BRAVERMAN - GAITSGORY THEORY AND THE PBW THEOREM

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Learing seminar
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FILTERED ALGEBRAS :

V = finite dim'l vector space

$T = \oplus T^i$ = tensor algebra over V

P = some defining set of relations

$I = (P)$ = two sided ideal of relations

$$A = T/I$$

- If I is homogeneous, then A is graded.
- If I is non-homogeneous, then A is filtered with i^{th} component

$$F^i(A) = F^i(T/I) = (F^i(T)+I)/I$$

Example ①

Let V = the Lie algebra $\mathfrak{sl}_2$

$\mathfrak{sl}_2$ has basis $\{e, f, h\}$ with product

$$[e,f] = h \quad , \quad [h,f] = -2f \quad , \quad [h,e] = 2e$$

- Set $Q = \{x \otimes y - y \otimes x \mid x, y \in \mathfrak{sl}_2\}$

  then $I = (Q)$ is homogeneous and

  $A = \dfrac{T(\mathfrak{sl}_2)}{(Q)} = \mathrm{Sym}(\mathfrak{sl}_2)$ = symmetric alg on 3 generators

- Set $P = \{x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{sl}_2\}$

  then $I = (P)$ is non-homogeneous and

  $A = \dfrac{T(\mathfrak{sl}_2)}{(P)} = \mathcal{U}(\mathfrak{sl}_2)$ = Universal envelopping algebra of $\mathfrak{sl}_2$

HOMOGENEOUS VERSIONS OF FILTERED ALGEBRAS:

For a non-homogeneous ideal $I \subset T(V)$, to the filtered algebra $A = T(V)/I$ we can associate two different graded algebras

Version 1: Only keep the highest degree terms in the generating set P of relation to get $LH(P)$. Then,

$$A = T(V)/(LH(P)) \text{ is graded}$$

Version 2: Take the associated graded algebra $gr(T/(P))$

Definition:
When
$$\frac{T(V)}{(LH(P))} \cong gr(T/(P))$$
as graded algebra, we call $A = T/(P)$ a PBW deformation of its homogeneous version $T(V)/(LH(P))$

Example: (Continuing example ①)

$V = \mathfrak{sl}_2$ and $P = \{ x \otimes y - y \otimes x - [x,y] \mid x,y \in \mathfrak{sl}_2 \}$

then,

$$LH(P) = \{ x \otimes y - y \otimes x \mid x,y \in \mathfrak{sl}_2 \}$$

Then $$\frac{T(\mathfrak{sl}_2)}{(LH(P))} \cong Sym(\mathfrak{sl}_2) \cong gr\left(\frac{T(\mathfrak{sl}_2)}{(P)}\right) = gr(\mathcal{U}(\mathfrak{sl}_2))$$

Thus, $\mathcal{U}(\mathfrak{sl}_2)$ is a PBW deformation of $Sym(\mathfrak{sl}_2)$.

GOAL OF TALK:
Understand the PBW deformations of a Koszul algebra

Some groundwork needed ...

GRADED DEFORMATIONS

Let A be an $\mathbb{N}$-graded algebra. Let t be a formal parameter & set degree of $t = |t| = 1$.

Defn: A graded deformation of A over $\mathbb{k}[t]$ is a graded algebra A_t over $\mathbb{k}[t]$ satisfying

(i) $A_t = A[t]$ as a $\mathbb{k}$ vector space

(ii) $\frac{A_t}{(t)} \cong A$ as a graded algebra

REMARKS:

- (ii) $\Longleftrightarrow$ $\left[\begin{array}{l} \text{the product } * \text{ on } A_t \text{ satisfies} \\ \quad a * b = \sum_{i \geq 0} \mu_i(a \otimes b)\, t^i \quad \forall a, b \in A \\ \text{where } \mu_0(a \otimes b) = ab \end{array}\right]$

- Since the product $*$ is graded,
  $\deg(\mu_i) = -i$

- (ii) $\Longrightarrow$ $A_t|_{t=0} \cong A$

Defn: A i^{th}- level graded deformation of A is a graded associative algebra A_i over $\mathbb{k}[t]/(t^{i+1})$ that is isomorphic to $A[t]/(t^{i+1})$ as a $\mathbb{k}$-vector space, with multiplication given by

$$a * b = ab + \mu_1(a\otimes b)t + \cdots + \mu_i(a\otimes b)t^i$$

$\rightarrow$ We say that $(i-1)^{st}$ level graded deformation $A_{i-1} \xrightarrow{\text{lifts to}} A_i$ i^{th} level graded deformation

if $\exists$ a map $\mu_i : A\otimes A \rightarrow A$ such that

$$a * b = ab + \mu_1(a\otimes b)t + \cdots + \mu_i(a\otimes b)t^i$$

turns $A_i = A\otimes \mathbb{k}[t]/(t^{i+1})$ into an associative graded algebra over $\mathbb{k}[t]/(t^{i+1})$.

