QA Learing group

Oct 20, 2021

TALK 5: Endofunctors related to α-induction (FFRS §3.1)

(by Harshit Yadav)

Quick summary of last few talks:

Setting: RIBBON CATEGORY

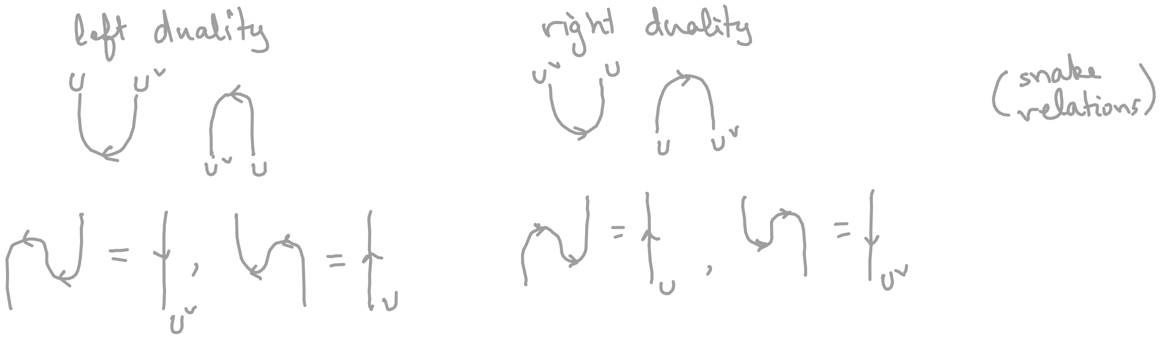
RIBBON CATEGORY =

- $\Bbbk$-linear : $\mathrm{Hom}_{\Bbbk}(X,Y) \in \Bbbk$, has $\oplus$
- monoidal : $\otimes$, $\mathbb{1}$
- braided : $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$

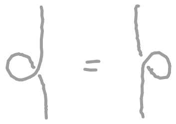
- rigid : have left and right duals
- pivotal : left dual = right dual

denote $X^{\vee}$ = dual of X

$f^{\vee}$ = dual of f



- ribbon : left twist = right twist



→ Idempotent : morphism $p: U \rightarrow U$ satisfying $p^2 = p$.

→ We say p splits if $\exists$ maps $r: U \rightarrow S$, $e: S \rightarrow U$ such that $U \xrightarrow{r} S \xrightarrow{e} U = p$ and $S \xrightarrow{e} U \xrightarrow{r} S = \mathrm{id}_S$

Graphically,

$e =$  , $r =$  satisfying  $= id_S$,  $= p$

The triple (S, e, r) is called the retract of p.

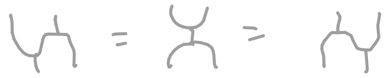
A category $\mathcal{C}$ is called Karoubian if all idempotents split. In this talk, the ribbon categories are not necessarily Karoubian.

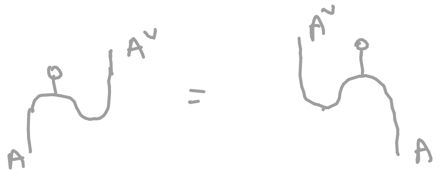
Objects of study : SYMMETRIC SPECIAL FROBENIUS ALGEBRAS

It is a 5-tuple $(A, m, u, \Delta, \varepsilon)$ such that

① $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$ 

② $(A, \Delta, \varepsilon) \in \mathrm{Coalg}(\mathcal{C})$ 

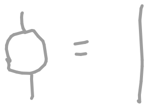
③ Frobenius condition 

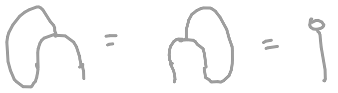
④ Symmetric 

⑤ Special $\varepsilon \circ \eta =$  $= \beta_{\mathbb{1}} \, id_{\mathbb{1}}$, $m \circ \Delta =$  $= \beta_A \, id_A$

for some $\beta_{\mathbb{1}}, \beta_A \in \mathbb{k} \setminus \{0\}$

⑥ When A is symmetric special, we can normalize Δ, ε to get the relations.

