\colon X\otimes Y \xrightarrow{\sim} Y\otimes X$ (called *braidings*) that are compatible with tensor product in the sense that

$$c_{X\otimes Y,Z} = (c_{X,Z}\otimes \mathsf{Id}_Y)(\mathsf{Id}_X\otimes c_{Y,Z}) \text{ and } c_{X,Y\otimes Z} = (\mathsf{Id}_Y\otimes c_{X,Z})(c_{X,Y}\otimes \mathsf{Id}_Z).$$

Tensor categories equipped with braidings are called *braided tensor categories*. A *symmetric tensor category* is a braided tensor category satisfying $c_{Y,X}\circ c_{X,Y} = \mathsf{Id}_{X\otimes Y}$.

Let $\mathcal{G}$ be an affine group scheme over $\Bbbk$. Then $\mathsf{Rep}(\mathcal{G})$ is a symmetric tensor category.

**Question**: Are all symmetric tensor categories of the form $\mathsf{Rep}(\mathcal{G})$?

[Saavedra-Rivano (1972)] [Deligne-Milne (1982)] If a symmetric tensor category $\mathcal{C}$ admits a fiber functor $\omega : \mathcal{C}\to \mathsf{Vec}_{\Bbbk}$, then yes.

[Deligne (1990)] Provided an intrinsic characterization of those symmetric tensor categories that admit fiber functors in $\mathsf{char}(\Bbbk) = 0$.

[Deligne (2002)] Constructed symmetric tensor categories $\mathsf{Rep}(GL_t)$, $\mathsf{Rep}(S_t)$ for $t\in\mathbb{C}$ that do not admit fiber functors when $t\notin\mathbb{N}$.

# Why go beyond symmetry? From symmetric to braided

Physics suggests *braiding* rather than symmetry.

- In 1980s, exactly solvable 1D models (Lieb, Baxter, Yang) led to quantum groups (Drinfeld, Jimbo).
- Quantum groups $U_q(\mathfrak{g})$ are *deformations* of universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$.
- Their representation categories are *braided* but not symmetric.

**Braids $\rightarrow$ knots/links:** Every knot is the closure of a braid (Alexander).

- For a *knot invariant*, it suffices to interpret braids algebraically.
- A braided rigid tensor category assigns morphisms to braids; closing gives *link polynomials*.
- In particular, the **Jones polynomial** arises from $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$ (via the standard 2D module and its $R$-matrix).

# Let us add a twist...

**Smooth 3-manifolds via surgery on framed links**

- Every closed, oriented smooth $3$-manifold arises by *Dehn surgery* on a **framed link** in $S^3$ (Lickorish-Wallace).
- To encode framing algebraically, we must upgrade from *braided* to **ribbon** categories by adding a twist.

A *ribbon category* is a braided tensor category $(\mathcal{C}, \otimes, \mathbb{1}, c)$ equipped with a natural isomorphism (twist) $\theta_X : X \to X$ satisfying:

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}, \quad \theta_{X^*} = (\theta_X)^*.$$

# Modular fusion categories

For a braided tensor category $\mathcal{C}$, define

$$\mathcal{C}' = \{X \in \mathcal{C} \mid c_{X,Y} \circ c_{Y,X} = \mathrm{Id}_{X \otimes Y}\}.$$

We call $\mathcal{C}$ *nondegenerate* if $\mathcal{C}' \simeq \mathrm{Vec}_{\Bbbk}$.

A *Modular Tensor Category* (MTC) is a ribbon tensor category that is nondegenerate and a finite tensor category. A semisimple MTC is called a *Modular Fusion Category* (MFC).

i) $\mathcal{C} = \mathrm{Rep}(u_q(\mathfrak{sl}_2))$ where $q = e^{2\pi i/n}$ with $n$ odd is a MTC.
ii) Certain quotients of categories of tilting modules of $U_q(\mathfrak{g})$ at roots of unity are MFCs (Andersen, Reshetikhin, et al.).

Why care about modular fusion categories?

