

Oct 20, 2021

TALK 5: Endofunctors related to α -induction (FFRS §3.1)

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Quick summary of last few talks:

Setting: RIBBON CATEGORY

RIBBON CATEGORY =

- \mathbb{k} -linear: $\text{Hom}_{\mathbb{k}}(X, Y) \in \mathbb{k}$, has \oplus - monoidal: $\otimes, \mathbb{1}$ - braided: $C_{x,y}: X \otimes Y \rightarrow Y \otimes X$

$$C_{x,y} = \begin{array}{c} y \quad x \\ \diagdown \quad / \\ x \quad y \end{array} \quad C_{y,x}^{-1} = \begin{array}{c} x \quad y \\ / \quad \diagdown \\ y \quad x \end{array}$$

- rigid: have left and right duals

- pivotal: left dual = right dual

denote $X^v = \text{dual of } X$ $f^v = \text{dual of } f$

left duality

$$\begin{array}{c} u \quad u^v \\ \downarrow \quad \uparrow \\ u^v \quad u \end{array} \quad \begin{array}{c} u^v \quad u \\ \uparrow \quad \downarrow \\ u \quad u^v \end{array}$$

$$\begin{array}{c} \downarrow \\ \uparrow \\ u^v \end{array} = \begin{array}{c} \downarrow \\ \uparrow \\ u \end{array}, \quad \begin{array}{c} \downarrow \\ \uparrow \\ u \end{array} = \begin{array}{c} \downarrow \\ \uparrow \\ u^v \end{array}$$

right duality

$$\begin{array}{c} u^v \quad u \\ \downarrow \quad \uparrow \\ u \quad u^v \end{array} \quad \begin{array}{c} u \quad u^v \\ \uparrow \quad \downarrow \\ u^v \quad u \end{array}$$

$$\begin{array}{c} \downarrow \\ \uparrow \\ u \end{array} = \begin{array}{c} \downarrow \\ \uparrow \\ u^v \end{array}, \quad \begin{array}{c} \downarrow \\ \uparrow \\ u^v \end{array} = \begin{array}{c} \downarrow \\ \uparrow \\ u \end{array}$$

(snake relations)

- ribbon: left twist = right twist

$$q = p$$

→ Idempotent: morphism $p: u \rightarrow u$ satisfying $p^2 = p$.→ We say p splits if \exists maps $r: u \rightarrow s$, $e: s \rightarrow u$ such that $u \xrightarrow{r} s \xrightarrow{e} u = p$ and $s \xrightarrow{e} u \xrightarrow{r} s = \text{id}_s$

Graphically,

$$e = \begin{array}{c} u \\ \downarrow \\ \circlearrowleft \\ \downarrow \\ s \end{array}, \quad r = \begin{array}{c} s \\ \downarrow \\ \circlearrowright \\ \downarrow \\ u \end{array} \quad \text{satisfying} \quad \begin{array}{c} s \\ \downarrow \\ \circlearrowleft \\ \downarrow \\ s \end{array} = id_s, \quad \begin{array}{c} u \\ \downarrow \\ \circlearrowright \\ \downarrow \\ u \end{array} = id_u$$

The triple (S, e, r) is called the retract of f .

A category \mathcal{C} is called Karoubian if all idempotents split. In this talk, the ribbon categories are not necessarily Karoubian.

Objects of study: **SYMMETRIC SPECIAL FROBENIUS ALGEBRAS**

It is a 5-tuple $(A, m, u, \Delta, \varepsilon)$ such that

① $(A, m, u) \in Alg(\mathcal{C})$ $\begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array}, \quad \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array} = | = \begin{array}{c} \circlearrowright \\ \circlearrowright \end{array}$

② $(A, \Delta, \varepsilon) \in Coalg(\mathcal{C})$ $\begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array}, \quad \begin{array}{c} \circlearrowright \\ \circlearrowright \end{array} = | = \begin{array}{c} \circlearrowleft \\ \circlearrowleft \end{array}$

③ Frobenius condition $\begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array}$

④ Symmetric $\begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array}$

⑤ Special $\varepsilon \circ \eta = \int = \beta_{\mathbb{1}} id_{\mathbb{1}}, \quad m \circ \Delta = \oint = \beta_A id_A$
for some $\beta_{\mathbb{1}}, \beta_A \in \mathbb{K} \setminus \{0\}$

⑥ When A is symmetric special, we can normalize Δ, ε to get the relations.

