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LOCAL INDUCTION

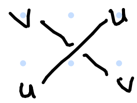
[FFRS, §4]

PART 1: RECAP OF RESULTS / DEFINITIONS FROM § 2, 3

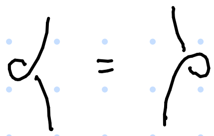
(i) SETTING : Ribbon categories, Modular categories

A Ribbon category = an abelian category that is

- $\Bbbk$-linear
- monoidal $(\mathcal{C}, \otimes, \mathbb{1})$
- rigid : $(\,)^\vee, {}^\vee(\,)$
- $\mathbb{1}$ is simple, i.e. $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \Bbbk$
- pivotal : left dual = right dual
- braided : $c_{u,v} : u \otimes v \to v \otimes w$

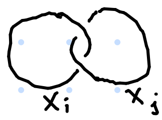
  $c_{u,v} =$ 

- ribbon : left twist = right twist

   $= \theta_x$

A Modular category = a ribbon category that is

- semisimple,
- has finitely many isomorphism classes of simple objects, and
- the S-matrix $S = [s_{ij}]$ is invertible, where

  $s_{i,j} =$ 

  for X_i, X_j simple objects of $\mathcal{C}$

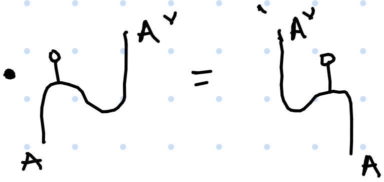
Remark: From here on, $\mathcal{C}$ will denote a ribbon category.

(ii) OBJECTS OF STUDY:

a) Algebras $(A, m =$  $, u =$  $)$ — $Alg(\mathcal{C})$

b) Coalgebras $(A, \Delta =$  $, \varepsilon =$  $)$ — $Coalg(\mathcal{C})$

c) Frobenius Algebras $(A, m, u, \Delta, \varepsilon)$ such that — $FrobAlg(\mathcal{C})$
- $(A, m, u) \in Alg(\mathcal{C})$
- $(A, \Delta, \varepsilon) \in Coalg(\mathcal{C})$
- 

d) Symmetric Frobenius Algebras — $SymFrobAlg(\mathcal{C})$
- $(A, m, u, \Delta, \varepsilon) \in FrobAlg(\mathcal{C})$
- 

$: A \xrightarrow{\sim} {}^\vee A$

e) Special Symmetric Frobenius Algebras — $SpeSymFrobAlg(\mathcal{C})$
- $(A, m, u, \Delta, \varepsilon) \in SymFrobAlg(\mathcal{C})$
- $\varepsilon \circ u =$  $= \beta_{\mathbb{1}}\, id_{\mathbb{1}}$,

  $m \circ \Delta =$  $= \beta_A\, id_A$

  for some $\beta_{\mathbb{1}}, \beta_A \in \Bbbk \setminus \{0\}$

f) For $(A, m, u) \in Alg(\mathcal{C})$, we call it

→ Commutative if  — $ComAlg(\mathcal{C})$

→ Haploid (connected) if $\dim_{\Bbbk} Hom_{\mathcal{C}}(\mathbb{1}, A) = 1$

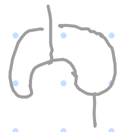
→ Simple if $\dim_{\Bbbk} Hom_{{}_A\mathcal{C}_A}(A, A) = 1$

(iii) KEY CONSTRUCTIONS

(a) Constructions of new algebras from old ones

① LEFT / RIGHT CENTRE

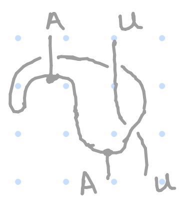
- $A \in \mathrm{SpeSymFrobAlg}(\mathcal{C}) \rightsquigarrow C^{\ell/r}(A) \in \mathrm{ComSymFrobAlg}(\mathcal{C})$

  $C^{\ell}(A) := \mathrm{im}(P^{\ell}_A)$ where $P^{\ell}_A =$ 

- A simple $\Longrightarrow$ $C^{\ell}(A), C^{r}(A)$ are simple
- If $C^{\ell/r}(A)$ is simple then
  $C^{\ell/r}(A)$ is special $\Longleftrightarrow$ $\dim(C^{\ell/r}(A)) \neq 0$

