Tensor functors with isomorphic left and right adjoints

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based on arXiv:2501.16978, joint with David Jaklitsch

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Note: The object χ_F is equipped with a half-braiding making it an object in $\mathcal{Z}({}_F\mathcal{D}_F)$.

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- Denote by ${}_F\mathcal{D}_F$ the $\mathcal{C}$-bimodule category $\mathcal{D}$ with action

$$X \triangleright Y \triangleleft X' := F(X) \otimes Y \otimes F(X') \qquad (X, X' \in \mathcal{C},\ Y \in \mathcal{D}).$$

- With this, $F : \mathcal{C} \to {}_F\mathcal{D}_F$ is a (strong) $\mathcal{C}$-bimodule functor.
- As $\mathcal{C}$ is rigid, its adjoints F^{ra} and F^{la} are also (strong) $\mathcal{C}$-bimodule functors.

Definition

A tensor functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *$\otimes$-Frobenius* if $F^{\mathrm{la}} \cong F^{\mathrm{ra}}$ as $\mathcal{C}$-bimodule functors.

Moreover, we can define the center of $F\colon \mathcal{C} \to \mathcal{D}$

$$\mathcal{Z}(F) : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}({}_F\mathcal{D}_F)$$

where $\mathcal{Z}(\mathcal{C})$ is the center of $\mathcal{C}$ and $\mathcal{Z}({}_F\mathcal{D}_F)$ is the relative center of the bimodule category ${}_F\mathcal{D}_F$. $\mathcal{Z}({}_F\mathcal{D}_F)$ is also a finite tensor category.

<u>Note</u>: The object χ_F is equipped with a half-braiding making it an object in $\mathcal{Z}({}_F\mathcal{D}_F)$.

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Theorem

Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a perfect tensor functor and $\mathcal{M}$ is a unimodular (or pivotal) $\mathcal{D}$-module category. Then ${}_F\mathcal{M}$ is a unimodular (or pivotal) $\mathcal{C}$-module category if and only if F is Frobenius with respect to $\mathcal{M}$.

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Given a tensor functor $F\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{D}$-module category $\mathcal{M}$, we get a functor

$$F^*_{\mathcal{M}} : \mathcal{D}^*_{\mathcal{M}} \to \mathcal{C}^*_{\mathcal{M}}$$

We say that F is *Frobenius with respect to* $\mathcal{M}$ if $F^*_{\mathcal{M}}$ is Frobenius, i.e., it has a left and right adjoint that are isomorphic functors.

Theorem

Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a perfect tensor functor and $\mathcal{M}$ is a unimodular (or pivotal) $\mathcal{D}$-module category. Then ${}_F\mathcal{M}$ is a unimodular (or pivotal) $\mathcal{C}$-module category if and only if F is Frobenius with respect to $\mathcal{M}$.

If F is $\otimes$-Frobenius, then for every $\mathcal{C}$-module category $\mathcal{M}$, F Frobenius with respect to $\mathcal{M}$.

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Definition

Let $\mathsf{f}\colon H' \to H$ be a perfect bialgebra map and L an exact H'-comodule algebra. An f-*Frobenius element* of L is an invertible element $\mathsf{a} \in L$ satisfying

$$\mathsf{a}\, l\, \mathsf{a}^{-1} = \chi_{\mathsf{f}}(l_{-1}) l_0, \qquad (\forall\, l \in L)$$

$$\mathsf{f}(\mathsf{a}_{-1}) \otimes \mathsf{a}_0 = \mathsf{f}(\mathsf{g}_{H'})\, \overline{\mathsf{g}_H} \otimes \mathsf{a}.$$

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Theorem

The tensor functor F_{f} is Frobenius with respect to $\mathsf{Rep}(L)$ *if and only if L admits an* f*-Frobenius element.*