- Ribbon categories $\mathcal{C}$ yields invariants for 3-manifolds. If $\mathcal{C}$ is a MFC, we get a $3$-dimensional Topological Quantum Field Theory.
- MFCs also model anyons in topological quantum computation (Kitaev, Freedman, Nayak, Wang, et al.).

# Operator algebraic and Number theoretic connections

The main sources of MFCs are quantum groups, vertex operator algebras, and *factors/operator algebras*.

- a *factor* $M$ is a Von Neumann algebra with trivial center.

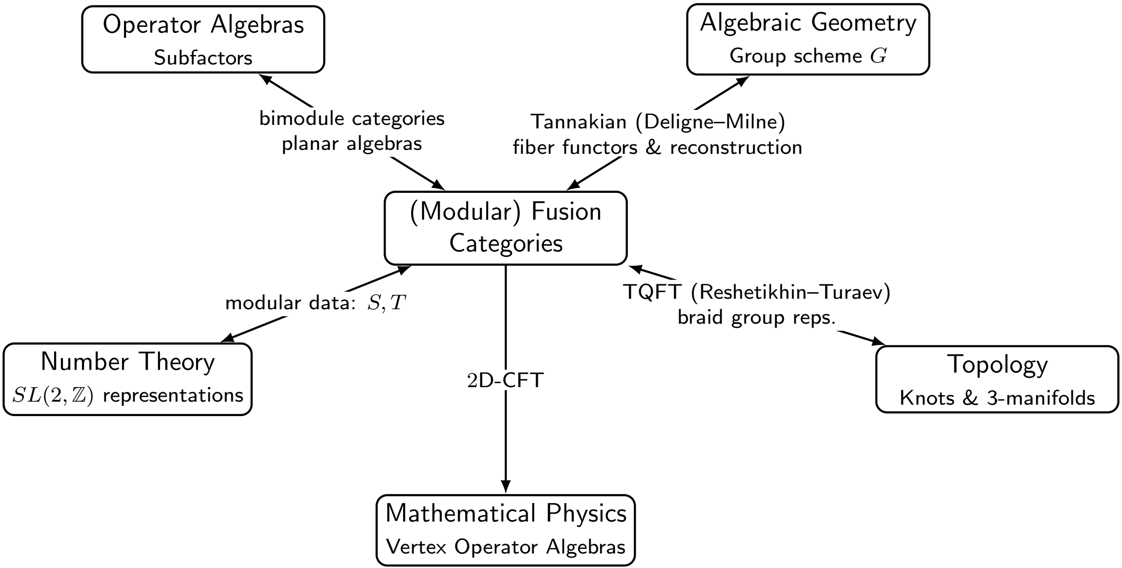
$$M \rightsquigarrow {}_M\mathrm{Bim}_M \text{ is fusion} \rightsquigarrow \mathcal{Z}({}_M\mathrm{Bim}_M) \text{ is a MFC.}$$

  Important examples include the Haagerup fusion categories.

To a MFC of rank $r$, one can associate matrices $S, T \in GL_r(\mathbb{C})$ (called *modular data*) satisfying the relations of $SL(2, \mathbb{Z})$, thereby giving a (projective) representation.

- The matrix entries are algebraic integers. Congruence/positivity phenomena force arithmetic restrictions.
- [Bruillard-Ng-Rowell-Zhang (2015)] For each fixed rank $r$, there are only finitely many MTCs (up to equivalence) of rank $r$.

# Tensor Categories Across Mathematics (at a glance)



# Part 2: Modular tensor categories and Vertex Operator Algebras

Emphasis in the last 3 decades has been on the semisimple setting (MFCs) but last decade has seen a surge in interest in non-semisimple MTCs:

- manifold invariants obtained using non-semisimple MTCs being better,
- connections to topological quantum computation, and
- connections to logarithmic $2$D-conformal field theories (log-CFTs)

Thus, as for the semisimple case, it is important to:

- construct families of MTCs, and
- study constructions for obtaining new MTCs from old ones.

