QA reading group
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[FFRS, §7] Correspondences of tensor categories

(by Harshit Yadav)

What is correspondence?

- Consider a class of mathematical objects for which cartesian product is defined.

  Example: Groups

- Given two such objects X, Y, a correspondence is a subobject R of the product $X \times Y$.

  Example: Take groups G_1, G_2
  Correspondence = subgroup $R \leq G_1 \times G_2$

  Subexample: For every group homomorphism $f: G_1 \to G_2$, we get a correspondence $\Gamma_f = \{(g_1, f(g_1)) \mid g_1 \in G_1\} \leq G_1 \times G_2$

Hence, we can think of correspondence as generalizing the notion of morphisms between two mathematical objects.

GOAL

get info about Y using X and R

explicit example

describe the category $Rep(G_2)$ in terms of $Rep(G_1)$ and $Rep(R)$

Since we are interested in application to quantum groups, vertex algs., QFT, etc., we rephrase the goal in tensor categorical language

$\mathrm{Rep}(G_1)$	$\mathcal{C}$ braided $\otimes$ cat
$\mathrm{Rep}(G_2)$	$\mathcal{D}$ braided $\otimes$ cat
$\mathrm{Rep}(G_1 \times G_2) \triangleq \mathrm{Rep}(G_1) \boxtimes \mathrm{Rep}(G_2)$	$\mathcal{C} \boxtimes \mathcal{D}$
$\mathrm{Rep}(R) \subseteq \mathrm{Rep}(G_1 \times G_2)$	commutative alg $A_R \in \mathcal{C} \boxtimes \mathcal{D}$
	$\mathcal{D} = (\mathcal{C} \boxtimes \mathcal{D})_{A_R}$

For a group G, there is a bijection

$$\left\{\begin{matrix}\text{subgroups} \\ \text{of } H\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{commutative} \\ \text{algebras in} \\ \mathrm{Rep}(G)\end{matrix}\right\}$$

$$H \longmapsto \mathbb{C}[G/H]$$

$$\mathrm{Rep}(G)_A \longleftarrow A$$

Thus

$R \longleftrightarrow A_R$ commutative alg in $\mathrm{Rep}(G_1 \times G_2)$

GOAL: • Given $\mathcal{C}$, A_R a comm. alg. in $\mathcal{C} \boxtimes \mathcal{D}$
Find $\mathcal{C}$ in terms of $\mathcal{D}$ and A_R, that is,
- construct a (braided) tensor cat. $\mathcal{E}$
- construct $B \in \mathrm{ComAlg}(\mathcal{E})$

such that $\mathcal{C}$ is of the form $\mathcal{E}_B^{loc}$

This paper achieves this goal for ribbon categories.

> Take $\mathcal{C}, \mathcal{D}$ ribbon categories with $\mathcal{D}$ modular and A a nice comm alg. in $\mathcal{C} \boxtimes \mathcal{D}$. Set $\mathcal{E} = (\mathcal{C} \boxtimes \mathcal{D})_A^{loc}$
> Then under nice conditions,
> $\exists$ a nice comm. alg $B \in \mathcal{E} \boxtimes \bar{\mathcal{D}}$ such that
> $$\mathcal{C} \cong (\mathcal{E} \boxtimes \bar{\mathcal{D}})_B^{loc}$$

The precise result is the following:

[THEOREM 7.6, FFRS] (simplified version)

Suppose that we have the following data

- $\mathcal{C}$ ribbon category (assume Karoubian)
- $\mathcal{D}$ modular tensor category

  ($\Rightarrow$ $\mathcal{D}$ is trivializable, i.e., $(\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{T} \simeq \mathrm{Vec}_{\Bbbk}$
  with $T = \bigoplus_{i\in I} U_i \boxtimes \overline{U_i}$
  where $\{U_i\}_{i\in I}$ are simples in $\mathcal{D}$)

- A commutative symmetric special Frobenius algebra in $\mathcal{C}\boxtimes\mathcal{D}$

such that

(i) $\mathcal{C}\boxtimes\mathcal{D}\boxtimes\overline{\mathcal{D}}$ is separable, and

(i.e. p idempotent $+$ $tr(p)=0$ $\Rightarrow$ $p=0$)

(ii) A is $\mathcal{C}$-haploid.