(i) $\int = 0$ (ii) $\oint = |$

(iii) $\begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array} = |$ (iv) $\begin{array}{c} \cup \\ \cup \end{array} = \begin{array}{c} \cup \\ \cup \end{array} = |$

Remark about graphical calculus for symmetric Frobenius algebras in ribbon categories

The symmetric, Frobenius and (co)associativity conditions together imply that we can move multiplication and comultiplication past each other in all possible arrangements.

Lemma:

$$\text{Cup with arrows on left} = \text{Cup with arrows on right} \quad \text{--- (7)}$$

Proof:

Then,



Last time, we had defined left and right central idempotents of A

$$P_A^l = \text{diagram: a cup with a vertical line from its center to a cap labeled 'A' above it, and another vertical line from the center of the cup to a cap labeled 'A' below it. The cup is open to the left and right.$$

$$P_A^r = \text{diagram: a cap with a vertical line from its center to a cup labeled 'A' above it, and another vertical line from the center of the cap to a cup labeled 'A' below it. The cap is open to the left and right.$$

Next, we constructed the

- ① $\left[\begin{array}{l} \cdot \text{left centre } C_l(A) = \text{retract of } P_A^l \\ \cdot \text{right centre } C_r(A) = \text{retract of } P_A^r \end{array} \right.$

and showed that

C_l and C_r are commutative symmetric Frobenius algebras in \mathcal{C} , and

- ② $\left[\begin{array}{l} \text{there are natural bijections} \\ \text{Hom}(C_l(A) \otimes U, V) \cong \text{Hom}_{A|A}(\alpha_A^+(U), \alpha_A^+(V)) \\ \cong \text{Hom}(U, C_l(A) \otimes V) \\ \text{Hom}(C_r(A) \otimes U, V) \cong \text{Hom}_{A|A}(\alpha_A^-(U), \alpha_A^-(V)) \\ \cong \text{Hom}(U, C_r(A) \otimes V) \end{array} \right.$

where $\alpha_A^\pm : \mathcal{C} \rightarrow \mathcal{C}_{A|A}$ are the α -induction functors

$$\alpha_A^+ : \mathcal{C} \mapsto (A \otimes U, \text{diagram: } \int_A^{A \otimes U} := \text{cup}, \text{diagram: } \int_{A \otimes U}^A := \text{cap}, \text{diagram: } P_r^+ := \text{diagram with } A \text{ and } U \text{ labels})$$

$$\alpha_A^- : \mathcal{C} \mapsto (A \otimes U, \text{diagram: } \int_A^{A \otimes U} := \text{cup}, \text{diagram: } \int_{A \otimes U}^A := \text{cap}, \text{diagram: } P_r^- := \text{diagram with } A \text{ and } U \text{ labels})$$

Today's talk is about generalizing the construction ① and the result ②

§ 3.1 Endofunctors related to α -induction

Consider the endomorphisms

$$P_A^{\ell}(U) = \text{[Diagram: A cup with a vertical line from the right side of the cup to the top, labeled U. The left side of the cup is labeled A. The bottom of the cup is labeled A and U.]}$$

$$P_A^{\lambda}(U) = \text{[Diagram: A cup with a vertical line from the left side of the cup to the top, labeled U. The right side of the cup is labeled A. The bottom of the cup is labeled A and U.]}$$

Clearly $P_A^{\lambda/\ell}(\mathbb{1}) = P_A^{\lambda/\ell}$.

Just like $P_A^{\lambda/\ell}$, the endomorphisms $P_A^{\lambda/\ell}(U)$ are also idempotent.

Defn: A special Frobenius algebra A in a ribbon category \mathcal{C} is called **centrally split** if the idempotents $P_A^{\lambda/\ell}(U)$ are split for every $U \in \mathcal{C}$.

Since our ribbon categories are not necessarily Karoubian, we make the following assumption.

Declaration: From here on, every special Frobenius algebra will be assumed to be centrally split.

