GAUSS SUMS AND CENTRAL CHARGES

CONTENTS

§1 HISTORY

- Gauss (in 1801) introduced the sums

$$\boxed{\tau_n(k) = \sum_{j=0}^{k-1} e^{2\pi i n j^2/k}}$$

and used these to prove many reciprocity laws.

(reciprocity laws ≡ when are certain quadratic, cubic equations solvable mod p
e.g. $x^3 \equiv p \pmod q$)

- These sums are now called (quadratic) Gauss sums.

- Consider the abelian group $G = \mathbb{Z}_k$ and the function

$$q : \mathbb{Z}_k \longrightarrow \mathbb{C}^\times$$
$$j \longmapsto e^{2\pi i n j^2/k}$$

notice that $q(j) = q(k-j)$

$\boxed{k-j = j^{-1} \text{ in } \mathbb{Z}_k}$

& the symmetric function $b(j_1,j_2) = \dfrac{q(j_1+j_2)}{q(j_1)q(j_2)}$

satisfies $b(j_1 j_2, j_3) = b(j_1,j_3)\, b(j_2,j_3)$

$\therefore (\mathbb{Z}_k, q)$ is a pre-metric group.

- Now, notice that

$$\sum_{j \in \mathbb{Z}_k} q(j) = \tau_n(k)$$

- Thus, we define Gauss sums of a pre-metric group (G,q) as

$$\boxed{\tau^{\pm}(G,q) = \sum_{a\in G} q(a)^{\pm}}$$

- We say that (G,q) is orthogonal direct sum of (G_1,q_1) and (G_2,q_2) if

a) $G = G_1 \oplus G_2$

b) and $\forall\ g=(g_1,g_2)\in G$, $q(g) = q(g_1)\,q(g_2)$

then
$$\tau^{\pm}(G,q) = \sum_{g\in G} q(g)^{\pm}$$
$$= \tau^{\pm}(G_1,q_1)\,\tau^{\pm}(G_2,q_2)$$

§2 GAUSS SUMS FOR A PRE-MODULAR CATEGORY

- Given a premetric group (G, q), we can define a pre-modular category $\mathcal{C}(G, q)$ with

  → objects : $\delta_g \quad \forall g \in G$

  → tensor : $\delta_g \otimes \delta_h = \delta_{gh}$

  → Homs : $\mathrm{Hom}(\delta_g, \delta_h) = \begin{cases} \mathbb{k} & \text{if } g = h \\ 0 & \text{else} \end{cases}$

  This category has a ribbon structure θ such that

  $$\theta_{\delta_g} = q(g)\, \mathrm{Id}_{\delta_h}$$

  and with respect to this ribbon structure,

  $$\dim(\delta_g) = 1.$$

- We want to define Gauss sums such that for $\mathcal{C}(G,q)$, they coincide with $\tau^{\pm}(G,q)$.

- The Gauss sums of $\mathcal{C}$ are defined by

$$\boxed{\tau^{\pm}(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \theta_X^{\pm 1} \dim(X)^2}$$

Clearly, for $\mathcal{C} = \mathcal{C}(G,q)$, $\tau^{\pm}(\mathcal{C}) = \tau^{\pm}(G,q)$

$$\left(\tau^{+} \leftrightarrow \tau_{1} \quad , \quad \tau^{-} \leftrightarrow \tau_{-1} \right)$$

notation..