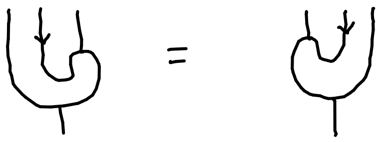
(i)  (ii) 

(iii)  (iv) 

Remark about graphical calculus for symmetric Frobenius algebras in ribbon categories

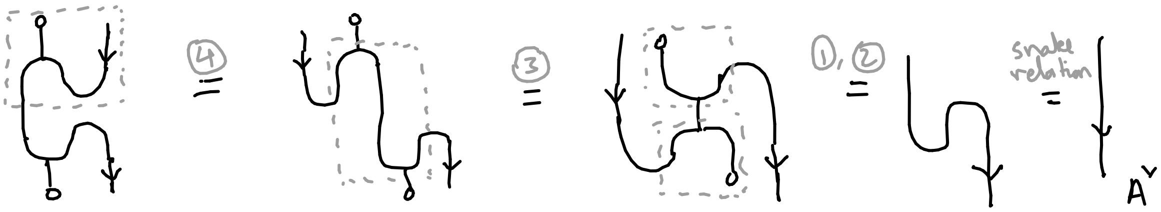
The symmetric, Frobenius and (co) associativity conditions together imply that we can move multiplication and or comultiplication past each other in all possible arrangements.

Lemma :

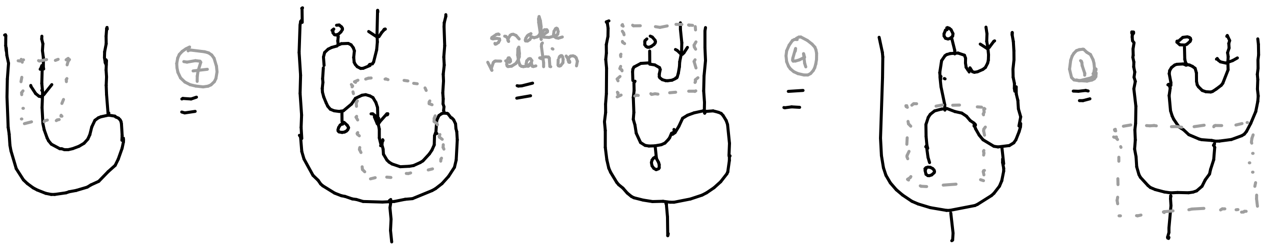


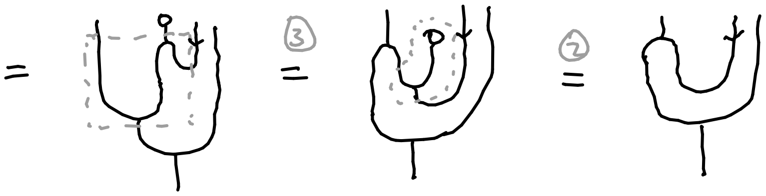
— ⑦

Proof :



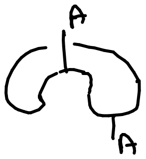
Then,

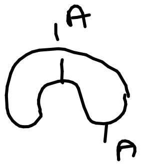




■

Last time, we had defined left and right central idempotents of A

$P_A^l =$ 

$P_A^r =$ 

Next, we constructed the

① [· left centre $C_l(A)$ = retract of P_A^l
· right centre $C_r(A)$ = retract of P_A^r

and showed that

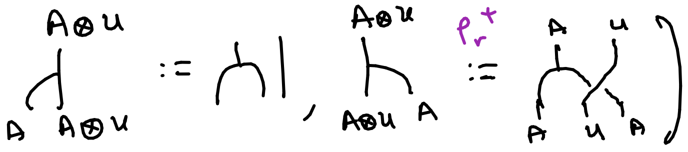
C_l and C_r are commutative symmetric Frobenius algebras in $\mathcal{C}$, and