Defn: $E_A^{\ell}(U) := \text{Im}(P_A^{\ell}(U))$, $E_A^{\lambda}(U) := \text{Im}(P_A^{\lambda}(U))$

Then by definition, we have maps

$$e : E_A^{\lambda/\ell}(U) \rightarrow A \otimes U, \quad \iota : A \otimes U \rightarrow E_A^{\lambda/\ell}(U)$$

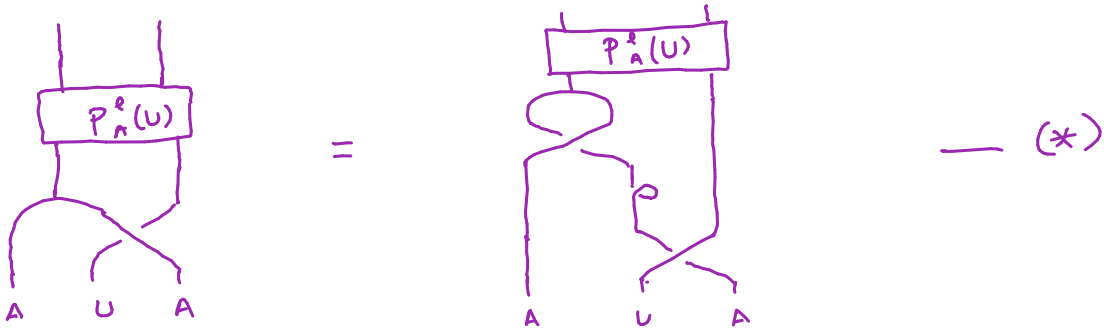
For $f \in \text{Hom}(U, V)$, define

$$E_A^{\ell}(f) := \text{[Diagram: A vertical line from a box labeled f to a cup labeled e. The top of the cup is labeled \iota. The bottom of the cup is labeled E_A^{\ell}(U). The top of the line is labeled E_A^{\ell}(V).]}$$

$$E_A^{\lambda}(f) = \text{[Diagram: A vertical line from a box labeled f to a cup labeled e. The top of the cup is labeled \iota. The bottom of the cup is labeled E_A^{\lambda}(U). The top of the line is labeled E_A^{\lambda}(V).]}$$

Lemma: Let p_a^\pm denote the right A -action on $A \otimes U$, then

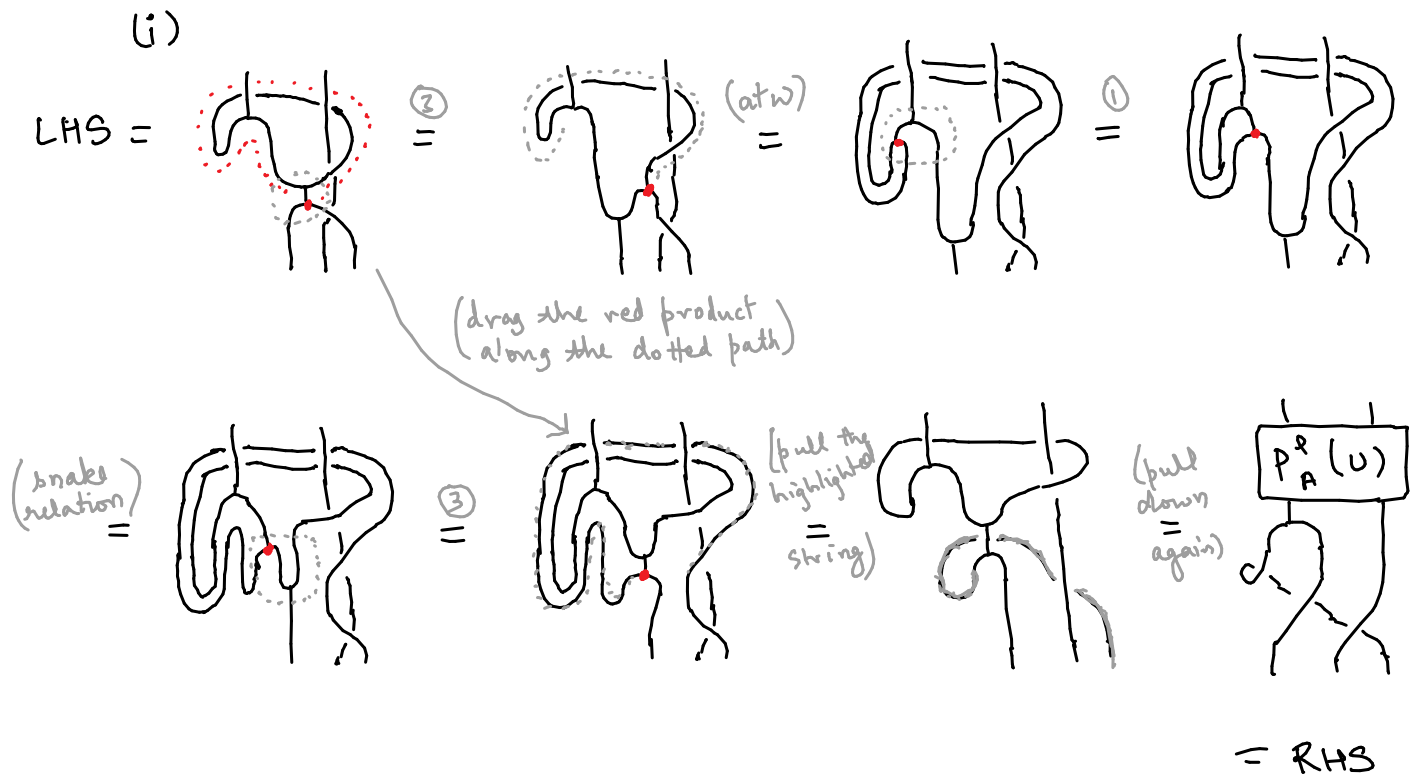
$$(i) P_A^e(U) \circ p_r^- = P_A^e(U) \circ ([m \circ c_{A,A} \circ (id \otimes \theta_A)] \otimes id_U) \circ (id_A \otimes c_{U,A})$$