§ 3 SOME PROPERTIES

[Lemma 8·15·3, EGNO]
Let $\mathcal{C}$ be a pre-modular category. Then for any $Y \in \mathcal{O}(\mathcal{C})$ we have:

$$\sum_{X \in \mathcal{O}(\mathcal{C})} \theta_X \dim(X) s_{XY} = \dim(Y)\, \theta_Y^{-1} \tau^{+}(\mathcal{C}) \qquad (\alpha).$$

<u>Proof</u>: Recall, we saw last time that

(Prop 8-13-8)
$$s_{XY} = \theta_X^{-1} \theta_Y^{-1} \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} \theta_Z \dim(X)$$

$$\therefore \text{LHS} = \theta_Y^{-1} \sum_{X \in \mathcal{O}(\mathcal{C})} \dim(X) \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} \theta_Z \dim(X)$$

(but $N_{XY}^{Z} = \tau(V_X \otimes V_Y \otimes V_{Z^*}) = \tau(V_Y \otimes V_{Z^*} \otimes V_{X^{**}}) = N_{Y,Z^*}^{X^*}$
and $\dim(X) = \dim(X^*)$)

$$= \theta_Y^{-1} \sum_{Z \in \mathcal{O}(\mathcal{C})} \left(\sum_{X \in \mathcal{O}(\mathcal{C})} N_{ZY^*}^{X} \dim(X^*) \right) \theta_Z \dim(Z)$$

but $\sum_X N_{ZY^*}^X \dim(X) = \dim(Z \otimes Y^*)$
$= \dim(Z)\dim(Y^*) = \dim(Z)\dim(Y))$

$$\therefore \text{LHS} = \theta_Y^{-1} \sum_{Z \in O(\mathcal{C})} \left(\dim(Z)\dim(Y)\right) \theta_Z \dim(Z)$$

$$= \theta_Y^{-1} \dim(Y) \sum_Z \theta_Z \dim(Z)^2$$

$$= \theta_Y^{-1} \dim(Y) \quad \tau^+(\mathcal{C})$$

$$= \text{RHS}$$

[PROP 8.15.4, EGNO]
For a modular category $\mathcal{C}$, we have
$$\tau^+(\mathcal{C})\,\tau^-(\mathcal{C}) = \dim(\mathcal{C})$$

PROOF: Recall from Prop 8.15.3

$$\sum_{X\in\mathcal{O}(\mathcal{C})} \dim(X)\,\theta_X\, s_{XY} = \dim(Y)\,\theta_Y^{-1}\,\tau^+(\mathcal{C})$$

$\rightsquigarrow$ multiply both sides with $\dim(Y)$ to get

$$\sum_{X\in\mathcal{O}(\mathcal{C})} \underline{\dim(Y)}\,\dim(X)\,\theta_X\, s_{XY} = \underline{\theta_Y^{-1}\dim(Y)^2}\,\tau^+(\mathcal{C})$$

$\rightsquigarrow$ sum over all $Y\in\mathcal{O}(\mathcal{C})$ to get

$\underline{\text{RHS}} = \tau^+(\mathcal{C})\,\tau^-(\mathcal{C})$

$$\underline{\text{LHS}} = \sum_{X\in\mathcal{O}(\mathcal{C})}\Big(\sum_{Y\in\mathcal{O}(\mathcal{C})} s_{XY}\dim(Y)\Big)\dim(X)\,\theta_X = \sum_{X\in\mathcal{O}(\mathcal{C})}\Big(\sum_{Y\in\mathcal{O}(\mathcal{C})} s_{XY}\, s_{Y\mathbb{1}}\Big)\dim(X)\,\theta_X$$

$$= \sum_{X\in\mathcal{O}(\mathcal{C})} (s^2)_{X,\mathbb{1}}\,\dim(X)\,\theta_X$$

$$= \dim(\mathcal{C})$$

since $s^2 = \dim(\mathcal{C})\,E$
$\therefore\ s^2_{X,\mathbb{1}} = \delta_{X,\mathbb{1}}\dim(\mathcal{C})$

[Cor 8.15.5, EGNO] Let $\mathcal{C}$ be a modular category. Then for any $Y \in \mathcal{O}(\mathcal{C})$ we have:

$$\sum_{X \in \mathcal{O}(\mathcal{C})} \theta_X^{-1} \dim(X) \, s_{XY} = \dim(Y) \, \theta_Y \, \tau^{-}(\mathcal{C})$$

<u>Proof</u>: Follows using lemma 8.15.3 and the same argument as Prop 8.15.4

§4 Central Charge

- For a metric group (G, q), it is known that
$$\tau(G,q) = \zeta\sqrt{|G|}$$
for some 8^{th}-root of unity ζ.