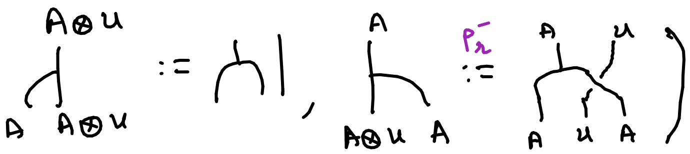
② there are natural bijections

$$\mathrm{Hom}(C_l(A) \otimes U, V) \cong \mathrm{Hom}_{A|A}(\alpha_A^+(U), \alpha_A^+(V))$$
$$\cong \mathrm{Hom}(U, C_l(A) \otimes V)$$

$$\mathrm{Hom}(C_r(A) \otimes U, V) \cong \mathrm{Hom}_{A|A}(\alpha_A^-(U), \alpha_A^-(V))$$
$$\cong \mathrm{Hom}(U, C_r(A) \otimes V)$$

where $\alpha_A^\pm : \mathcal{C} \longrightarrow \mathcal{C}_{A|A}$ are the α-induction functors

$\alpha_A^+ : U \longmapsto ($A⊗U, $)$

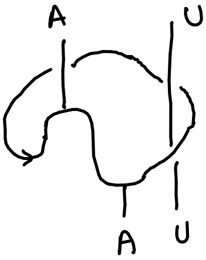
$\alpha_A^- : U \longmapsto ($A⊗U, $)$

Today's talk is about generalizing the construction ① and the result ②

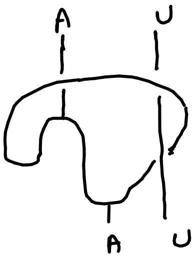
§ 3.1 Endofunctors related to α-induction

Consider the endomorphisms

$P^l_A(U) =$



$P^r_A(U) =$



Clearly $P^{l/r}_A(\mathbb{1}) = P^{l/r}_A$.

Just like $P^{l/r}_A$, the endomorphisms $P^{l/r}_A(U)$ are also idempotent.

Defn: A special Frobenius algebra A in a ribbon category $\mathcal{C}$ is called centrally split if the idempotents $P^{l/r}_A(U)$ are split for every $U \in \mathcal{C}$.

Since our ribbon categories are not necessarily Karoubian, we make the following assumption.

Declaration: From here on, every special Frobenius algebra will be assumed to be centrally split.

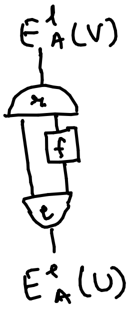
Defn: $E^l_A(U) := \mathrm{Im}(P^l_A(U))$, $E^r_A(U) := \mathrm{Im}(P^r_A(U))$

Then by definition, we have maps

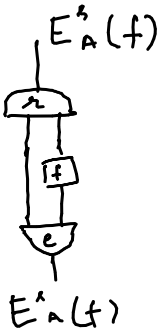
$$e : E^{l/r}_A(U) \longrightarrow A \otimes U \ , \quad r : A \otimes U \longrightarrow E^{l/r}_A(U)$$