(+ similar result for $P_A^a(U)$)


(ii) If A is commutative, then
 $P_A(U) \circ p_r^+ = P_A(U) \circ p_r^-$
 for $P_A(U) \equiv P_A^{e/n}(U)$.

Proof: (ii) It follows immediately from (i) by using that
 $\alpha = \cap$, $\beta = |$.



Recall that $P_A^r(u) =$ 

$$\therefore r \circ P_A^r(u) = \text{tree diagram} = \text{tree diagram with } E_A^r(u) \text{ at the root}$$

Thus, postcomposing (*) with , gives us that

$$\text{tree diagram with } E_2^A(u) \text{ at the root} = \text{tree diagram with } E_2^A(u) \text{ at the root} \quad - (**)$$

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be a pair of functors. We call G right adjoint of F if there exists a natural family of isomorphisms

$$\phi_{X,Y}: \text{Hom}_{\mathcal{C}}(F(X), Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}}(X, G(Y))$$

for $X \in \mathcal{C}, Y \in \mathcal{D}$

Recall the functors $\alpha_A^{\pm}: \mathcal{C} \rightarrow \mathcal{C}_{A|A}$

Suppose that these have right adjoints denoted as $(\alpha_A^{\pm})^{ra}$ and left adjoints denoted by $(\alpha_A^{\pm})^{la}$. Then by defn, we get natural isomorphisms.

(set $X = \alpha_A^+(U), Y = V, F = (\alpha_A^-)^{ra}, G = \alpha_A^-$)

$$\rightarrow \text{Hom}((\alpha_A^-)^{la} \circ \alpha_A^+(U), V) \cong \text{Hom}_{A|A}(\alpha_A^+(U), \alpha_A^-(V)) \cong \text{Hom}(U, (\alpha_A^+)^{ra} \circ \alpha_A^-(V))$$

(similarly for $\text{Hom}_{A|A}(\alpha_A^-(U), \alpha_A^+(V))$)

\rightarrow FFRS show that $E_A^{\lambda/\lambda}$ can be regarded as compositions of the functors α_A^{\pm} and their adjoints by showing the following results.

