Unimodular module categories

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- Frobenius monoidal functors from coHopf adjunctions, arxiv:2209.15606
- On unimodular module categories, arxiv:2302.06192

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What is unimodularity?

Unimodularity

- An $n \times n$ matrix A with integer entries is called *unimodular* if $\det(A) = \pm 1$.
- This idea is used to define unimodularity of
  - bilinear forms
  - lattices
  - Poisson algebras
  - Hopf algebras
  - Finite multi-tensor categories
- Fun fact: Unimodular lattices (E_8 and Leech lattice) were used in the work of Viazovska[1] to obtain efficient sphere packings in dimensions 8 and 24.

[1] Viazovska, "The sphere packing problem in dimension 8".

Unimodular Hopf algebras

Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite dimensional Hopf algebra.

- A *left integral* is an element $\Lambda^l \in H$ satisfying

$$h\Lambda^l = \varepsilon(h)\Lambda^l \qquad \forall\, h \in H$$

- A *right integral* is an element $\Lambda^r \in H$ satisfying

$$\Lambda^r h = \varepsilon(h)\Lambda^r \qquad \forall\, h \in H$$

- For any left integral Λ^l, there exists $\alpha : H \to \Bbbk$ (*distinguished character*) such that

$$\Lambda^l h = \alpha(h)\Lambda^l \qquad \forall\, h \in H$$

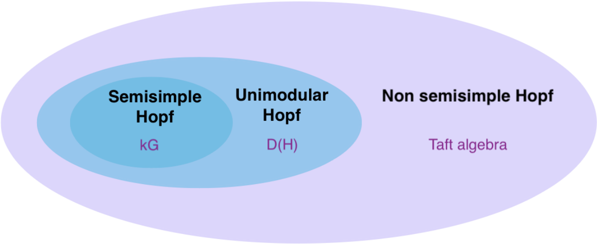
Definition

H is called *unimodular* if the following equivalent conditions are satisfied:

- H admits a two-sided integral $\iff$ $\alpha = \varepsilon$.

Examples

1. Let $G =$ finite group with $\text{char}(\Bbbk) \nmid |G|$. Then, $\Lambda = \sum_{g \in G} g$ is a left and right integral of the $\Bbbk G$.
2. Semisimple Hopf algebras are unimodular ($\Lambda^l \Lambda^r$ is a two-sided integral).
3. When $\text{char}(\Bbbk)$ divides $|G|$, $\Bbbk G$ is not semisimple but still unimodular.
4. Let $H_{2,i} = \frac{\Bbbk<g,x>}{(g^2=1, x^2=0, gx=-xg)}$. Then H does not admit a two-sided integral.



Unimodular tensor categories

A finite multi-tensor category is a finite, abelian, rigid, monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$.

Hopf algebras	Finite tensor categories
(H, m, u) algebra	$\mathsf{Rep}(H)$ abelian, $\Bbbk$-linear
Δ, ε are algebra maps	$(\mathsf{Rep}(H), \otimes_{\Bbbk}, \Bbbk_{\varepsilon})$ monoidal
S bijective	$\mathsf{Rep}(H)$ admits duals, is rigid
H unimodular ($\alpha = \varepsilon$)	$D := \Bbbk_{\alpha}$, then $D \cong \mathbb{1}$

In fact[2], $D \cong \int_{X \in \mathcal{C}} \mathsf{Hom}_{\mathcal{C}}(X, \mathbb{1}) \cdot X$.

Definition

A finite multi-tensor category $\mathcal{C}$ is called *unimodular* if $D \cong \mathbb{1}$.

Example: Drinfeld centers of finite tensor categories are unimodular.

[2] Shimizu, "On unimodular finite tensor categories".

Unimodular $\mathcal{C}$-Module categories

Let $\mathcal{C}$ be a finite multi-tensor category. A left $\mathcal{C}$-module category is a finite abelian category $\mathcal{M}$ along with an exact functor $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and coherent natural isomorphisms:

$$(X \otimes Y) \triangleright M \cong X \triangleright (Y \triangleright M), \qquad \mathbb{1} \triangleright M \cong M.$$

A $\mathcal{C}$-module category $\mathcal{M}$ is called *exact* is for all projective objects $X \in \mathcal{C}$ and any $M \in \mathcal{M}$, $X \triangleright M \in \mathcal{M}$ is projective.

Definition

An exact $\mathcal{C}$-module category $\mathcal{M}$ is called *unimodular*[a] if the multi-tensor category $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular.