For $f \in \mathrm{Hom}(U, V)$, define

$E^l_A(f) :=$

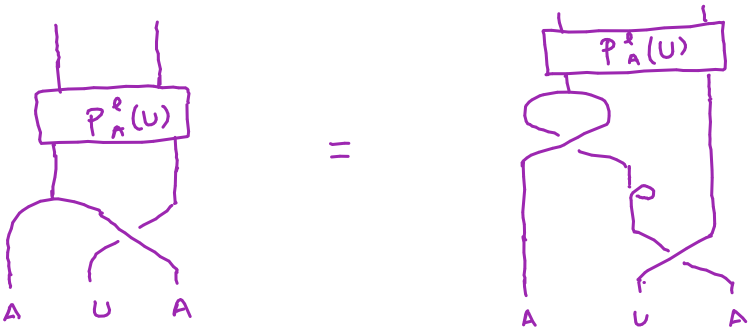


, $E^r_A(f) =$



Lemma: Let $\rho_r^{\mp}$ denote the right A-action on $A \otimes U$, then

(i) $P_A^\ell(U) \circ \rho_r^- = P_A^\ell(U) \circ \big([m \circ c_{A,A} \circ (id \otimes \Theta_A)] \otimes id_U \big) \circ (id_A \otimes c_{U,A})$



— (*)

(+ similar result for $P_A^r(U)$)

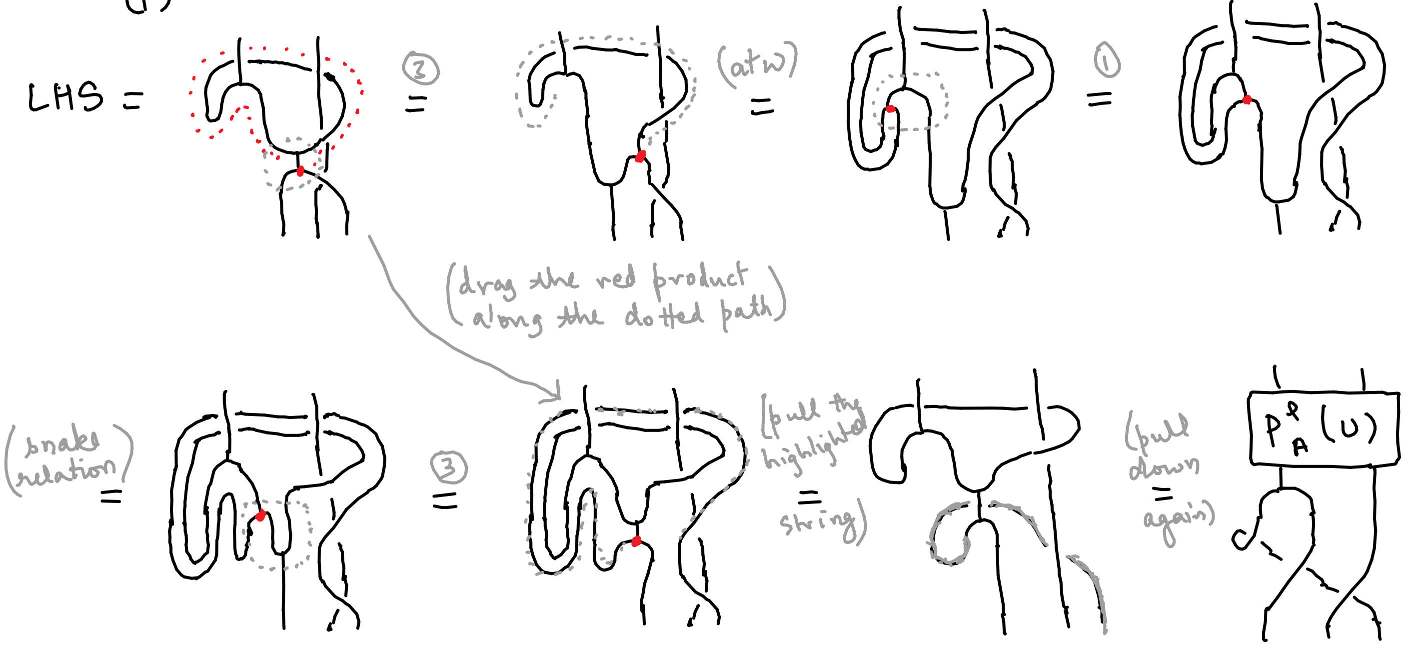
(ii) If A is commutative, then

$$P_A(U) \circ \rho_r^+ = P_A(U) \circ \rho_r^-$$

for $P_A(U) \equiv P_A^{\ell/r}(U)$.