Proof: For $U, V \in \mathcal{C}$, there are natural bijections

$$\text{Hom}(E_A^{\rho}(U), V) \cong \text{Hom}_{A|A}(\alpha_A^{-1}(U), \alpha_A^{+}(V)) \cong \text{Hom}(U, E_A^{\rho}(V))$$

and

$$\text{Hom}(E_A^{\lambda}(U), V) \cong \text{Hom}_{A|A}(\alpha_A^{+}(U), \alpha_A^{-}(V)) \cong \text{Hom}(U, E_A^{\lambda}(V))$$

Proof: We will discuss the natural isomorphism

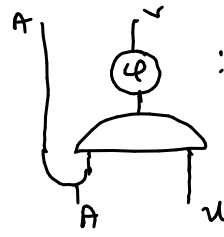
$$\text{Hom}(E_A^{\rho}(U), V) \xrightleftharpoons[\Psi_{U,V}]{\Phi_{U,V}} \text{Hom}_{A|A}(\alpha_A^{-1}(U), \alpha_A^{+}(V))$$

Other cases have similar proofs.

Naturality amounts to showing that $\forall U \xrightarrow{f} U', V \xrightarrow{g} V'$

$$\begin{array}{ccc} \text{Hom}(E_A^{\rho}(U), V) & \xrightarrow{\Phi_{U,V}} & \text{Hom}_{A|A}(\alpha_A^{-1}(U), \alpha_A^{+}(V)) & \text{---} (\#) \\ \downarrow \text{Hom}(E_A^{\rho}(f), g) & & \downarrow \text{Hom}_{A|A}(\alpha_A^{-1}(f), \alpha_A^{+}(g)) & \\ \text{Hom}(E_A^{\rho}(U'), V') & \xrightarrow{\Phi_{U',V'}} & \text{Hom}_{A|A}(\alpha_A^{-1}(U'), \alpha_A^{+}(V')) & \text{--- commutes} \end{array}$$

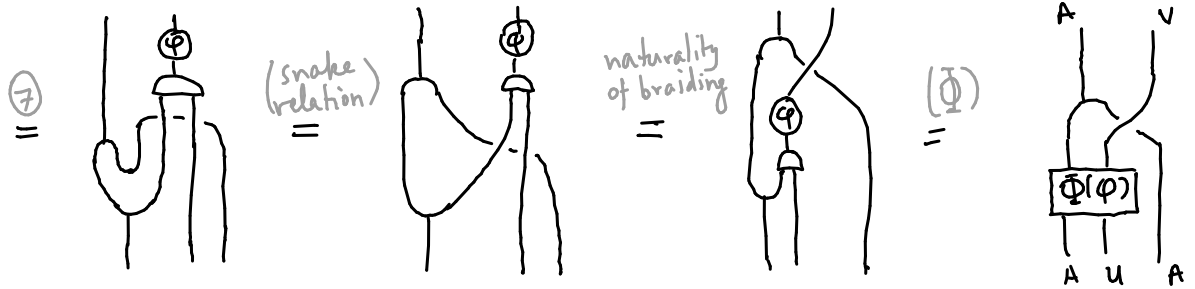
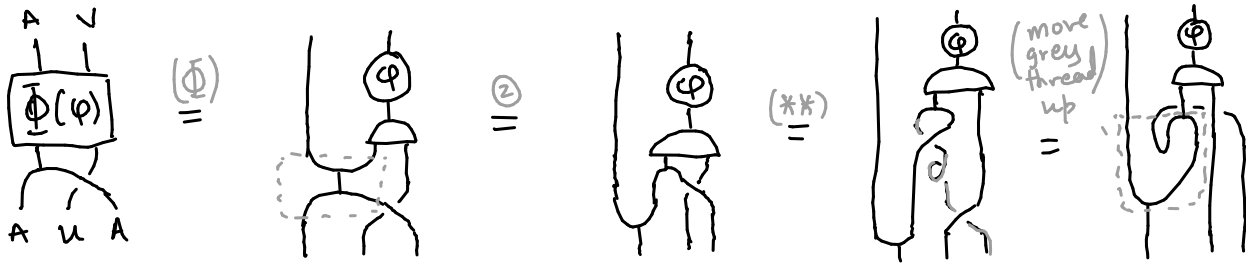
Definition of Φ :

• For $\varphi: E_A^{\rho}(U) \rightarrow V$, define $\Phi_{U,V}(\varphi) =$  : $A \otimes U \rightarrow A \otimes V$
 $\alpha_A^{-1}(U)$ $\alpha_A^{+}(V)$
 --- (Φ)

