ON UNIMODULAR MODULE CATEGORIES

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Based on the works:

[Y1] Frobenius monoidal functors from (co)Hopf adjunctions (arXiv: 2209.15606)

[Y2] On unimodular module categories (in preparation)

MOTIVATION

Consider the following categories:

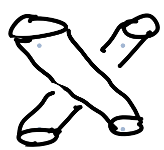
nCob	**$Vec_{\mathbb{C}}$**
Objects : closed $(n-1)$ diml. mflds/$\sim$	Objects : f.d. $\mathbb{C}$-vector spaces
Morphisms : n dimensional bordisms/$\sim$	Morphisms : $\mathbb{C}$-linear maps
$\otimes$: disjoint union	$\otimes$: $X \otimes_{\mathbb{C}} Y$
unit object : empty manifold	unit object : $\mathbb{C}$
braiding : Flip	braiding : $X \otimes Y \longrightarrow Y \otimes X$ $x \otimes y \longmapsto y \otimes x$

EXAMPLE: 2 Cob

Objects : disjoint unions of 

Morphisms : 

$\otimes$: 

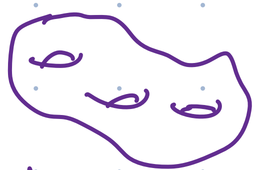
braiding : 

MOTIVATION

DEFINITION: An nD Topological Quantum Field Theory (TQFT) is a symmetric monoidal functor

$$F : n\mathrm{Cob} \longrightarrow \mathrm{Vec}_{\mathbb{C}}$$

- n-D TQFTs yield invariants of closed n-D manifolds

F() : $F(\phi) \longrightarrow F(\phi)$, with $F(\phi) = \mathbb{C}$ and $F(\phi) = \mathbb{C}$

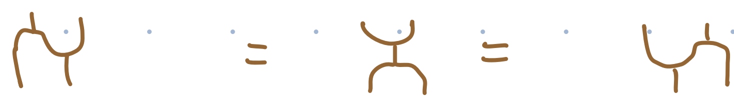
$$\phi \xrightarrow{M} \phi$$

i.e. $F(M)$ is a scalar

EXAMPLE : $\{2D \text{ TQFTs}\} \longleftrightarrow \{\text{commutative Frobenius algebras}\}$

$$F \longmapsto F(\bigcirc)$$

[A Frobenius algebra is a 5-tuple $(A, m, u, \Delta, \varepsilon)$ such that (A, m, u) is an algebra, (A, Δ, ε) is a coalgebra and m, Δ satisfy $(id \otimes m)(\Delta \otimes id) = \Delta m = (m \otimes id)(id \otimes \Delta)$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MODULAR TENSOR CATEGORIES

- The story of TQFTs gets much more interesting in 3D

$$\{\text{Modular tensor categories}\} \longrightarrow \{\text{3D TQFTs}\}$$

(not neccesarily semisimple)

DEFINITION: A Modular Tensor category (MTC) $\mathcal{C}$ is a braided finite tensor category that is
- ribbon
- non-degenerate

- Various classical knot and 3-manifold invariants are obtained using 3D TQFTs

  e.g. $\mathcal{C} = \overline{\mathrm{Rep}(\mathcal{U}_q(\mathfrak{sl}_2))}$ yields the Jones polynomial

- We want more invariants of 3D-manifolds

  $\Rightarrow$ want more 3D-TQFTs

  $\Rightarrow$ want more examples of MTCs

MODULAR TENSOR CATEGORIES

COMMON SOURCES OF **MTCs**

- Representation categories of quantum groups
- Representation categories of VOAs
- Drinfeld centers of spherical categories

WAYS OF GETTING NEW **MTCs** FROM OLD

- Deligne product $\mathcal{C} \boxtimes \mathcal{D}$
- graded extensions
- Zesting
- Take nice Frobenius alg A in $\mathcal{C}$ and form the category of local modules $\mathcal{C}_A^{\circ}$.