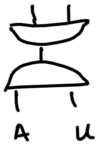
Proof: (ii) It follows immediately from (i) by using that

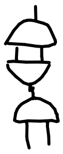
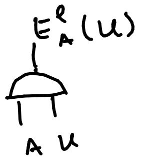
(i)



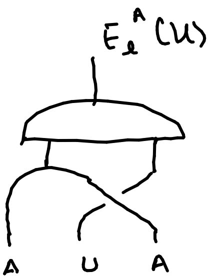
= RHS

∎

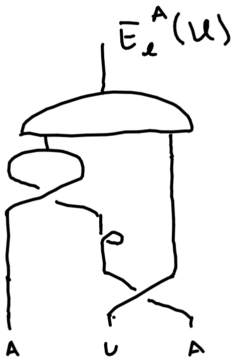
Recall that $P^{\ell}_{A}(u) =$ 

$\therefore\ r \circ P^{\ell}_{A}(u) =$  $=$ 

Thus, postcomposing $(*)$ with , gives us that



$=$



— $(**)$

Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ be a pair of functors. We call G right adjoint of F if there exists a natural family of isomorphisms

$$\phi_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(X, G(Y))$$

for $X \in \mathcal{C}$, $Y \in \mathcal{D}$

Recall the functors $\alpha^{\pm}_{A} : \mathcal{C} \to \mathcal{C}_{A|A}$

Suppose that these have right adjoints denoted as $(\alpha^{\pm}_{A})^{ra}$ and left adjoints denoted by $(\alpha^{\pm}_{A})^{la}$. Then by defn, we get natural isomorphisms.

(set $X = \alpha^{+}_{A}(U)$, $Y = V$, $F = (\alpha^{-}_{A})^{la}$, $G = \alpha^{-}_{A}$)

→ $\mathrm{Hom}((\alpha^{-}_{A})^{la} \circ \alpha^{+}_{A}(U), V) \cong \mathrm{Hom}_{A|A}(\alpha^{+}_{A}(U), \alpha^{-}_{U}(V)) \cong \mathrm{Hom}(U, (\alpha^{+}_{A})^{ra} \circ \alpha^{-}_{A}(V))$

(similarly for $\mathrm{Hom}_{A|A}(\alpha^{-}_{A}(U), \alpha^{+}_{A}(V))$)

→FFRS show that $E^{\ell/r}_{A}$ can be regarded as compositions of the functors $\alpha^{\pm}_{A}$ and their adjoints by showing the following results.

Prop: For $U, V \in \mathcal{C}$, there are natural bijections

$$\mathrm{Hom}(E^l_A(U),V) \cong \mathrm{Hom}_{A|A}(\alpha^-_A(U), \alpha^+_A(V)) \cong \mathrm{Hom}(U, E^r_A(V))$$

and

$$\mathrm{Hom}(E^r_A(U),V) \cong \mathrm{Hom}_{A|A}(\alpha^+_A(U), \alpha^-_A(V)) \cong \mathrm{Hom}(U, E^l_A(V))$$

Proof: We will discuss the natural isomorphism

$$\begin{tikzcd}
\mathrm{Hom}(E^l_A(U),V) \arrow[r, shift left, "\Phi_{U,V}"] & \mathrm{Hom}_{A|A}(\alpha^-_A(U), \alpha^+_A(V)) \arrow[l, shift left, "\Psi_{U,V}"]
\end{tikzcd}$$

Other cases have similar proofs.