- This fact generalizes to modular categories with $|G|$ replaced by its categorical analogue, $\dim(\mathcal{C})$.

- It was proved in Ch 7, EGNO that $\dim(\mathcal{C})$ is a totally positive element of $\mathbb{k}_{alg} \subset \mathbb{k}$. Totally positive means $\sigma(\dim(\mathcal{C})) > 0$ for any embedding $\mathbb{k}_{alg} \hookrightarrow \mathbb{C}$. Fix any such embedding.

Further, $\mathcal{C}$ is fusion $\Rightarrow$ $\dim(\mathcal{C}) \neq 0$ (thm. of ENO)

So, we define $\xi(\mathcal{C})$ by

$$\tau^+(\mathcal{C}) = \xi(\mathcal{C}) \sqrt{\dim(\mathcal{C})}$$

- $\xi(\mathcal{C}) = \dfrac{\tau^+(\mathcal{C})}{\sqrt{\dim(\mathcal{C})}}$ is called the multiplicative central charge.

- $$\xi(\mathcal{C})^2 = \frac{\tau^+(\mathcal{C})^2}{\dim(\mathcal{C})} = \frac{\tau^+(\mathcal{C})^2}{\tau^+(\mathcal{C})\,\tau^-(\mathcal{C})} = \frac{\tau^+(\mathcal{C})}{\tau^-(\mathcal{C})}$$

By a result of Anderson-Moore, $\tau^+(\mathcal{C})/\tau^-(\mathcal{C})$ is a root of unity.

$\therefore$ $\xi(\mathcal{C})$ is a root of unity.

§5 Higher generalizations

- For a MTC $\mathcal{C}$, higher Gauss sums are defined as

$$\boxed{\tau_n(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \theta_X^{\,n} \dim(X)^2} \qquad n \in \mathbb{Z}$$

- Higher central charge for $\mathcal{C}$ is defined as

$$\boxed{\xi_n(\mathcal{C}) = \frac{\tau_n(\mathcal{C})}{|\tau_n(\mathcal{C})|}}$$

note that for $n=1$, $|\tau_1(\mathcal{C})| = \sqrt{\dim(\mathcal{C})}$

$\therefore$ $\xi_1 = \xi$ defined earlier

(this generalization is due to Ng-Schopieray-Wang.)

§ 6 Some applications

→ Gauss sums & central charges are one of the many tools used to study / classify modular tensor categories.

<u>HOW?</u>

They behave nicely w.r.t. constructions of getting new MTC from old ones

(i) <u>Deligne product</u> : If $\mathcal{C} = \mathcal{D} \boxtimes \mathcal{E}$, then

$$\tau^{\pm}(\mathcal{C}) = \tau^{\pm}(\mathcal{D})\,\tau^{\pm}(\mathcal{E})$$

(ii) <u>G-(De)Equivariantization</u>

(iii) <u>Drinfeld center</u> $\quad \xi(\mathcal{Z}(\mathcal{C})) = 1$

→ Gauss sums are useful in classification of MTCs. They can help us identify if a MTC is obtained using smaller MTCs using some constructions.

→ <u>Witt invariants</u>:

Since understanding all MTCs is difficult, they are studied modulo Drinfeld centres.

- Two nondeg. braided tensor cats $\mathcal{C}, \mathcal{D}$ are Witt equivalent if $\exists$ tensor cats A, B s.t.

$$\mathcal{C} \boxtimes Z(A) \simeq_{\mathrm{Br}\otimes} \mathcal{D} \boxtimes Z(B)$$

- The set of all Witt eq. classes of nondeg. BTC under Deligne product form a group called the <u>Witt group</u>.

- It is interesting to get invariants of the Witt groups.

- ξ_n higher central charges (for certain n) are Witt invariants