PROBLEM: Construct nice Frobenius algebras in the Drinfeld center

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STRATEGY: Construct Frobenius monoidal functors
$$F : \boxed{} \longrightarrow \mathcal{Z}(\mathcal{C})$$

DEFINITION: A Frobenius monoidal functor between monoidal categories $\mathcal{C}, \mathcal{D}$ is a 5-tuple (F, F_2, F_0, F^2, F^0) where

- $F : \mathcal{C} \to \mathcal{D}$ is a functor
- (F, F_2, F_0) is a lax monoidal functor
- (F, F^2, F^0) is an oplax monoidal functor
- F_2, F^2 satisfy a compatibility relation

$$\left(\begin{tikzcd} F(X \otimes Y) \otimes F(Z) \arrow[d, "F_2"] \\ F(X \otimes Y \otimes Z) \arrow[d, "F^2"] \\ F(X) \otimes F(Y \otimes Z) \end{tikzcd}\right) = \left(\begin{tikzcd} F(X \otimes Y) \otimes F(Z) \arrow[d, "F^2 \otimes \mathrm{id}"] \\ F(X) \otimes F(Y) \otimes F(Z) \arrow[d, "\mathrm{id} \otimes F_2"] \\ F(X) \otimes F(Y \otimes Z) \end{tikzcd}\right) + \text{similar diagrams}$$

- Strong monoidal functors (F_2, F_0 are iso) are Frobenius monoidal.
- If $A \in \mathrm{FrobAlg}(\mathcal{C})$, then $F(A) \in \mathrm{FrobAlg}(\mathcal{D})$.

How to get Frobenius monoidal functors with target $\mathcal{Z}(\mathcal{C})$?

Use right adjoints of strong monoidal functors

(e.g. $\mathcal{U}: \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$)

THEOREM 1 [Y1] Let $\mathcal{U}: \mathcal{C} \to \mathcal{D}$ be a strong monoidal functor between abelian monoidal categories such that

(i) $\mathcal{U}$ admits a right adjoint R.

(ii) R is exact, faithful

(iii) $\mathcal{U} \dashv R$ is a coHopf adjunction → (true if $\mathcal{C}, \mathcal{D}$ are rigid)

Then,

a) R is a Frobenius monoidal functor $\iff$ $R(\mathbb{1}_{\mathcal{D}})$ is a Frobenius algebra in $\mathcal{D}$.

b) Further if $\mathcal{C}, \mathcal{D}$ are pivotal and $F: \mathcal{C} \to \mathcal{D}$ is a pivotal functor then

R is a pivotal Frobenius monoidal functor $\iff$ $R(\mathbb{1}_{\mathcal{D}})$ is a symmetric Frobenius algebra in $\mathcal{D}$

∎

- Part (a) follows from earlier work of Balan.
- The notion of 'pivotal Frobenius monoidal functor' above is a generalization of 'pivotal functors' introduced by Ng-Schauenburg.

How to get Frobenius monoidal functors with target $\mathcal{Z}(\mathcal{C})$?

- Let $(\mathcal{M}, \triangleright)$ be an indecomposable exact left $\mathcal{C}$-module category.
- Set $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$ = category of $\mathcal{C}$-module endofunctors of $\mathcal{M}$.

Shimizu studied the following functor

$$\psi : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$$
$$(c, \sigma) \longmapsto (c \triangleright -, s^{\sigma})$$

→ ψ is a strong monoidal functor

→ ψ satisfies the conditions (i)–(iii) of Theorem 1.

THEOREM 2 [Y1] a) ψ^{ra} is a Frobenius monoidal functor $\iff$ $\psi^{ra}(\mathrm{id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$

b) Let $\mathcal{C}$ be pivotal and $\mathcal{M}$ a pivotal left $\mathcal{C}$-module category. Then
ψ^{ra} is a pivotal Frobenius functor $\iff$ $\psi^{ra}(\mathrm{id}_{\mathcal{M}})$ is a symmetric Frobenius alg. in $\mathcal{Z}(\mathcal{C})$. ■

When is $\psi^{ra}(\mathrm{id}_{\mathcal{M}})$ is Frobenius monoidal?