$\rightarrow \Phi_{U,V}(\varphi)$ is a morphism of left A -modules because

$$\begin{array}{ccccccc} \begin{array}{c} A \quad V \\ | \quad | \\ \boxed{\Phi(\varphi)} \\ | \quad | \\ A \quad A \quad U \end{array} & \stackrel{(\Phi)}{=} & \begin{array}{c} | \quad | \\ | \quad | \\ \text{---} \\ | \quad | \\ A \quad A \quad U \end{array} & \stackrel{(\Phi)}{=} & \begin{array}{c} | \quad | \\ | \quad | \\ \text{---} \\ | \quad | \\ A \quad A \quad U \end{array} & \stackrel{(\Phi)}{=} & \begin{array}{c} A \quad V \\ | \quad | \\ \boxed{\Phi(\varphi)} \\ | \quad | \\ A \quad A \quad U \end{array} \end{array}$$

→ $\Phi_{u,v}(\varphi)$ is a morphism of right A -modules because



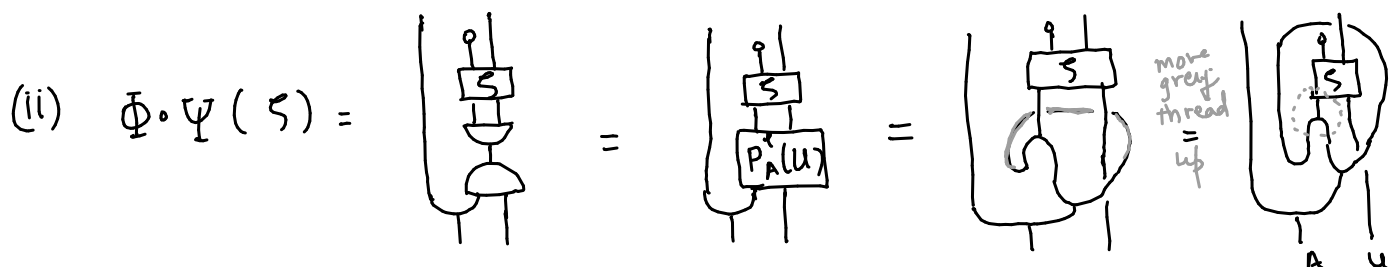
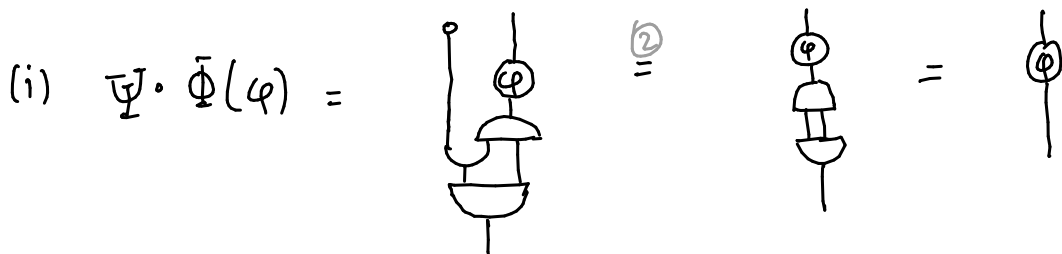
Definition of Ψ :