② ENDOFUNCTORS $E^{\ell/r}_A$

- $A \in \mathrm{SpeSymFrobAlg}(\mathcal{C}) \rightsquigarrow E^{\ell/r}_A : \mathcal{C} \longrightarrow \mathcal{C}$

  $E^{\ell}_A(u) := \mathrm{im}(P^{\ell}_A(u))$ where $P^{\ell}_A(u) =$ 

  $E^{\ell/r}_A(\mathbb{1}) \cong C^{\ell/r}(A)$

- $E^{\ell/r}_A$ is an endofunctor on
  - $\mathrm{FrobAlg}(\mathcal{C})$
  - $\mathrm{SymFrobAlg}(\mathcal{C})$
  - $\mathrm{ComFrobAlg}(\mathcal{C})$

- If $B \in \mathrm{SpeSymFrobAlg}(\mathcal{C})$, $\dim_{\Bbbk}(\mathrm{Hom}_{\mathcal{C}}(B, C^{r/\ell}_A(\mathbb{1}))) = 1$, $\dim(E^{\ell/r}_A(B)) \neq 0$ $\Rightarrow$ $E^{\ell}_A(B)$ is special, haploid

- If A is commutative, then $E^{\ell}_A = E^{r}_A$

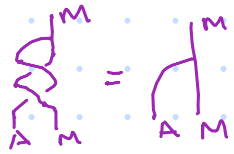
(b) Constructions of new categories from old ones

Take $A \in \mathrm{ComSpeSymFrobAlg}(\mathcal{C})$

$M = (M, \rho_M : A \otimes M \to M) \in {}_A\mathcal{C}$

Defn
- M is called **local** if $\rho_M \circ P_A^{l/A}(M) = \rho_M$

(equivalently, M is local if $\rho_M \circ c_{M,A} \circ c_{A,M} = \rho_M$)



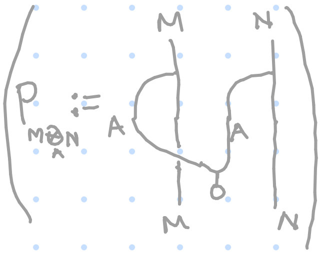
- ${}_A\mathcal{C}^{loc}$:= category local left A-modules

For A as above and M local,

(i) ${}_A\mathcal{C}^{loc}$ is ribbon

$\mathbb{1}$: (A, m)

$\otimes$: $(M, \rho_M) \otimes (N, \rho_N) = (M \otimes_A N, \rho_{M \otimes_A N})$, where

$M \otimes_A N = \mathrm{im}\left(p_{M \otimes_A N} :=\right.$  $\left.\right)$, $\rho_{M \otimes_A N} =$ 

braiding : $c^A_{M,N} :=$ 

twist : $\theta^A_M := \theta_M$

duality : same as duality on $\mathcal{C}$

(ii) $\mathcal{C}$ semisimple $\Rightarrow$ ${}_A\mathcal{C}^{loc}$ is semisimple

(iii) $\mathcal{C}$ modular, A simple $\Rightarrow$ ${}_A\mathcal{C}^{loc}$ modular

PART 2 : LOCAL INDUCTION

Recall the endofunctors $E_A^{\ell/r}$ for $A \in \mathrm{SpSymFrobAlg}(\mathcal{C})$.
(from here on we only discuss the results for left version E_A^{ℓ})

$$E_A^{\ell}(\mathcal{U}) := \mathrm{im}\left(p_A^{\ell}(\mathcal{U})\right) = \mathrm{im}\Bigg($$  $$\Bigg)$$

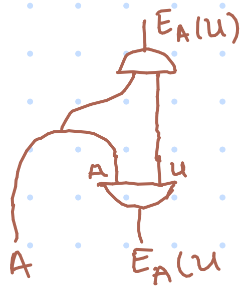
Next, we will see that $E_A^{\ell}(\mathcal{U})$ is a local $C^{\ell}(A)$ module

