ON UNIMODULAR MODULE CATEGORIES

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§1 MOTIVATION

a) TQFT is a symm. monoidal functor $F: n\mathrm{Cob} \to \mathrm{Vect}$

$\{2D\text{-TQFT}\} \leftrightarrow \left\{\begin{matrix}\text{commutative}\\ \text{Frobenius}\\ \text{algebras}\end{matrix}\right\}$

(Story about how I got interested) ....

$\{3D\text{-TQFTs}\} \longleftarrow \left\{\begin{matrix}\text{modular tensor}\\ \text{categories}\end{matrix}\right\}$

including non-semisimple ones

very important

yield invariants of mflds.
- Jones polynomial
- Reshetikhin–Turaev invariants
- ...

Hence we want more TQFT
$\Rightarrow$ want more MTCs

There are many ways of producing new MTC

Common Sources of MTCs
- Reps of Quantum groups, VOAs
- Drinfeld centers of spherical categories

Processes of getting new MTCs from old ones
- Deligne product $\mathcal{C} \boxtimes \mathcal{D}$
- graded extension
- Zesting
- category of local modules $\mathcal{C}_A^0$ where A = nice Frob. algebra

$\rightarrow$ My talk focuses on
Drinfeld centers + local modules using nice Frobenius algebras in $Z(\mathcal{C})$

From CFT perspective, MTCs encode the chiral data of some CFT

→ In the formulation of Fuchs-Runkel-Schweigert,
Frob algs A $\longleftrightarrow$ boundary fields
A-A' bimodules $\longleftrightarrow$ defects

∴ From the TQFT + CFT motivations, it is important to understand Frobenius algebras in MTCs

PROBLEM : Construct Frobenius algebras (functorially) in the Drinfeld center.

STRATEGY : Construct Frobenius monoidal functors

$$\boxed{} \longrightarrow Z(\mathcal{C})$$

preserve Frobenius algebras

RESULTS

① Consider a strong monoidal functor $F : \mathcal{C} \to \mathcal{D}$ such that

rigid

- ~~$F \dashv F^{ra}$ is a coHopf adjunction~~
- F^{ra} is exact + faithful

then get that

F^{ra} is	$\Longleftrightarrow$	$F^{ra}(\mathbb{1}_{\mathcal{D}})$ is
(known) • Frob. monoidal		Frob. algebra
• separable Frob. mon		separable Frob
• pivotal		symmetric Frob.
⋮		⋮

② Apply ① to $\varphi : Z(\mathcal{C}) \longrightarrow \mathrm{Fun}_{\mathcal{C}}(\mathcal{M})$

$(c, \sigma) \longmapsto (c \triangleright - , s^{\sigma})$

category of right exact $\mathcal{C}$-module endofunctors of a module category $\mathcal{M}$.

Get that

$\varphi^{ra} : \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}) \to Z(\mathcal{C})$ is $\iff$ $\varphi^{ra}(\mathrm{id}_{\mathcal{M}})$ is

• Frobenius monoidal	• Frobenius alg
• pivotal	• symm. Frob. alg.
⋮	⋮

③ Defn: Exact left $\mathcal{C}$-module category is called unimodular if $\mathrm{Rex}_{\mathcal{C}}(\mathcal{M})$ is unimodular finite tensor category (i.e. $D \cong \mathbb{1}$).

Thm: The following are equivalent
- $\mathcal{M}$ is unimodular module cat.
- $\mathbb{S}_{\mathcal{M}} \circ \mathbb{N}_{\mathcal{M}} \cong \mathrm{id}_{\mathcal{M}}$ as $\mathcal{C}$-module functors
- $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M})$ is unimodular FTC

$\mathcal{M}$ indec: $\begin{cases} \bullet\ \varphi^{ra}(\mathrm{id}_{\mathcal{M}}) \text{ is Frob. algebra in } Z(\mathcal{C}) \\ \bullet\ \varphi^{ra} \text{ is a Frob. monoidal functor} \end{cases}$

also able to reprove result of Fuchs-Schweigert.

Thm: Let $\mathcal{C}$ be pivotal, $\mathcal{M}$ unimodular + pivotal
Then $\varphi^{ra}(\mathrm{id}_{\mathcal{M}})$ is symm. Frobenius
$\Rightarrow \varphi^{ra}$ is pivotal Frob. mon. functor.

④ Classify unimodular module categories over $\mathcal{C} = \mathrm{Rep}(H)$.
→ Module categories are of the form $\mathcal{M} = \mathrm{Rep}(L)$ where L is a left H-comodule algebra.

Thm: $\underset{\mathrm{Rep}(L)}{\overset{\shortparallel}{\mathcal{M}}}$ is unimodular $\iff$ L admits a unimodular element
- some invertible element of L
- answers Q of Shimizu