[EGNO, Sections 5.2, 5.3, 9.9], [TV, Section 6.2]

In the last talk, we saw that for a bialgebra $A = (A, m, u, \Delta, \varepsilon)$, the category mod_A is monoidal.

Now suppose A is a bialgebra in Vec (category of finite dim. vector spaces) and consider the subcategory $\mathrm{Rep}(A)$ of mod_A whose objects are f.d. modules over A. Then $\mathrm{Rep}(A)$ is

locally finite + abelian + $\Bbbk$-linear + monoidal + $\mathrm{End}(\mathbb{1}) = \Bbbk$ + $\otimes$ is biexact, bilinear

i.e. a ring category

Further $\mathrm{Rep}(A)$ is finite (since it is the cat. of f.d. modules over a f.d. bialgebra). Thus, $\mathrm{Rep}(H)$ is a finite ring category.

Furthermore, we have a forgetful functor

$$\mathrm{Forget} : \mathrm{Rep}(A) \longrightarrow \mathrm{Vec}$$

which forgets the left A-action. The functor Forget is exact, faithful & monoidal.

Such a functor is called a Fiber functor.

Thus, starting with A, we get a finite ring category $\mathcal{C} = \mathrm{Rep}(A)$ equipped with a fiber functor $F : \mathrm{Rep}(H) \longrightarrow \mathrm{Vec}$.

Q Can we reverse this process starting from $(\mathcal{C}, F)$? YES!

Observe that in the case, $\mathcal{C} = \mathrm{Rep}(A)$ & $F = \mathrm{Forget}$, $\forall\ a \in A$ & $X \in \mathrm{Rep}(A)$, we get a map of vector spaces

$$a\cdot : X \to X, \quad x \mapsto a \cdot x$$

We can reformulate this statement as $\forall\ a \in A$, the left action of a gives a natural transformation $\tilde{a}$

$$\begin{tikzcd} \mathrm{Rep}(H) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \tilde{a}\cdot"] & \mathrm{Vec} \end{tikzcd}$$

Further the natural transformation corresponding to left action of aa' is the composition of the nat. trans. for a, a', i.e. $\widetilde{aa'} = \tilde{a} \circ \tilde{a}'$

Also left action of unit 1_A corresponds to the identity natural transformation on F.

$(\tilde{a} \circ \tilde{1}_A = \tilde{a} = \tilde{1}_A \circ \tilde{a})$

This encourges us to consider in general

$$\mathrm{End}(F) = \left\{ \eta \;\middle|\; \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathrm{Vec} \end{tikzcd} \text{ natural transformations} \right\}$$

End(F) is an algebra with unit the identity natural transformation of F.

But we want a coalgebra, so continue ...

Given $a \in A$, $\Delta(a) \in A \otimes A$. Given $X, Y \in \mathrm{Rep}(A)$

a acts on $X \otimes Y$ componentwise using $\Delta(a)$

$$A \otimes X \otimes Y \longrightarrow X \otimes Y$$
$$a \otimes x \otimes y \longmapsto a_{(1)}x \otimes a_{(2)}y \qquad [\Delta(a) = a_{(1)} \otimes a_{(2)}]$$

Thus $\Delta(a)$ yield a vector space map

$$X \otimes Y \longrightarrow X \otimes Y$$

We have the associated natural transformation

$\widetilde{\Delta(a)} \in \mathrm{End}(F) \otimes \mathrm{End}(F)$

yielding a coproduct on End (A)

This works more generally

<u>Thm 5.2.3:</u> The assignments

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}(F), \quad H \mapsto (\mathrm{Rep}\, H, \mathrm{Forget})$$

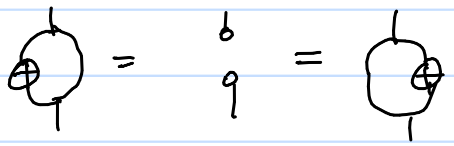
are mutually inverse bijections between

(1) finite ring categories $\mathcal{C}$ with a fiber functor $F : \mathcal{C} \longrightarrow \mathrm{Vec}$, up to monoidal equivalence & iso. of monoidal functors and

(2) iso. classes of f.d. bialgebras H over $\mathbb{k}$.

[6.2.1, TV]

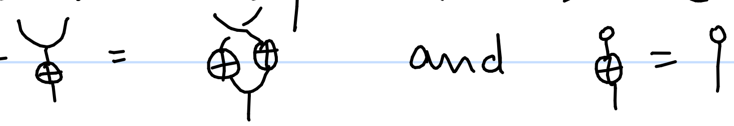
Let $A = (A, m, u, \Delta, \varepsilon)$ be a bialgebra in a braided category $\mathcal{C}$. An antipode of A is a morphism $S : A \to A$ denoted s.t.



$$m(S \otimes id)\Delta = u\varepsilon = m(id \otimes S)\Delta \qquad (*)$$

→ Condition (*) is equivalent to saying that S is a two-sided inverse of id_H in the convolution monoid $Hom_{\mathcal{C}}(A, A)$. As a consequence, if a antipode exists, then it is unique.