# Category of local modules

Given a braided tensor category $\mathcal{C}$, a *commutative algebra* in $\mathcal{C}$ is an object $A \in \mathcal{C}$ equipped with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$ satisfying the usual associativity, unitality and commutativity axioms.

A *right $A$-module* in $\mathcal{C}$ is a pair $(M, \rho_M^r)$ where $M \in \mathcal{C}$ and $\rho_M^r : M \otimes A \to M$ satisfying the usual axioms.

In a similar manner, the category of $A$-bimodules ${}_A\mathcal{C}_A$ can be defined.

- The category ${}_A\mathcal{C}_A$ is monoidal with tensor product $\otimes_A$ defined via coequalizers.
- The category of right $A$-modules in $\mathcal{C}$ (denotes as $\mathcal{C}_A$) is a monoidal subcategory of ${}_A\mathcal{C}_A$.

To get a braided monoidal category, Pareigis introduced the following:

A right $A$-module $(M, \rho_M^r)$ is called *local* if $\rho_M^r \circ c_{A,M} \circ c_{M,A} = \rho_M^r$.

[Pareigis (1995)] The category $\mathcal{C}_A^{\mathrm{loc}}$ of local $A$-modules is braided monoidal.

# Modularity of the category of local modules

Q: Provide sufficient conditions on $\mathcal{C}$ and $A$ to ensure $\mathcal{C}_A^{\mathrm{loc}}$ is rigid, ribbon, modular, etc.

[Kirillov-Ostrik (2003)]: Provided an answer in the semisimple case.

An algebra $A$ in a tensor category $\mathcal{C}$ is called *exact* if for any object $X \in \mathcal{C}$, and a projective right $A$-module $P$, the object $X \otimes P$ is projective right $A$-module.

[Etingof-Ostrik (2003)]: Introduced exact algebras and proved: $A$ exact $\implies {}_A\mathcal{C}_A$ is rigid.

**Theorem (Shimizu-Y.)**

*If $\mathcal{C}$ is a braided finite tensor category and $A$ a commutative algebra in $\mathcal{C}$. Then,*

1. *If $A$ is exact, then $\mathcal{C}_A^{\mathrm{loc}}$ is a braided finite tensor category.*
2. *If $\mathcal{C}$ is a MTC and $A$ is an exact symmetric Frobenius algebra, then $\mathcal{C}_A^{\mathrm{loc}}$ is a MTC.*

We also provided ways to construct examples of commutative exact (symmetric Frobenius) algebras in MTCs.

This has applications to VOAs as we will see next.

# Vertex Operator Algebras (VOAs)

A *vertex operator algebra* is $\mathbb{Z}$-graded vector space $V = \prod_{n\in\mathbb{Z}} V_{(n)}$ along with

- a vector $|0\rangle \in V_{(0)}$,
- a map $Y(\cdot, z) : V \to (\operatorname{End} V)\{z\}$, and
- a vector $\omega \in V_{(2)}$. We write $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

satisfying certain conditions like grading restriction, Jacobi identity, etc.

An *(ordinary) VOA module* is a pair $(W, Y_M)$ where

- $W$ is a vector space, and
- $Y_M(\cdot, z) : V \to (\operatorname{End} W)\{z\}$ is a map

such that $W = \prod_{h\in\mathbb{C}} W_{[h]}$ is a $\mathbb{C}$-graded vector space and each $W_{[h]}$ is a $L(0)$-eigenspace of eigenvalue $h$. We also require this data to satisfying certain conditions like grading restriction, Jacobi identity, etc.

# VOAs: foundational examples

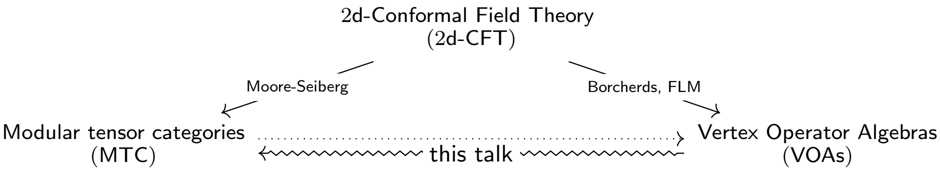
There are two important classes of VOAs:

- A $\mathbb{Z}_{\geq 0}$-graded, simple, self-contragredient and $C_2$-cofinite VOA is called *strongly finite*.
- A strongly finite and with all modules semisimple is called *strongly rational*.