[a] Fuchs, Schaumann, and Schweigert, "Eilenberg-Watts calculus for finite categories and a bimodule Radford S^4 theorem".

Examples

1. $\mathcal{C}$ = finite tensor cat., $\mathcal{M} = \mathcal{C}$
   Then, $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M}) \cong \mathcal{C}^{\mathsf{rev}}$.
   $\implies$ $\mathcal{M}$ is unimodular if and only if $\mathcal{C}$ is unimodular tensor cat.

2. $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\mathsf{rev}}$, $\mathcal{M} = \mathcal{C}$
   Then, $\mathsf{Rex}_{\mathcal{D}}(\mathcal{M}) \cong \mathcal{Z}(\mathcal{C})$. But the $\mathcal{Z}(\mathcal{C})$ is always unimodular.
   $\implies$ $\mathcal{M} = \mathcal{C}$ is a unimodular $\mathcal{C} \boxtimes \mathcal{C}^{\mathsf{rev}}$ module category.

3. $\mathcal{C}$ = $\mathsf{Rep}(H)$, $\mathcal{M}$ = Vec via the fiber functor $F : \mathsf{Rep}(H) \to \mathsf{Vec}$
   Then, $\mathsf{Rex}_{\mathsf{Rep}(H)}(\mathsf{Vec}) \cong \mathsf{Rep}(H^*)$.
   $\implies$ Vec is a unimodular if and only if H^* is unimodular.

4. $\mathcal{C}$ = fusion cat. of dim$\neq 0$, $\mathcal{M}$ = any semisimple $\mathcal{C}$-module cat.
   Then, $\mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ is also fusion[3], and hence, unimodular.
   $\implies$ $\mathcal{M}$ is a unimodular $\mathcal{C}$-module category.

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[3] Etingof, Nikshych, and Ostrik, "On fusion categories".

Characterization of unimodularity

Understanding unimodularity

Let $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor. Shimizu[4] proved that:

$$\mathcal{C} \text{ is unimodular } \iff U^{\mathsf{ra}}(\mathbb{1}) \in \mathsf{Frob}(\mathcal{C})$$

For $(\mathcal{M}, \triangleright)$ an exact, left $\mathcal{C}$-module category, consider the functor[5]:

$$\Psi : \mathcal{Z}(\mathcal{C}) \to \mathsf{Rex}_{\mathcal{C}}(\mathcal{M}), \qquad (X, \sigma) \mapsto (X \triangleright -, s^{\sigma})$$

As a corollary of Shimizu's result, we obtained that[6]:

Corollary

$\mathcal{M}$ is unimodular $\iff \Psi^{\mathsf{ra}}(\mathbb{1}) \in \mathsf{Frob}(\mathsf{Rex}_{\mathcal{C}}(\mathcal{M}))$.

[4] Shimizu, "On unimodular finite tensor categories".
[5] Shimizu, "Further results on the structure of (co) ends in finite tensor categories".
[6] Yadav, "On unimodular module categories".

Enhancing Shimizu's result

Definition

A Frobenius monoidal functor is a tuple (F, F_0, F_2, F^2, F^0) where $(F, F_2, F_0) : \mathcal{C} \to \mathcal{D}$ is a lax monoidal functor, $(F, F^2, F^0) : \mathcal{C} \to \mathcal{D}$ is an oplax monoidal functor and F^2, F_2 satisfy a certain 'Frobenius relation'.

- Strong monoidal functors are Frobenius monoidal.
- Crucially, if $A \in \mathsf{Frob}(\mathcal{C})$, then $F(A) \in \mathsf{Frob}(\mathcal{D})$.
- Frobenius monoidal functors preserve duals, i.e., $F(X^*) \cong F(X)^*$. This yields a natural iso $\xi_X^F : F(X^{**}) \to F(X)^{**}$.

Definition

A Frobenius monoidal functor $F : (\mathcal{C}, \mathfrak{p}) \to (\mathcal{D}, \mathfrak{q})$ is called *pivotal* if it satisfies: $\xi_X^F \circ F(\mathfrak{p}_X) = \mathfrak{q}_{F(X)}$ for all $X \in \mathcal{C}$.

- Pivotal Frobenius monoidal functors preserve symmetric Frobenius algebras.

Frobenius monoidal functors coHopf adjunctions

Theorem ([a] [b])

[a] Balan, "On Hopf adjunctions, Hopf monads and Frobenius-type properties".
[b] Yadav, "Frobenius monoidal functors from (co) Hopf adjunctions".