(i.e. only retract of A of the form $U\boxtimes 1\!\!1_{\mathcal{D}}$ is $1\!\!1_{\mathcal{C}}\boxtimes 1\!\!1_{\mathcal{D}}$)

set $\mathcal{E} = (\mathcal{C}\boxtimes\mathcal{D})^{loc}_{A}$ and $B := \ell\text{-}\mathrm{Ind}_{A\times 1_{\overline{\mathcal{D}}}}(1\!\!1_{\mathcal{C}}\times T)$

then $\mathcal{C} \simeq (\mathcal{E}\boxtimes\overline{\mathcal{D}})^{loc}_{B}$

PROOF SKETCH:

$\boxed{\text{STEP 1}}$ $\mathcal{C} \simeq (\mathcal{C}\boxtimes\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{1_{\mathcal{C}}\times T}$

$$
\begin{aligned}
\mathcal{C} &\simeq \mathcal{C}\boxtimes\mathrm{Vec}_{\Bbbk} && \text{(Lemma 6.7 + } \mathcal{C} \text{ Karoubi closed)}\\
&\simeq \mathcal{C}\boxtimes(\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{T} && \text{(since } \mathcal{D} \text{ is a MTC)}\\
&\simeq \mathcal{C}^{loc}_{1_{\mathcal{C}}}\boxtimes(\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{T} && (1_{\mathcal{C}} = \text{unit} \Rightarrow \text{every object is local module over it})\\
&\simeq (\mathcal{C}\boxtimes\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{1_{\mathcal{C}}\times T} && \text{(Prop 6.11)}
\end{aligned}
$$

$\boxed{\text{STEP 2}}$ Take $F = (1_{\mathcal{C}} \times T) \otimes (A \times 1_{\bar{\mathcal{D}}}) \in \mathcal{C} \boxtimes \mathcal{D} \boxtimes \bar{\mathcal{D}}$

$1_{\mathcal{C}} \times T$ and $A \times 1_{\bar{\mathcal{D}}}$ are commutative algebras

$\Rightarrow C_l(F) \cong E_{1_{\mathcal{C}} \times T}(A \times 1_{\bar{\mathcal{D}}})$ & $C_r(F) \cong E_{A \times 1_{\bar{\mathcal{D}}}}(1_{\mathcal{C}} \times T)$

(by Proposition 3.14)

Also, $(\mathcal{C} \boxtimes \mathcal{D} \boxtimes \bar{\mathcal{D}})^{loc}_{C_l(F)} \cong (\mathcal{C} \boxtimes \mathcal{D} \boxtimes \bar{\mathcal{D}})^{loc}_{C_r(A)}$

(by Theorem 5.20)

$\boxed{\text{STEP 3}}$ $\underset{(a)}{E_{1_{\mathcal{C}} \times T}(A \times 1_{\bar{\mathcal{D}}})} \cong \underset{(b)}{1_{\mathcal{C}} \times T}$

Both these objects are defined as retracts of certain idempotents.

To show that $(a) \cong (b)$, we show that the two idempotents are the same

(see details on pages 319, 320)

$\boxed{\text{STEP 4}}$ $(\mathcal{C} \boxtimes \mathcal{D} \boxtimes \bar{\mathcal{D}})^{loc}_{C_l(F)} \cong \mathcal{C}$

$$
\begin{aligned}
(\mathcal{C} \boxtimes \mathcal{D} \boxtimes \bar{\mathcal{D}})^{loc}_{C_l(F)} &\cong (\mathcal{C} \boxtimes \mathcal{D} \boxtimes \bar{\mathcal{D}})^{loc}_{E_{1_{\mathcal{C}} \times T}(A \times 1_{\bar{\mathcal{D}}})} && \text{(Step 2)} \\
&\cong (\mathcal{C} \boxtimes \mathcal{D} \boxtimes \bar{\mathcal{D}})^{loc}_{1 \times T} && \text{(Step 3)} \\
&\cong \mathcal{C} && \text{(Step 1)}
\end{aligned}
$$

$\boxed{\text{STEP 5}}$ $\quad (\mathcal{C}\boxtimes\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{Cr(F)} \cong (\mathcal{E}\boxtimes\mathcal{D})^{loc}_{B}$

$$
\begin{aligned}
(\mathcal{C}\boxtimes\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{Cr(F)} &\cong (\mathcal{C}\boxtimes\mathcal{D}\boxtimes\overline{\mathcal{D}})^{loc}_{E_{A\times 1_{\overline{\mathcal{D}}}}(1_{\mathcal{C}}\times T)} && \text{(Step 2)}\\
&\cong \left[((\mathcal{C}\boxtimes\mathcal{D})\boxtimes\overline{\mathcal{D}})^{loc}_{A\times 1_{\overline{\mathcal{D}}}}\right]^{loc}_{\ell\text{-Ind}_{A\times 1_{\overline{\mathcal{D}}}}(1_{\mathcal{C}}\times T)} && \text{(Prop 4.16)}\\
&\cong \left[(\mathcal{C}\boxtimes\mathcal{D})^{loc}_{A}\boxtimes\overline{\mathcal{D}}\right]^{loc}_{\ell\text{-Ind}_{A\times 1_{\mathcal{D}}}(1_{\mathcal{C}}\times T)} && \text{(Prop. 6.11)}\\
&\cong (\mathcal{E}\boxtimes\overline{\mathcal{D}})^{loc}_{B}
\end{aligned}
$$

Now, by steps 4, 5 the claim follows.

$\blacksquare$