first some background

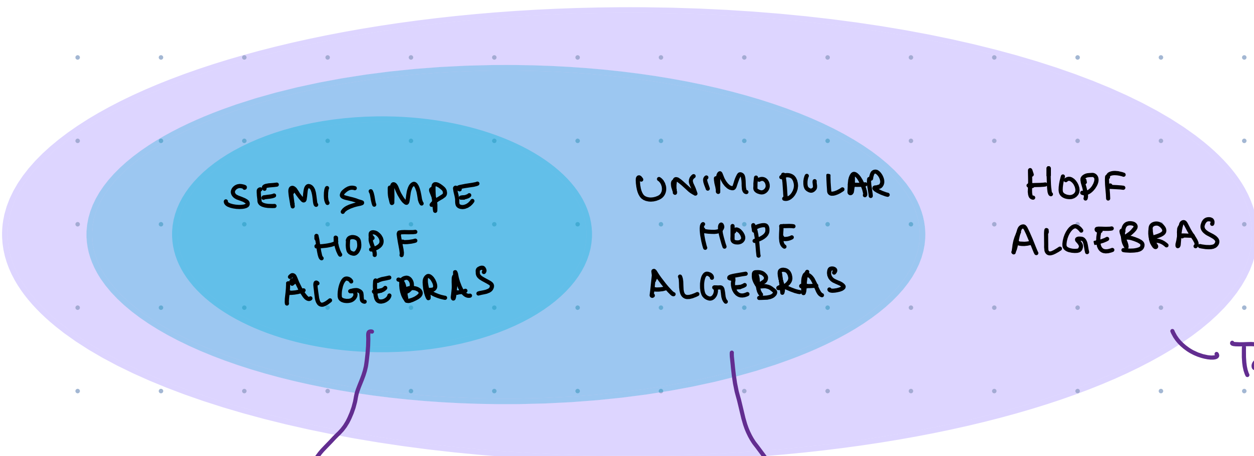
UNIMODULAR HOPF ALGEBRAS

(rewrite)

- Let $(H, m, u, \Delta, \varepsilon, S)$ be a finite-dimensional Hopf algebra
- A left integral is an element $\Lambda \in H$ satisfying
$$h\Lambda = \varepsilon(h)\Lambda \quad \forall \quad h \in H.$$
- Left integrals always exist. Fix any one.
- $\exists$ $\alpha \in H^*$ satisfying $\Lambda h = \alpha(h)\Lambda$ $\forall$ $h \in H$. α is called the distinguished grouplike element of H^*.

used in Radford's S^4 formula

- H is called unimodular if $\alpha = \varepsilon$.



Taft algebras

$\Bbbk G$ with $\mathrm{char}(\Bbbk) \nmid |G|$

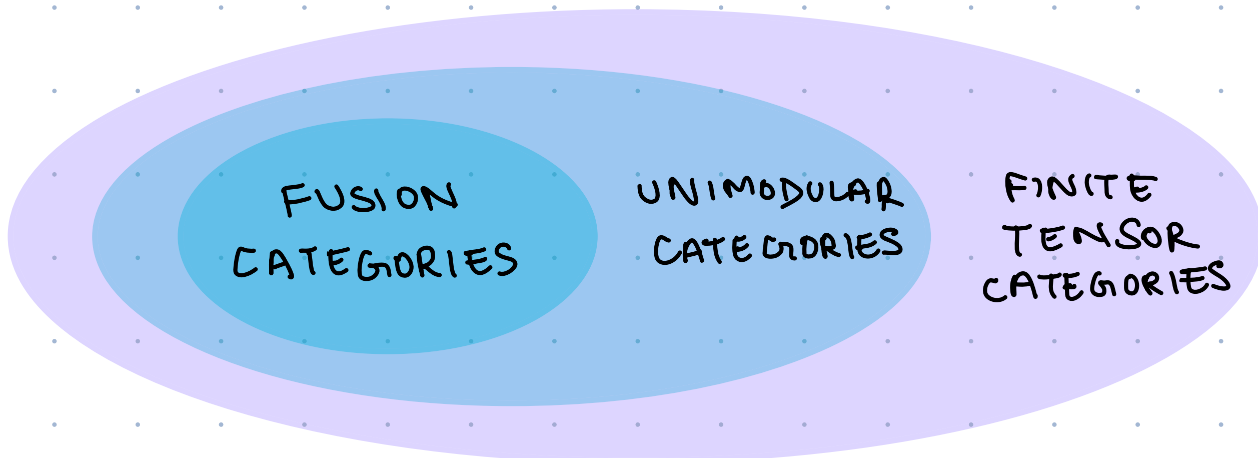
$\Bbbk G$ with $\mathrm{char}(\Bbbk) \mid |G|$

UNIMODULAR TENSOR CATEGORIES

- For H Hopf, define the object $D \in \mathrm{Rep}(H)$
  $D = \Bbbk$ as vector space, $h \cdot c = \alpha(h)c \quad \forall\, c \in D, h \in H$
- Then $D \cong \mathbb{1}_{\mathrm{Rep}(H)} \iff \alpha = \varepsilon$

Etingof–Ostrik defined an analogue of the object D in any finite tensor category $\mathcal{C}$.