$\mathcal{C}, \mathcal{D}$ abelian monoidal categories,
$U : \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor admitting a right adjoint R, and R is exact, faithful and the adjunction $U \dashv R$ is coHopf. Then,

$$R \text{ is a } \circledast \text{ monoidal functor } \iff R(\mathbb{1}_{\mathcal{D}}) \text{ is a } \divideontimes \text{ algebra in } \mathcal{C}.$$

Input			$R(\mathbb{1}_{\mathcal{D}})$	R
$\mathcal{C}$	$\mathcal{D}$	U	$\divideontimes$	$\circledast$
$\otimes$	$\otimes$	strong $\otimes$	Frob.	Frob. monoidal
$\otimes$	$\otimes$	strong $\otimes$	separable Frob.	separable Frob.
$\otimes$	$\otimes$	strong $\otimes$	special Frob.	special Frob.
pivotal	pivotal	pivotal, strong $\otimes$	symmetric Frob.	pivotal Frob.
ribbon	ribbon	ribbon, strong $\otimes$	symmetric Frob.	ribbon Frob.

Theorem

Let $\mathcal{C}$ be a finite tensor category and $\mathcal{M}$ and indecomposable, exact left $\mathcal{C}$-module category. Then, the following conditions are equivalent:

- *$\mathcal{M}$ is unimodular.*
- *$\Psi^{\mathsf{ra}}(\mathsf{Id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.*[a]
- *Ψ^{ra} is a Frobenius monoidal functor.*[b]
- *$\mathbb{S}_{\mathcal{M}}\mathbb{N}_{\mathcal{M}} \cong_{\mathcal{C}} \mathsf{Id}_{\mathcal{M}}$.*[c] [d]

[a] Shimizu, "On unimodular finite tensor categories".
[b] Yadav, "On unimodular module categories".
[c] Fuchs et al., "Spherical Morita contexts and relative Serre functors".
[d] Shimizu, "Nakayama functor for monads on finite abelian categories".

For a pivotal category $\mathcal{C}$, an exact left $\mathcal{C}$-module category is called *pivotal* if $\mathbb{S}_{\mathcal{M}} \cong_{\mathcal{C}} \mathsf{Id}_{\mathcal{M}}$ holds.

Theorem

If $\mathcal{C}$ is pivotal and $\mathcal{M}$ is a pivotal, unimodular left $\mathcal{C}$-module category, then Ψ^{ra} is a pivotal Frobenius monoidal functor.

Application/Questions

1. Internal natural transformation[7]:For $F, G \in \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$, we have $\underline{\mathsf{Nat}}(F, G) = \Psi^{\mathsf{ra}}(G \circ F^{\mathsf{ra}})$.Then, $\mathcal{M}$ pivotal, unimodular $\implies \underline{\mathsf{Nat}}(F, F) = \Psi^{\mathsf{ra}}(F \circ F^{\mathsf{ra}})$ is a symmetric Frobenius alg in $\mathcal{Z}(\mathcal{C})$.

Some questions:

1. What structure do internal homs of unimodular module cats. have?
2. For $\mathcal{C}$ braided, we can carry out the above analysis with $\Theta : \mathcal{C} \to \mathsf{Rex}_{\mathcal{C}}(\mathcal{M})$ to obtain commutative symmetric Frobenius algebras in $\mathcal{C}$. For $\mathcal{M} = \mathcal{C}_A$, define

$$Z_A(X) := \Theta^{\mathsf{ra}}(X \triangleright -)$$

How is the endofunctor Z_A is related to the endofunctors $E_A^{l/r}$ constructed in FFRS[8] via idempotents?

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[7]Fuchs and Schweigert, "Internal natural transformations and Frobenius algebras in the Drinfeld center".

[8]Fröhlich et al., "Correspondences of ribbon categories".

Examples from Hopf algebras

Unimodular H-comodule algebras

Definition

A left H-comodule algebra is an algebra A with a left H-comodule structure $\rho : A \to H \otimes A$ such that ρ is an algebra map.

$$\rho(aa') = a_{(-1)}a'_{(-1)} \otimes a_{(0)}a'_{(0)}, \qquad \rho(1_A) = 1_H \otimes 1_A.$$

Using A, we can define a functor $\triangleright$: Rep$(H) \times$ Rep$(A) \to$ Rep(A) where,

- $X \triangleright M := X \otimes_{\Bbbk} M$ as a vector space, and
- $a \cdot (x \otimes_{\Bbbk} m) := a_{(-1)} \cdot x \otimes a_{(0)} \cdot m$.