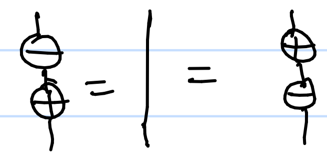
Further A is anti multiplicative, i.e.



and anti-comultiplicative, i.e.

→ to prove this, one can show that both  and  are convolution inverses of m in the monoid $Hom_{\mathcal{C}}(A \otimes A, A)$, thus they are equal because of the uniqueness of antipode.

When S is invertible, we denote $S^{-1} : A \to A$ by  & this satisfies



6.2.2 HOPF ALGEBRA:

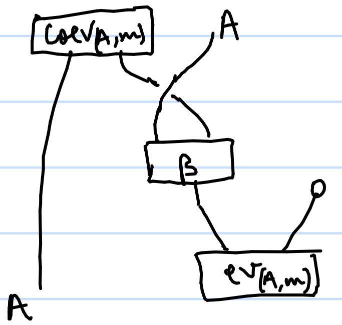
- A <u>Hopf algebra</u> is a bialgebra $(A, m, u, \Delta, \varepsilon)$ equipped with an invertible antipode S.

- A Hopf algebra morphism between two Hopfs is just a bialgebra morphism between them. Given $f : A \longrightarrow A'$ Hopf alg. morphism, one has $f \circ S = S' \circ f$

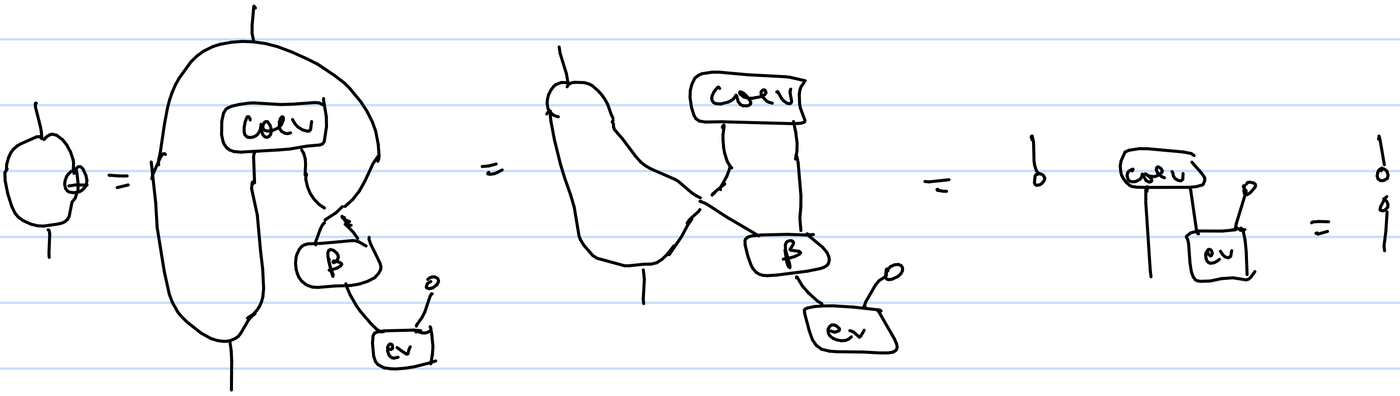
Recall that for $\mathcal{C}$ braided and A a bialgebra in $\mathcal{C}$, the category mod_A is a monoidal category
left A-modules

<u>Lemma 6.1</u>: Let A be a bialgebra in a braided ~~category~~ rigid category $\mathcal{C}$. The monoidal category mod_A is rigid if and only if A is a Hopf algebra.
Moreover, if A is a Hopf algebra, then any left/right duality in $\mathcal{C}$ has a unique lift in mod_A along the forgetful functor $\mathrm{mod}_A \longrightarrow \mathcal{C}$.

Proof: $(\Rightarrow)$ Suppose mod_A is rigid.
Since $(A, m) \in \mathrm{mod}_A$, pick a left dual ${}^\vee(A, m)$ with $ev_{(A,m)}$ & $coev_{(A,m)}$
${}^\vee(A,m)$ comes equipped with a left A-action β

Define $S =$ 

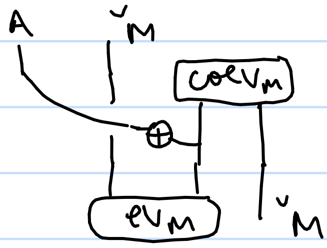
Similarly can define S^{-1} using right dual $(A,m)^\vee$.