- A finite tensor category $\mathcal{C}$ is called unimodular if $D \cong \mathbb{1}_{\mathcal{C}}$.



PROBLEM: Construct nice Frobenius algebras in $\mathcal{Z}(\mathcal{C})$

- Used $\varphi : \mathcal{Z}(\mathcal{C}) \longrightarrow \mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$
- $\varphi^{ra} : \mathrm{Rex}_{\mathcal{C}}(\mathcal{M}) \to \mathcal{Z}(\mathcal{C})$ is Frobenius monoidal $\Longleftrightarrow$ $\varphi^{ra}(\mathrm{id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$

Q When is $\varphi^{ra}(\mathrm{id}_{\mathcal{M}})$ Frobenius?

THEOREM 3 [Y2] The following are equivalent

a) φ^{ra} is Frobenius monoidal

b) $\varphi^{ra}(\mathrm{id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.

c) $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$ is a unimodular finite tensor category.

d) $\mathbb{S}^r \circ \mathbb{N}^r \cong \mathrm{id}_{\mathcal{M}}$

[Here, $\mathbb{S}^r : \mathcal{M} \to \mathcal{M}$ is the relative Serre functor
$\mathbb{N}^r : \mathcal{M} \to \mathcal{M}$ is the Nakayama functor]

DEFINITION: If any of the above four equivalent conditions is satisfied, we call $\mathcal{M}$ a unimodular module category.

Remarks
- Theorem 3 essentially follows from work of Shimizu.
- For part (d), we use the formula for $D_{\mathrm{Rex}_{\mathcal{C}}(\mathcal{M})}$ provided in the work of Fuchs-Galindo-Jaklitsch-Schweigert.

BACK TO HOPF ALGEBRAS ...

- Consider $\mathcal{C} = \mathrm{Rep}(H)$, then $Z(\mathcal{C}) = {}^H_H\mathcal{YD}$ (Yetter-Drinfeld modules)
- By results of Andruskiewitsch-Mombelli, every left $\mathcal{C}$-module category is of the form $\mathcal{M} = \mathrm{Rep}(L)$ for L a left H-comodule algebra.
- Then $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M}) \cong {}^H_L\mathfrak{M}_L$ (L-bimodules with compatible H coaction)

Using a description of $\mathbb{S}^r$ (from Shimizu's work) and $\mathbb{N}^r$, we are able to explicitly characterize the unimodular module categories over $\mathrm{Rep}(H)$.

THEOREM 4 [Y2] $\mathcal{M} = \mathrm{Rep}(L)$ is unimodular $\iff$ L admits a unimodular element

[this answers a question of Shimizu)

UNIMODULAR ELEMENT

- By Skyrabin's result, L is a Frobenius algebra.
- Let λ_L denote the Frobenius form
- $\nu_L =$ Nakayama automorphism of L
- $g_H =$ distinguished grouplike element of H

DEFINITION A unimodular element of L is an invertible element $\bar{g} \in L$ satisfying

(i) $\bar{g}\, \ell\, \bar{g}^{-1} = \bar{\nu}(\ell) \quad \forall\, \ell \in L$

(ii) $1_H \otimes \bar{g} = \mathcal{J} \cdot \delta(\bar{g})$

Here, $\bar{\nu}(\ell) = \langle \alpha_H, S(a_{-1}) \rangle\, \nu_L^2(a_0)$

$\mathcal{J} = \langle \lambda_A, a_0^i \rangle \langle \lambda_A, a_0^j \rangle\, g_H\, S^{-3}(a_{-1}^j)\, S^{-1}(a_{-1}^i) \otimes \nu(b_j b_i) \in H \otimes_{\Bbbk} L$

This definition simplifies a lot, if we assume that the form λ_L is grouplike.

- Let L be a left H-comodule algebra
- A linear form $\lambda : H \longrightarrow \Bbbk$ is called g_L-grouplike if there exists a grouplike element $g_L \in H$ such that

$$l_{(-1)} \langle \lambda, l_{(0)} \rangle = \langle \lambda, l \rangle g_L$$

PROPOSITION [YZ] Let L be a left H-comodule algebra whose Frobenius form λ_L is g_L-grouplike. Then an unimodular element of L is an invertible $\bar{g} \in L$ satisfying

(i) $\bar{g}\, l\, \bar{g}^{-1} = \nu(l)$ (ii) $g_H^{-1} g_L \otimes \bar{g} = \delta(\bar{g})$