An H-comodule algebra A is called is *exact* if the Rep(H)-module category Rep(A) is exact.

Definition

An exact H-comodule algebra is called *unimodular* if the Rep(H)-module category Rep(A) is unimodular.

Hopf algebras $\leftrightarrow$ tensor categories

Hopf algebras

- (H, m, u) algebra
- Δ, ε are algebra maps
- S bijective
- H unimodular $(\alpha = \varepsilon)$
- H-comodule algebra (A, ρ)
- exact H-comodule algebra A
- unimodular H-comodule algebra A

$\vdots$

Finite tensor categories

- Rep(H) abelian, $\Bbbk$-linear
- (Rep$(H), \otimes_{\Bbbk}, \Bbbk_{\varepsilon}$) monoidal
- Rep(H) admits duals, is rigid
- $D := \Bbbk_{\alpha}$, then $D \cong \mathbb{1}$
- Rep(H)-module category Rep(A)
- Rep(A) is exact module category
- Rep(A) is unimodular module category

$\vdots$

Main result

Theorem

An exact H-comodule algebra A is unimodular if and only if A admits a unimodular element.[a]

[a] Yadav, "On unimodular module categories".

By results of Skryabin's[9], we have:

- Exact H-comodule algebras are Frobenius. Let λ be a Frobenius form on A and $\{a^i\}, \{b_i\}$ dual basis satisfying $\langle \lambda, a^i b_j \rangle = \delta_{i,j}$.
- Let ν denote the Nakayama automorphism of A.

Definition

A unimodular element of A is an invertible element $\tilde{g} \in A$ satisfying:

1. $\tilde{g} a \tilde{g}^{-1} = \langle \alpha, S(a_{(-1)}) \rangle \nu^2(a_{(0)})$
2. $1_h \otimes \tilde{g} = \tau \cdot \rho(\tilde{g})$

where $\tau := \langle \lambda_A, a_0^i \rangle \langle \lambda_A, a_0^j \rangle \; g_H S^{-3}(a_{-1}^j) S^{-1}(a_{-1}^i) \otimes \nu(b_j b_i) \in H \otimes_{\Bbbk} L$.

[9] Skryabin, "Projectivity and freeness over comodule algebras".

Potential application

Question: Is there a notion of integrals for H-comodule algebra? Can it be used to characterize unimodularity?

Kaplansky's 5^{th} conjecture: If H is semisimple or cosemisimple then $S^2 = \mathrm{Id}$.

Theorem[10]: H cosemisimple with $S^2 = \mathrm{Id}$ implies H is unimodular.

K5 conjecture + Theorem together imply the following conjecture:

Weak Kaplansky conjecture: Cosemisimple Hopf algebras H are unimodular.

Equivalently, in the language of unimodular module categories:

Conjecture: Let H be a semisimple Hopf algebra. Then $\mathsf{Vec} = \mathsf{Rep}(\Bbbk)$ is a unimodular $\mathcal{C} = \mathsf{Rep}(H)$ module category.

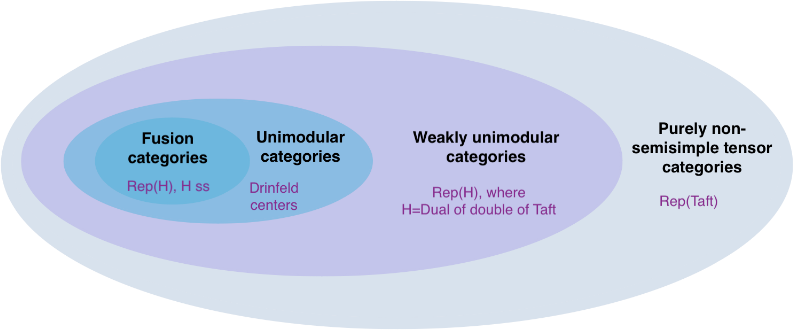
[10] Larson, "Characters of Hopf algebras".

Application

Theorem

The category $\mathcal{C} = \text{Rep}(Taft)$ is not categorically Morita equivalent to a unimodular tensor category.

A finite tensor category is called *weakly unimodular* is its categorical Morita equivalence class contains a unimodular category.



<u>Question</u>: Find a characterization of weakly unimodular categories.

Thank you!