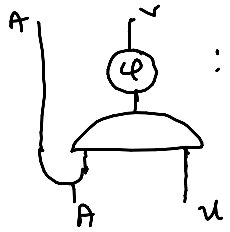
Naturality amounts to showing that $\forall\ U' \xrightarrow{f} U,\ V \xrightarrow{g} V'$

$$\begin{tikzcd}
\mathrm{Hom}(E^l_A(U),V) \arrow[r, "\Phi_{U,V}"] \arrow[d, "\mathrm{Hom}(E^l_A(f),g)"'] & \mathrm{Hom}_{A|A}(\alpha^-_A(U), \alpha^+_A(V)) \arrow[d, "\mathrm{Hom}_{A|A}(\alpha^-_A(f), \alpha^+_A(g))"] \\
\mathrm{Hom}(E^l_A(U'),V') \arrow[r, "\Phi_{U',V'}"'] & \mathrm{Hom}_{A|A}(\alpha^-_A(U'), V')
\end{tikzcd} \quad \text{——}(\#)$$

commutes

Definition of Φ:

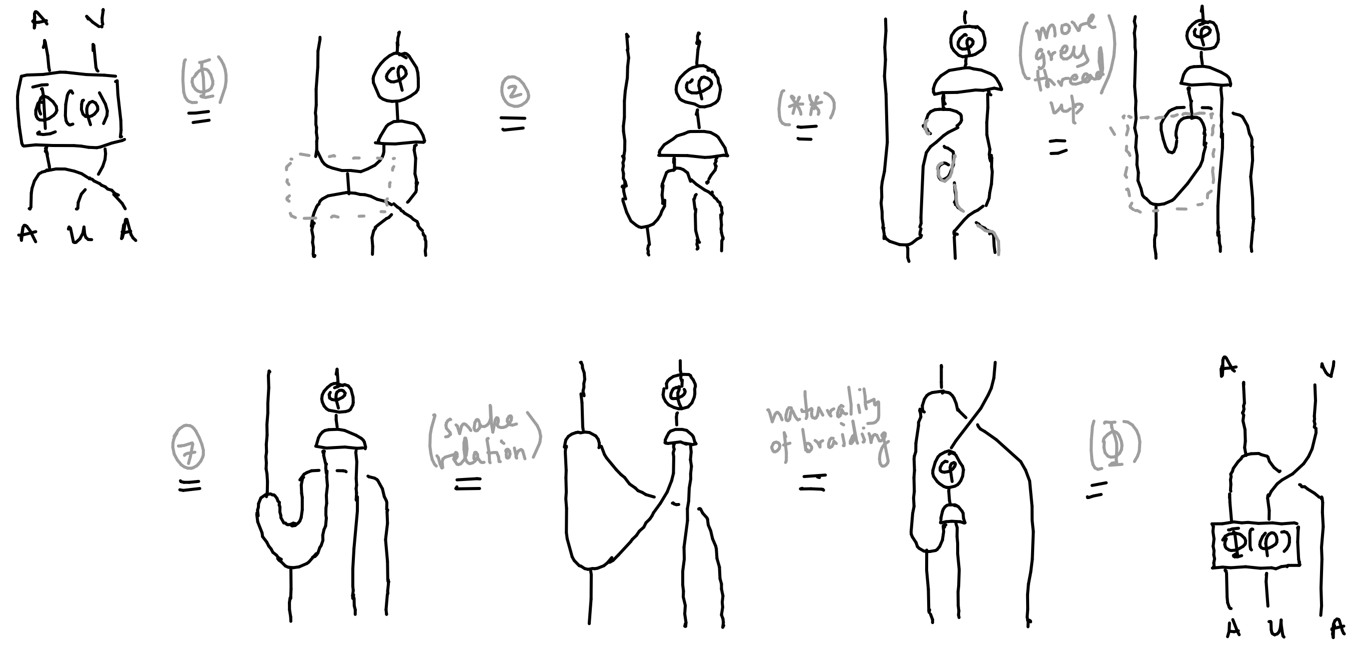
- For $\varphi : E^l_A(U) \to V$, define $\Phi_{U,V}(\varphi) =$