(Self note: try to prove  later)

($\Leftarrow$) Conversely suppose A is hopf with antipode S. Since $\mathcal{C}$ is rigid, fix left duality $\{({}^{\vee}X, ev_X, coev_X)\}_{X \in ob(\mathcal{C})}$

For any (M, γ) in mod_A, consider $(({}^{\vee}M, {}^{\circ}\gamma), ev_M)$ where

${}^{\circ}\gamma$ = 

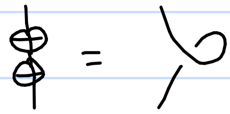
then $(({}^{\vee}M, {}^{\circ}\gamma), ev_M)$ is left dual of (M, γ).

Similar for right dual.

(See TV's book for more details)

[TV] 6.2.3: Involutory Hopf algebra

Let $\mathcal{C}$ be braided pivotal category. A Hopf algebra A in $\mathcal{C}$ is involutory if its antipode S satisfies $S^2 = \theta_A^{\vee}$ = right twist of $\mathcal{C}$



H involutory $\Rightarrow$ mod_A pivotal s.t. forgetful $mod_A \to \mathcal{C}$ is strictly pivotal

$\therefore$ left/right trace of f in mod_A = trace in $\mathcal{C}$

Further $\mathcal{C}$ spherical $\Longrightarrow$ mod_A is spherical

Examples ① Hopf algebras in $\mathrm{Vect}_{\Bbbk}$ are usual Hopf algebras

② A Hopf algebra in $\mathrm{proj}_{\Bbbk}$ (= projective modules) is involutory if and only if the antipode is an involution, i.e. $S^2 = \mathrm{Id}_H$.

Recall

tensor category = locally finite + $\Bbbk$-linear + abelian + rigid + monoidal + $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$

= ring category + rigid

$\therefore$ $\mathrm{Rep}(H)$ for H a Hopf algebra in Vec is a finite tensor category

We have the following reconstruction theorem for f.d. Hopf algebras

Thm 5.3.12 (EGNO): The assignments

$(\mathcal{C}, F) \mapsto H = \mathrm{End}(F)$, $H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$

are mutually inverse bijections between

(1) equivalence classes of finite tensor categories $\mathcal{C}$ with a Fiber functor F, up to tensor equivalence and isomorphism of tensor functors, and

(2) isomorphism classes of finite dimensional Hopf algebras over $\Bbbk$.

In order to get reconstruction for infinite dim. Hopf algebras, we have to use the coend construction for the category of comodules (See [EGNO, Section 5.4])

These results are great but not concrete enough because we don't have a hold of all f.d. Hopf algebras.
So we add more adjectives

Object		Its Category of modules
f.d. bialgebra	$\longleftrightarrow$	finite ring category with fiber functor (FF)
f.d. Hopf alg	$\longleftrightarrow$	finite tensor cat with FF
quasitri Hopf. alg.	$\longrightarrow$	finite braided tensor + FF
triangular Hopf alg	$\longrightarrow$	finite symm tensor + F.F
f.d. semisimple Hopf algebra	$\longrightarrow$	fusion cat + FF

Next, we will discuss Deligne's theorem for classification of symmetric tensor categories. Before that we need to understand an important symmetric tensor category.

$\boxed{\text{Rep}(G, z)}$ G: finite group, $z \in G$ central element of order 2 (also called finite super group)

$\hookrightarrow$ an irrep of G is odd if z acts by -1 ($\deg = 1$) & even if ($\deg = 0$) z acts by identity

Denote the degree of a simple object X by $|X| \in \{0, 1\}$, then the braiding is

$$c'_{X,Y}(X \otimes Y) = (-1)^{|X||Y|}\, Y \otimes X$$

The category Rep(G) equipped with the braiding c' described on last page is called $\mathrm{Rep}(G,z)$.

Deligne's theorem for general tensor category

Let $\mathcal{C}$ be a finitely $\otimes$ generated symm. tensor cat.
a) s.t. for any $X \in \mathcal{C}$ there is λ with $S_\lambda(X)=0$), or equivalently,
b) $\forall X \in \mathcal{C}$ $\exists$ $N \in \mathbb{N}$ s.t.

$$\text{length}(X^{\otimes n}) \leq N^n \quad \forall n \geq 0$$

($\mathcal{A}$ has subexponential growth)
Then $\mathcal{C}$ is equivalent as a tensor category to $\mathrm{Rep}(G,z)$ for some supergroup G.

- After this thm, Deligne sought for symm. tensor cats. that don't exhibit ~~sub~~exponential growth. In the process he discovered $\mathrm{Rep}(S_t)$, these don't have subexponential growth.

- Above thm reduces the problem of understanding symmetric tensor cats (a large class) to understanding supergroups (G,z) which is easier to do.

- Symmetric fusion categories exhibit subexponential group. So, the above thm applies to them.

- Deligne's theorem led to the classification of triangular Hopf algebras.