There are three foundational families of strongly rational VOAs:

1. **Affine (WZW) VOAs** [Frenkel-Zhu (1992) Duke Math. J.]
   - Denoted $V_k(\mathfrak{g})$ and constructed using a Lie algebra $\mathfrak{g}$ and a choice of level $k$.
   - Strongly rational for $k \in \mathbb{Z}_{\geq 0}$ (classical result).
   - [Arakawa (2016) Duke Math. J.] Proved rationality of $L_k(\mathfrak{g})$ at admissible fractional levels.
2. **Lattice VOAs** [Frenkel-Lepowsky-Meurman (1987)]
   - $V_L$ for even lattices $L$ (e.g. $E_8$, Leech).
   - [Dong (1996) Adv. Math.] Strongly rational.
   - category of modules is equivalent to $\mathrm{Vec}_A$ (the category of $A = L^*/L$-graded vector spaces).
3. **$W$-algebras** [Feigin-Frenkel (1992) Phys. Lett. B]
   - Denoted $W_k(\mathfrak{g}, f)$ for a Lie algebra $\mathfrak{g}$, level $k \in \mathbb{k}$ and a nilpotent element $f \in \mathfrak{g}$.
   - Defined as quantum Drinfeld-Sokolov reduction of affine VOAs.
   - [Arakawa (2015) Annals of Math.] Strongly rational for so called 'nondegenerate' levels $k$ and principal nilpotent $f$.

# VOA tensor categories




**Expectation**: $V$ a 'nice enough' VOA $\implies \mathcal{C} = \mathrm{Rep}(V)$ is a MTC.

Strategy:

1. First analyze key examples and prove this property.
2. Then use constructions (extensions, orbifolds, cosets, etc.) to get new examples.

# History of some key results

[Kazhdan-Lusztig (1993-1994) JAMS]: Constructed braided monoidal categories from representations of affine VOAs $V_k(\mathfrak{g})$ for irrational and certain negative integer levels.

[Huang-Lepowsky (1990s)]: Gave a general theory of tensor product of ordinary VOA-modules.
(⚠ Tensor product of $V$-modules $W_1, W_2$ is not based on vector space tensor product, defined using universal property and it is a certain subspace of $(W_1' \otimes W_2')^*$ .)

[Huang (2008) CCM]: Proved that for a strongly rational VOA $V$, its category $\mathcal{C} = \text{Rep}(V)$ of ordinary modules is a finite semisimple MTC.

**Beyond strongly rational VOAs**

[Huang-Lepowsky-Zhang (2010s)]: Developed logarithmic tensor product theory for non-semisimple categories of modules of VOAs satisfying certain finiteness and reductivity conditions.

[Huang (2008) CCM]: Proved that for a strongly finite VOA $V$ satisfying certain additional conditions, $\mathcal{C} = \text{Rep}(V)$ is a braided monoidal category.

[McRae (2021)]: Proved that if $V$ is a strongly finite VOA such that $\text{Rep}(V)$ is rigid, then it is a MTC.

Step 1: HLZ theory has been used to obtain many examples of non-rational VOAs whose categories of modules are braided monoidal and often modular (*cf.* works of Creutzig, McRae, Kanade, Ridout, Yang etc.)

Step 2: Prove properties of $\mathcal{C} = \text{Rep}(V)$ like rigidity and modularity are preserved under constructions like extensions, orbifolds, cosets, of VOAs.

An *(conformal) extension* is an injective map of VOAs $V \subset W$ with the same conformal vector.