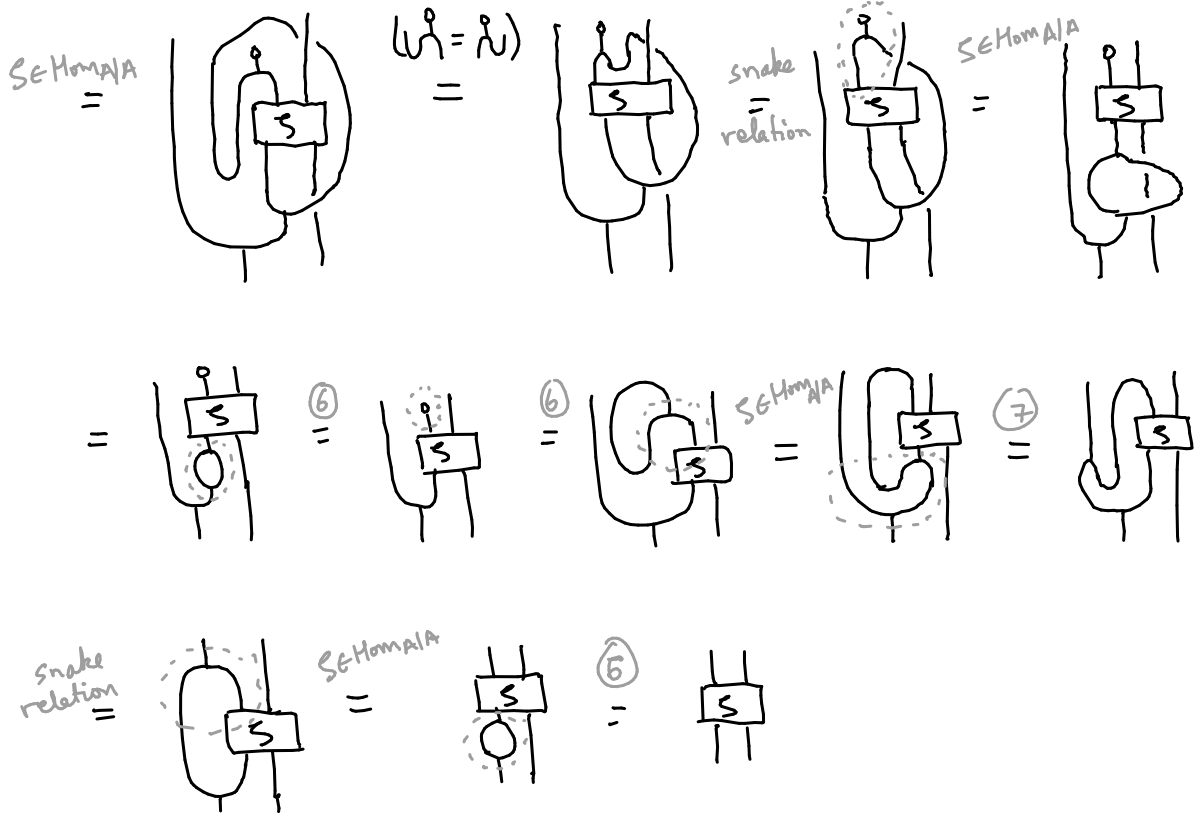
For $\zeta \in \text{Hom}_{A|A}(\alpha_A^{-1}(U), \alpha_A^{-1}(V))$, define

$$\Psi_{u,v}(\zeta) = (\eta \otimes \text{id}_v) \circ \zeta \circ e = \text{diagram} \quad (7)$$

• Next, we need to check that

(i) $\Psi \circ \Phi(\varphi) = \varphi$ and (ii) $\Phi \circ \Psi(\zeta) = \zeta$





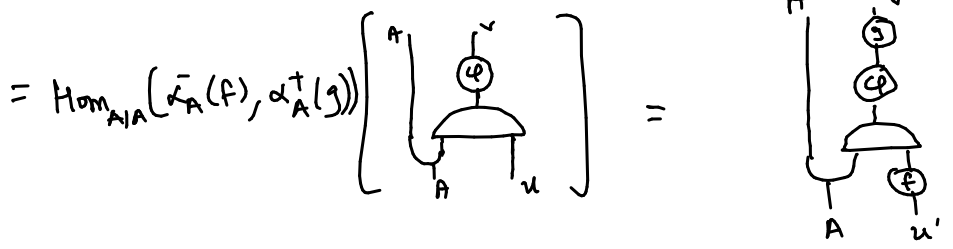
→ Finally we are left with checking that Φ is a natural isomorphism, that is, (using diagram #)

$$\text{Hom}_{A|A}(\alpha_A^-(f), \alpha_A^+(g)) \circ \Phi_{u,v} = \Phi_{u',v'} \circ \text{Hom}(E_A^p(f), g)$$

Let's evaluate both sides on $\varphi \in \text{Hom}(E_A^p(u), v)$

$u \xrightarrow{f} u'$
 $v \xrightarrow{g} v'$

LHS: $\text{Hom}_{A|A}(\alpha_A^-(f), \alpha_A^+(g)) \circ \Phi_{u,v}(\varphi)$



RHS = $\Phi_{u',v'} \circ \text{Hom}(E_A^p(f), g)(\varphi)$

= $\Phi_{u',v'}(g \circ \varphi \circ E_A^p(f))$