PROP 4.1 $E_A^{\ell}(\mathcal{U})$ is a local $C^{\ell}(A)$ module with action

$$\rho^{loc}_{C^{\ell}(A);\mathcal{U}} =$$

COROLLARY 4.2 For $A \in \mathrm{ComSpSymFrobAlg}(\mathcal{C})$ and $\mathcal{U} \in \mathcal{C}$
$E_A(\mathcal{U}) := E_A^{\ell}(\mathcal{U}) = E_A^{r}(\mathcal{U})$ is a local A-module with action

$$\rho^{loc}_{A;\mathcal{U}} :=$$



DEFN The local induction functors

$$\ell\text{-}\mathrm{Ind}_A^{\ell} : \mathcal{C} \longrightarrow \mathcal{C}^{loc}_{C^{\ell}(A)}$$

are defined by

$$\ell\text{-}\mathrm{Ind}_A^{\ell}(\mathcal{U}) := \left(E_A^{\ell}(\mathcal{U}),\ \rho^{loc}_{C^{\ell}(A);\mathcal{U}}\right),$$

$$\ell\text{-}\mathrm{Ind}_A^{\ell}(f) := E_A^{\ell}(f)$$

REMARK: The above defined functors are called local induction functors because when A is commutative, we have an embedding

$$\ell\text{-Ind}_A(\mathcal{U}) \hookrightarrow \text{Ind}_A(\mathcal{U}) = A \otimes \mathcal{U}$$

and, this embedding satisfies nice properties,

- $\text{Hom}_A(M, \ell\text{-Ind}_A(\mathcal{U})) \cong \text{Hom}_A(M, \text{Ind}_A(\mathcal{U}))$
- $\text{Hom}_A(\ell\text{-Ind}_A(\mathcal{U}), M) \cong \text{Hom}_A(\text{Ind}_A(\mathcal{U}), M)$

This allows us to use reciprocity theorems of ordinary induction when working with local induction.

Next, we discuss local induction of algebras, that is, do local induction functors send algebras to algebras.

PROP (= combination of 4.13, 4.14 in paper) $\ell\text{-Ind}_A^{\mathcal{C}}$ lifts to a functor from

a) $\text{FrobAlg}(\mathcal{C}) \longrightarrow \text{FrobAlg}({}_{C^{\mathcal{C}}(A)}\mathcal{C}^{loc})$

b) $\text{SymFrobAlg}(\mathcal{C}) \longrightarrow \text{SymFrobAlg}({}_{C^{\mathcal{C}}(A)}\mathcal{C}^{loc})$

c) $\text{ComFrobAlg}(\mathcal{C}) \longrightarrow \text{ComFrobAlg}({}_{C^{\mathcal{C}}(A)}\mathcal{C}^{loc})$

d) If $B \in \text{SpeSymFrobAlg}(\mathcal{C})$, $\dim_{\mathbb{k}}(\text{Hom}_{\mathcal{C}}(B, C_A^{\mathcal{C}}(\mathbb{1})) = 1$ and $\dim(E_A^{\mathcal{C}}(B)) \neq 0$, then $\ell\text{-Ind}_A^{\mathcal{C}}(B)$ is special, haploid

PROOF SKETCH OF (a)

Call $C_{\mathcal{C}}(A) = C$.

Recall that

$$\ell\text{-Ind}_A^{\mathcal{C}}(B) = (E_A^{\mathcal{C}}(B), \rho^{loc}_{C;B}) \in {}_C\mathcal{C}^{loc}$$

and the monoidal product on ${}_C\mathcal{C}^{loc}$ is $\otimes_C$.

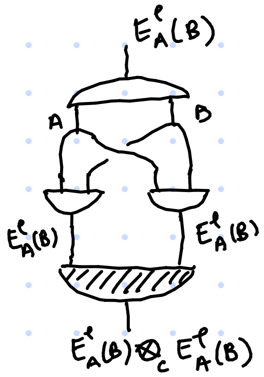
Thus, to define the multiplication on $\ell\text{-Ind}_A^\ell(B)$, we have to construct a map

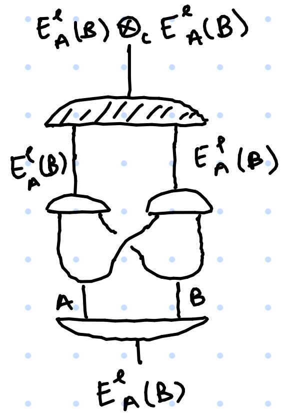
$$\tilde{m} : E_A^\ell(B) \otimes_C E_A^\ell(B) \longrightarrow E_A^\ell(B)$$

Also remember that $E_A^\ell(B) \otimes_C E_A^\ell(B)$ was defined as the image of an idempotent map

$$P : E_A^\ell(B) \otimes E_A^\ell(B) \longrightarrow E_A^\ell(B) \otimes E_A^\ell(B)$$

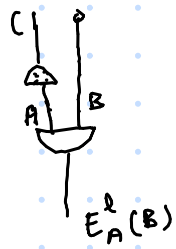
thus we can use the retraction & inclusion maps.