Given an extension of VOAs $V \subset W$ we can relate $\mathcal{C} = \text{Rep}(V)$ and $\mathcal{D} = \text{Rep}(W)$.

**Theorem**: (Kirillov-Ostrik, Huang-Kirillov-Lepowsky, Creutzig-Kanade-McRae)
If $\mathcal{C}$ is a braided monoidal category and $W \in \mathcal{C}$, then

- $A := W$ is a commutative algebra object in $\mathcal{C}$.
- $\mathcal{D} \cong_{\text{br}\otimes} \mathcal{C}_A^{\text{loc}}$ is the category of local $A$-modules in $\mathcal{C}$.

Upshot: Understanding the category $\mathcal{C}_A^{\text{loc}}$ helps us understand $\text{Rep}(W)$.

# Applications to VOA extensions

By applying our earlier theorem on rigidity of category of local modules, we obtain:

**Theorem (Creutzig-Mcrae-Shimizu-Y.)**

*If $V$ is a strongly finite VOA with Rep$(V)$ rigid and $V \subseteq A$ is a VOA extension with $A$ simple and $\mathbb{Z}_{\geq 0}$-graded, then*

- *If $A$ is strongly finite, then, Rep$(A)$ is rigid.*
- *If $V$ is a strongly rational VOA, then $A$ is strongly rational.*

**W-algebras**

Certain hook type $W$-superalgebras like

$$A = W_k(\mathfrak{so}_{2n+2m+1}, f_{\mathfrak{so}_{2m+1}})$$

are extensions of the strongly rational VOAs like $W_s(\mathfrak{sp}_{2r}) \otimes L_\ell(\mathfrak{so}_{2n})$. Also they are simple and $\mathbb{Z}_{\geq 0}$-graded. Thus, they are strongly rational.

We also obtain techniques for obtaining rigidity of Rep$(V)$ by leveraging extensions $V \subset W$ where Rep$(W)$ is known to be rigid.

Thank you!

# Other VOA construction

1. **Infinite extensions**. We considered extensions $V \subset W$ where $W$ is a commutative algebra in $\mathcal{C} = \text{Rep}(V)$. However, one can consider more general situations where $W$ is an infinite algebra, that is an algebra in the ind-completion of $\mathcal{C}$. In a work in preparation with Shimizu, we have generalized parts of the previous theorems to this setting.

2. **Orbifolds**. Given a VOA $V$ and a finite group $G$ of automorphisms of $V$, the orbifold VOA $V^G$ is the sub-VOA of $G$-invariant vectors.
The categories $\text{Rep}(V)$ and $\text{Rep}(V^G)$ are related by a two step process:

- First consider the category $\mathcal{D}$ of $G$-twisted modules of $V$. This is a $G$-graded category with trivial component $\text{Rep}(V)$.
- Then the category $\text{Rep}(V^G)$ is obtained as the $G$-equivariantization of $\mathcal{D}$.

In a work in preparation with others, we consider problem of existence of $G$-graded extensions of tensor categories.

3. **Coset construction**. Given a VOA $V$ and a sub VOA $U \subset V$, the coset VOA $\text{Com}(U, V)$ is the sub VOA of vectors commuting with $U$. In an ongoing work with Shimizu, we are studying the coset construction categorically.

# Example details

Consider the $W$-algebra

$$A = W_k(\mathfrak{so}_{2n+2m+1}, f_{\mathfrak{so}_{2m+1}})$$

for

$$\psi = k + 2m + 2n - 1 = \frac{2n + 2m + 2r + 1}{2(m + 1)}$$

and $r$ a positive integer such that $\gcd(2n + 2r - 1, m + 1) = 1$.

In this case, $A$ is a conformal extension of

$$W_s(\mathfrak{sp}_{2r}) \otimes L_\ell(\mathfrak{so}_{2n})$$

at the non-degenerate admissible level $s = -(r + 1) + \frac{2n+2m+2r+1}{2(2r+2n-1)}$ and the admissible level $\ell = -(2n - 2) + \frac{2n+2r-1}{2(m+1)}$.