An extended TQFT is a symm. monoidal 2-functor

$$\mathrm{Bord}_1^3 \xrightarrow{Z} \mathrm{LinCat}$$

We want to know:

① Given Z, what is the structure of $Z(S^1)$?
② To build Z, what all do we need?

→ So far we learned that:
$Z(S^1)$ is a braided tensor category

→ In this lecture, we learned that

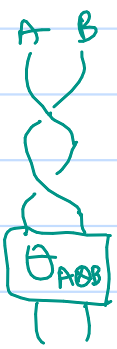
- $Z(S^1)$ has further structure
  → it is equipped with a $\boxed{\text{twist}}$ and is called $\boxed{\text{balanced}}$

Defn Given a braided tensor category $\mathcal{C}$, a twist is an (invertible) natural transformation

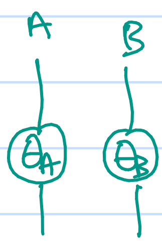
$$\theta : \mathrm{Id}_{\mathcal{C}} \Longrightarrow \mathrm{Id}_{\mathcal{C}}$$

satisfying

compatibility with c



$$= \boxed{\theta_{A\otimes B} \circ c_{B,A} \circ c_{A,B} = \theta_A \otimes \theta_B} =$$



compatibility with unit

$$\boxed{\theta_{\mathbb{1}} = \mathrm{Id}_{\mathbb{1}}}$$

→ A braided tensor category equipped with a twist is called balanced.

Q What structure on a quasitriangular Hopf algebra (A,R) do we need so that A-mod is balanced?

→ A twist on the category A-mod corresponds to the choice of an invertible elt. $\nu \in A$ such that

- $\nu \in Z(A)$
- $\varepsilon(\nu) = 1$
- $R\overline{R} = \Delta(\nu^{-1})(\nu \otimes \nu)$

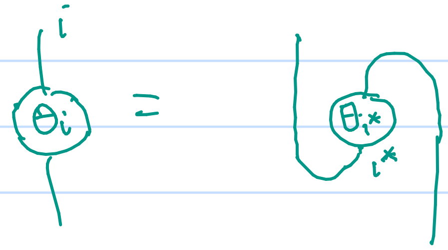
Such a ν is called a balancing element.

Given such a ν, get $\theta : Id_{A\text{-mod}} \longrightarrow Id_{A\text{-Mod}}$

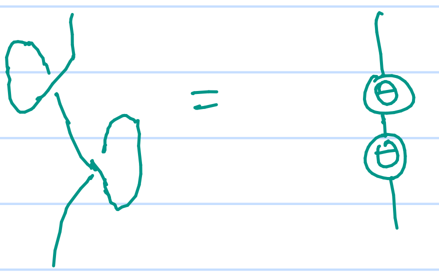
by $\theta_M : M \longrightarrow M$
$m \longmapsto \nu \cdot m$

Ribbon structure : A balanced tensor category is called ribbon if θ satisfies

(i)



(ii)

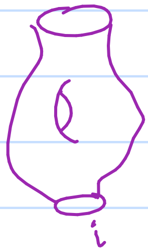


After this, the technology of internal string diagrams was introduced and used to prove the following important results.

<u>Thm</u>: $Z(S')$ is rigid, i.e., it has duals.

<u>Prop</u>: $Z(S')$ is ribbon

→ Internal string diagrams

Say $\Sigma =$ 

For $i \in \mathcal{C}$, the object $Z(\Sigma)(i)$ is understood by understanding the vector space $\mathrm{Hom}(j, Z(\Sigma)(i))$ for $j \in \mathcal{C}$

FACT: $\mathrm{Hom}(j, Z(\Sigma)(i)) = \{$formal linear combinations of internal string diagrams$\}$ / isotopy

ex: 

What's the point of this?

a natural trans $Z(\Sigma) \xrightarrow{Z(M^3)} Z(\Sigma')$

→ is determined by $Z(\Sigma)(i) \xrightarrow{Z(M^3)(i)} Z(\Sigma')(i)$

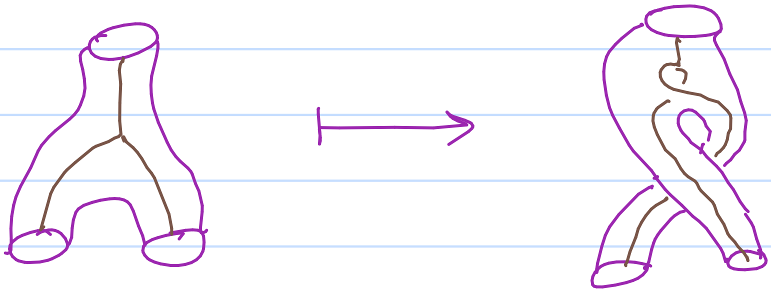
→ this induces a map

$$\mathrm{Hom}(j, Z(\Sigma)(i)) \longrightarrow \mathrm{Hom}(j, Z(\Sigma')(i)) \quad (\ast)$$

→ Furthermore, the map $(\ast)$ is determined by its effect on an internal string diagram on LHS

For example,

→ the braiding natural transformation is specified as



→ the twist is specified as