In this case, associativity of the product map $*$ for A_i is equivalent to:

$$\boxed{\delta_3^*(\mu_i)(a_1\otimes a_2\otimes a_3) = \underbrace{\sum_{j=1}^{i-1} \mu_j(\mu_{i-j}(a_1\otimes a_2)\otimes a_3) - \mu_j(a_1\otimes \mu_{i-j}(a_2\otimes a_3))}_{i^{th}\text{ obstruction}}} \quad (\dagger)$$

here δ_3^* is the Hochschild differential map.

- The RHS of equation $(\dagger)$ considered as a function from $A\otimes A\otimes A$ to A is called the i^{th}-obstruction.

- Set $HH^{n,i}(A) = \{ f : A^{\otimes n} \rightarrow A \in HH^n(A) \mid \deg(f) = i \}$

PROPOSITION :

(i) i^{th} obstruction $\in HH^{3,-i}(A)$

(ii) $(i-1)^{st}$ level graded deformation lifts to i^{th} level

$\Updownarrow$

i^{th} obstruction becomes 0 in cohomology

Proof: (i) straightfoward check

(ii) ($\Rightarrow$) then $\exists\ \mu_i$ satisfying $(\dagger)$

($\Leftarrow$) 0 in cohomology $\Rightarrow \exists\ \mu_i$ such that $d_3^*(\mu_i) =$ i^{th}-obstruction

hence $(\dagger)$ is satisfied.

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① i^{th}-level graded deformation $\longrightarrow$ graded deformation

- A is a 0^{th}-level graded deformation of itself.
- To construct a graded deformation A_t of A

  $\rightarrow$ start with $A = A_0$ and construct $\mu_1 : A \otimes A \longrightarrow A$ satisfying $(\dagger)$.

  $\rightarrow$ Then construct μ_2 satisfying $(\dagger)$ & so on

Finally get $A_t = A[t]$ with product $*$

$$a * b = \sum_{i \geq 0} \mu_i(a \otimes b)\, t^i$$

② graded deformations $\longrightarrow$ i^{th} level graded deformations

If A_t is a graded deformation of A, then μ_i satisfy equation $(\dagger)$ for all i and $A_t/(t^{i+1})$ is an i^{th}-level graded deformation of A.

KOSZUL ALGEBRAS

Consider a f.d. vector space V and $R \subset V \otimes V$.
Consider the following exact sequence $K_\bullet(A)$ for the quadratic algebra $A = \frac{T(V)}{(R)}$

$$\left[\cdots \xrightarrow{d_4} K_3(A) \xrightarrow{d_3} A \otimes R \otimes A \xrightarrow{d_2} A \otimes V \otimes A \xrightarrow{d_1} A \otimes A \xrightarrow{\pi} A \longrightarrow 0 \right] \quad (\star)$$

where for each $n \geq 2$,

$$K_n(A) = A \otimes K_n'(A) \otimes A \quad \text{with}$$

$$K_n'(A) = \bigcap_{i+j=n-2} \left(V^{\otimes i} \otimes R \otimes V^{\otimes j} \right)$$

write $K_0(A) = A \otimes A$ and $K_1(A) = A \otimes V \otimes A$,
then $K_i(A) \subset A^{\otimes i+2}$.
Thus we have a canonical embedding
$\iota_\bullet : K_\bullet(A) \hookrightarrow B_\bullet(A)$
and the maps d_i are obtained by restricting the differentials in $B(A)$ to $K_\bullet(A)$.

<u>Defn</u>: A quadratic algebra $A = T(V)/(R)$ is called Koszul if $K_\bullet(A)$ is a resolution of A as an A^e-module.

→ $K_\bullet(A)$ is called a Koszul bimodule resolution of A.

SETUP for Braverman-Gaitsgory result

Let $A = T(V)/(R)$ be a Koszul algebra.
Take two maps $\alpha : R \longrightarrow V$ and $\beta : R \longrightarrow \Bbbk$
and consider

$$P = P_{\alpha,\beta} = \{x - \alpha(x) - \beta(x) \mid x \in R\} \subset \Bbbk \oplus V \oplus (V \otimes V)$$

- Set $A' = T(V)/(P)$.
- Notice that when $\alpha = \beta = 0$, $A' = A$.