$m_{\ell\text{-Ind}_A^\ell(B)} =$ 

, $\Delta_{\ell\text{-Ind}_A^\ell(B)} =$ 

(Recall that the unit object of ${}_C\mathcal{C}^{loc}$ is (C, m_C). Thus, the unit map $u_{\ell\text{-Ind}_A^\ell(B)} : C \longrightarrow E_A^\ell(B)$)

$u_{\ell\text{-Ind}_A^\ell(B)} =$ 

, $\varepsilon_{\ell\text{-Ind}_A^\ell(B)} =$ 

From here one needs to check that

(i) $(\ell\text{-Ind}_A^\ell(B), m_{\ell\text{-Ind}_A^\ell(B)}, u_{\ell\text{-Ind}_A^\ell(B)})$ is an algebra.

(ii) $(\ell\text{-Ind}_A^\ell(B), \Delta_{\ell\text{-Ind}_A^\ell(B)}, \varepsilon_{\ell\text{-Ind}_A^\ell(B)})$ is a coalgebra.

(iii) Frobenius law holds for $m_{\ell\text{-Ind}_A^\ell(B)}, \Delta_{\ell\text{-Ind}_A^\ell(B)}$.

(iv) $m, \Delta, u, \varepsilon$ defined above are maps of left C-modules.

■

Finally, we get the following interesting result.

PROP 4.16: Let $A, B \in \mathrm{ComSpSymFrobAlg}(\mathcal{C})$. Suppose in addition that A is simple and the Frobenius algebra $E_A(B)$ is special. Then $\ell\text{-Ind}_A(B)$ is special, too, and we have an equivalence

$$\left({}_{\ell\text{-Ind}_A(B)}\left({}_A\mathcal{C}^{loc}\right)\right)^{loc} \simeq {}_{E_A(B)}\mathcal{C}^{loc} \qquad (*)$$

PROOF We know that $E_A(B) \in \mathrm{ComSymFrobAlg}(\mathcal{C})$ by Prop 3.8

A is simple $\Rightarrow \ell\text{-Ind}_A(B) \in \mathrm{ComSymFrobAlg}\left({}_{C(A)}\mathcal{C}^{loc}\right)$

Since $A, E_A(B), \ell\text{-Ind}_A(B)$ are all ComSymFrobAlgs the categories in $(*)$ are all ribbon.

To establish the equivalence $(*)$, we construct functor

$$F : {}_{\ell\text{-Ind}_A(B)}\left({}_A\mathcal{C}^{loc}\right)^{loc} \longrightarrow {}_{E_A(B)}\mathcal{C}^{loc}$$

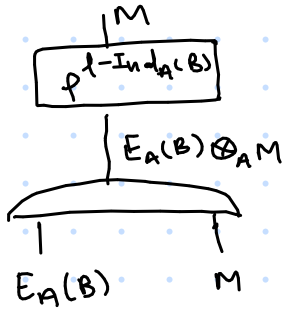
Take $M \in {}_{\ell\text{-Ind}(A)}\left({}_A\mathcal{C}^{loc}\right)^{loc}$. We can regard M as a triple $(M, \rho^A, \rho^{\ell\text{-Ind}_A(B)})$, with $M \in \mathcal{C}$, $(M, \rho^A) \in {}_A\mathcal{C}^{loc}$, and the whole triple in ${}_{\ell\text{-Ind}_A(B)}\left({}_A\mathcal{C}^{loc}\right)^{loc}$

where $\rho^A : A \otimes M \longrightarrow M$, and

$$\rho^{\ell\text{-Ind}_A(B)} : \ell\text{-Ind}_A(B) \otimes_A M \longrightarrow M$$

then define

$$\rho^{E_A(B)} : E_A(B) \otimes M \longrightarrow M$$



$E_A(B), M \in {}_A\mathcal{C}^{loc}$

We define $F((M, \rho^A, \rho^{l\text{-}Ind_A(B)})) = (M, \rho^{E_A(B)})$

- Next we construct a functor

$$G : {}_{E_A(B)}\mathcal{C}^{loc} \longrightarrow {}_{l\text{-}Ind_A(B)}({}_A\mathcal{C}^{loc})^{loc}$$

and show that G is inverse of F

- Finally show that F is a ribbon functor.

■