$: A \otimes U \to A \otimes V$, with $A \otimes U = \alpha^-_A(U)$ and $A \otimes V = \alpha^+_A(V)$ —— (Φ)

$\rightsquigarrow \Phi_{U,V}(\varphi)$ is a morphism of left A-modules becauses

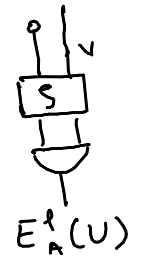
→ $\Phi_{u,v}(\varphi)$ is a morphism of right A-modules because



Definition of Ψ:

For $\varsigma \in \mathrm{Hom}_{A|A}(\alpha^-_A(U), \alpha^+_A(V))$, define

$$\Psi_{u,v}(\varsigma) = (\eta\otimes id_v)\circ\varsigma\circ e =$$

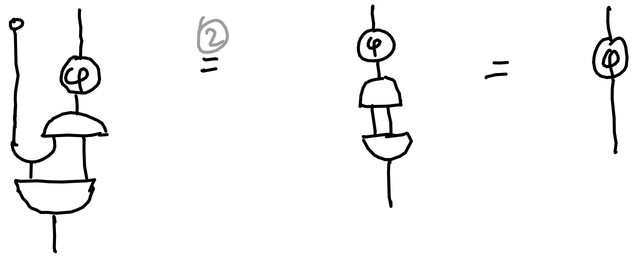


— (Ψ)

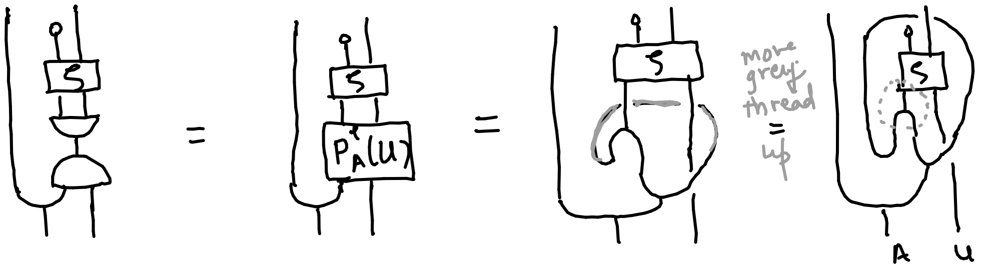
- Next, we need to check that

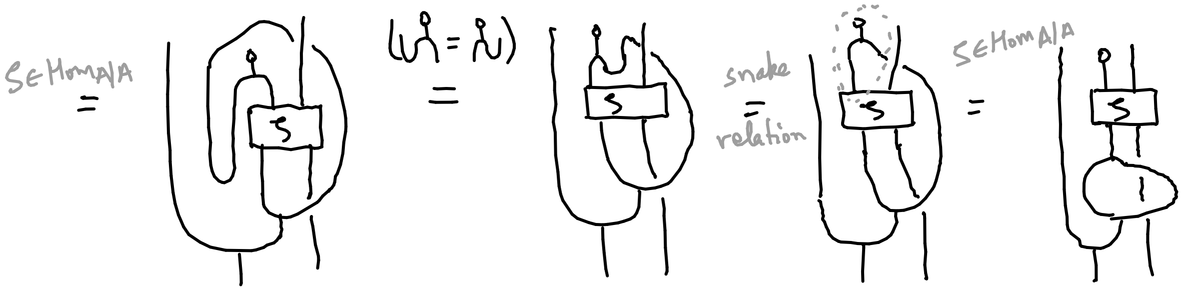
(i) $\Psi\circ\Phi(\varphi)=\varphi$ and (ii) $\Phi\circ\Psi(\varsigma)=\varsigma$