$\boxed{\text{BG-THEOREM}}$: Let $A = T(V)/(R)$ be a Koszul algebra.
The algebra $A' = T(V)/(P)$ is a PBW deformation of A if and only if the following conditions hold:

(i) $P \cap F_1(T(V)) = \{0\}$

(ii) $\mathrm{Im}(\alpha \otimes 1 - 1 \otimes \alpha) \subset R$

(iii) $\alpha(\alpha \otimes 1 - 1 \otimes \alpha) = -(\beta \otimes 1 - 1 \otimes \beta)$

(iv) $\beta(\alpha \otimes 1 - 1 \otimes \alpha) = 0$

where the maps $\alpha \otimes 1 - 1 \otimes \alpha$ and $\beta \otimes 1 - 1 \otimes \beta$ are defined on the subspace $K_0'(A) = (R \otimes V) \cap (V \otimes R)$ of $T(V)$.

Break ....

PROOF:

Let $P[t] = \{ x - \alpha(x)t - \beta(x)t^2 \mid x \in R \}$ a homogeneous subspace of $T(V)[t]$ of degree 2

Consider $A_t = \dfrac{T(V)[t]}{(P[t])}$

$$\begin{tikzcd}[column sep=small] & A_t \arrow[dl, squiggly, "t=0"'] \arrow[dr, squiggly, "t=1"] & \\ A_0 = A & & A_1 = A' \end{tikzcd}$$

Step 1: Reformulating the problem using graded deformations

(‡) $A' = A_1$ is a PBW deformation of A_0 $\Longleftrightarrow$ A_t is a graded deformation of A

($\Rightarrow$) Just need to check that $A_t \cong A \otimes \Bbbk[t]$ as a $\Bbbk[t]$-module.

Then, since A_t is an associative graded algebra with $A_t/(t) \cong A_0 \cong A$, A_t is a graded deformation of A

($\Leftarrow$) Clear

Step 2: Using that A is Koszul to extend α, β to maps $\mu_1, \mu_2 : A \otimes A \longrightarrow A$ extending them

Recall the Koszul bimodule resolution $K_\bullet$ of A.

- Let $\iota_\bullet : K_\bullet \hookrightarrow B_\bullet$
- Let $\psi_\bullet : B_\bullet \longrightarrow K_\bullet$ be the chain map lifting $id_A : A \longrightarrow A$

using comparison theorem because A is Koszul

$$\left(\begin{array}{ccccccccc} B_\bullet : & \cdots B_2 & \longrightarrow & B_1 & \longrightarrow & B_0 & \longrightarrow & A \\ \downarrow \psi & \downarrow \psi_2 & & \downarrow \psi_1 & & \downarrow \psi_0 & & \downarrow id_A \\ K_\bullet : & \cdots K_2 & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & A \end{array}\right)$$

- We can choose $\psi_\bullet$ so that $\psi_\bullet \iota_\bullet = id_{K_\bullet}$

Now, recall that $K_2 = A \otimes R \otimes A$

Since $\alpha : R \longrightarrow V$, $\beta : R \longrightarrow \Bbbk$ we can identity α, β as functions on K_2

$$\alpha(1 \otimes x \otimes 1) = \alpha(x), \quad \beta(1 \otimes x \otimes 1) = \beta(x) \quad \forall x \in R$$

and extend to A-bimodule homomorphism

Set $\mu_1 = \alpha \psi_2$, $\mu_2 = \beta \psi_2$

to get maps in $\mathrm{Hom}_{A^e}(A^{\otimes 4}, A) \cong \mathrm{Hom}_{\Bbbk}(A^{\otimes 2}, A)$

Back to proving the theorem

($\Longrightarrow$) Suppose that $A' = A_1$ is a PBW deformation of $A = A_0$

(i) First we show that (i) is necessary

$$\frac{T(V)}{(R)} = A \overset{\varphi}{\cong} gr(A') = gr\left(\frac{T(V)}{(P)}\right)$$

The isomorphism φ sends the generating space V to its image in $gr(A')$

$\therefore \quad \Bbbk \oplus V \subset gr(A')$ $\quad \Rightarrow \quad \boxed{P \cap \underbrace{(\Bbbk \oplus V)}_{= F_1(T(V))} = 0}$