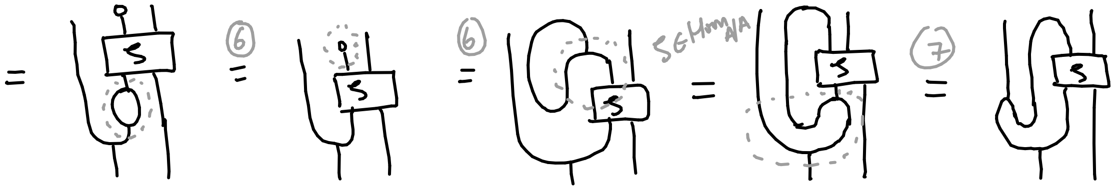
(i) $\Psi\cdot\Phi(\varphi)$ =

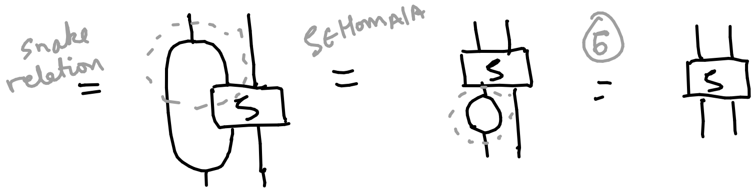


(ii) $\Phi\cdot\Psi(\varsigma)$ =









→ Finally we are left with checking that Φ is a natural isomorphism, that is, (using diagram (#))

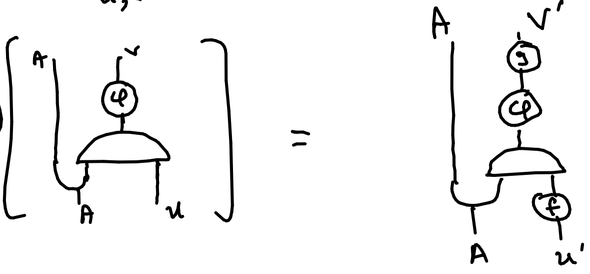
$$\mathrm{Hom}_{A|A}(\alpha_A^-(f), \alpha_A^+(g)) \circ \Phi_{u,v} = \Phi_{u',v'} \circ \mathrm{Hom}(E_A^\ell(f), g)$$

Let's evaluate both sides on $\varphi \in \mathrm{Hom}(E_A^\ell(u), V)$

$$\begin{tikzcd} u \arrow[r, "f"] & u' \\ V \arrow[r, "g"] & V' \end{tikzcd}$$

LHS : $\mathrm{Hom}_{A|A}(\alpha_A^-(f), \alpha_A^+(g)) \circ \Phi_{u,v}(\varphi)$

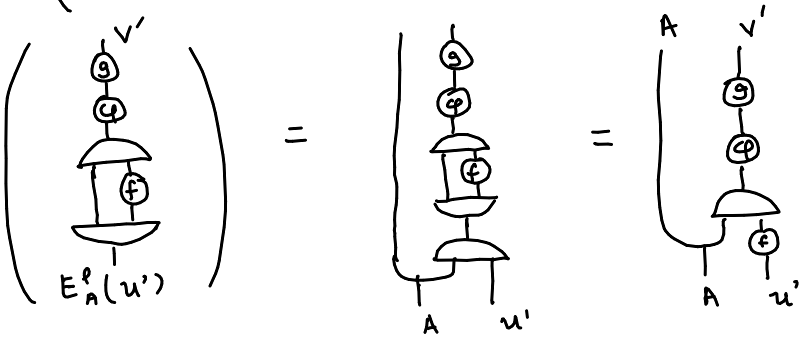
$= \mathrm{Hom}_{A|A}(\alpha_A^-(f), \alpha_A^+(g))$



RHS = $\Phi_{u',v'} \circ \mathrm{Hom}(E_A^\ell(f), g)(\varphi)$

$= \Phi_{u',v'}\left(g \circ \varphi \circ E_A^\ell(f)\right)$

$= \Phi_{u',v'}$



= LHS

■