STEP 3 : Use deformation theory

Now consider

$$\frac{A_t}{(t^2)} = \frac{T(V)[t]}{(P[t], t^2)} = \frac{T(V)[t]}{(x - \alpha(x), t^2 \mid x \in R)}$$

the product in $A_t/(t^2)$ is

$$a * b = ab + \mu_1(a \otimes b)$$

where ab = product in A

Thus, $A_t/(t^2)$ is a level 1 graded deformation

$\Downarrow$

μ_1 satisfies $(\dagger)$ for $i = 1$

that is $\mu_1(a_1 \otimes \mu_1(a_2 \otimes a_3)) = \mu_1(\mu_1(a_1 \otimes a_2) \otimes a_3)$

$\Downarrow$

(θ) .... $\mu_1(id \otimes \mu_1 - \mu_1 \otimes id)(a_1 \otimes a_2 \otimes a_3) = 0$ in $A = \frac{T(V)}{R}$

Take $a_1 \otimes a_2 \otimes a_3 \in (R \otimes V) \cap (V \otimes R)$

then $\mu|_{R \subset A \otimes A} = \alpha$

$\therefore$ $(\theta) \Rightarrow$ $\alpha(\mathrm{id} \otimes \alpha - \alpha \otimes \mathrm{id})(a_1 \otimes a_2 \otimes a_3) = 0$ in $\frac{T(V)}{R}$

$$\Rightarrow \boxed{\text{Image}(\mathrm{id} \otimes \alpha - \alpha \otimes \mathrm{id}) \subset R}$$

→ Similarly, $A_t/(t^2)$ is a level 2 graded deformation

$$\Rightarrow \boxed{\text{(iii) } \alpha(\alpha \otimes 1 - 1 \otimes \alpha) = -(\beta \otimes 1 - 1 \otimes \beta)}$$

→ Finally, $A_t/(t^3)$ is a level 3 graded deformation of A implies

$$\boxed{\text{(iv) } \beta(\alpha \otimes 1 - 1 \otimes \alpha) = 0}$$

$(\Leftarrow)$ Suppose that conditions (i) — (iv) are satisfied

Then we have

(ii) $\alpha(\mathrm{id} \otimes \alpha - \alpha \otimes \mathrm{id})(a_1 \otimes a_2 \otimes a_3) \in R$

for $a_1 \otimes a_2 \otimes a_3 \in (R \otimes V) \cap (V \otimes R)$

Since μ_1 is defined by extending α, we get

$\mu_1(\mathrm{id} \otimes \mu_1 - \mu_1 \otimes \mathrm{id}) = 0$

$\Rightarrow$ μ_1 is a Hochschild 2-cocycle

$\therefore$ $A_t/(t^2)$ is a level 1 graded deformation of A which product $*$, where

$$a * b = ab + \mu_1(a \otimes b)\, t$$

By similar arguments,

condition (iii) $\Rightarrow$ can lift $A_t/(t^2)$ to level 3 $A_t/(t^3)$

condition (iv) $\Rightarrow$ can lift $A_t/(t^3)$ to level 4 graded deformation

$\rightarrow$ Finally to lift $A_t/(t^4)$ to a full graded deformation, we need the i^{th}-obstructions (for $i \geq 4$) to vanish in $HH^*(A)$.

But i^{th}-deformation $\in HH^{3,-i}(A)$

and Braverman-Gaitsgory show that $HH^{3,-i}(A) = 0$ for $i \geq 4$.

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APPLICATION (Poincáre - Birkhoff - Witt theorem)

<u>Theorem</u> : Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\mathcal{U}(\mathfrak{g})$ be its universal envelopping algebra. Then

$$gr(\mathcal{U}(\mathfrak{g})) \cong S(\mathfrak{g})$$

<u>Proof</u>: Let $V = \mathfrak{g}$, $A = S(\mathfrak{g}) = T(V)/(R)$

where $R = \{ x\otimes y - y\otimes x \mid x, y \in V \} \subset V \otimes V$

Let $\alpha : R \longrightarrow \mathfrak{g}$

$$x\otimes y - y \otimes x \longmapsto [x, y]$$

$$\beta : R \longrightarrow \Bbbk \quad , \beta = 0$$

Then $P_{\alpha,\beta} = \{ x\otimes y - y\otimes x - [x,y] \mid x, y \in \mathfrak{g} \}$

and $T(V)/(P_{\alpha,\beta}) = \mathcal{U}(\mathfrak{g})$

Then , antisymmetry of $[\,,]$ $\Rightarrow$ (i)
Jacobi identity $\Rightarrow$ (iii)
since $\beta = 0$ $\Rightarrow$ (iv)

a little calculations shows (ii) also holds

$\therefore$ By BG- theorem,

$$gr(\mathcal{U}(\mathfrak{g})) = S(\mathfrak{g})$$

$\blacksquare$