Let $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ be a monoidal category.

Defn: A $\boxed{\text{(left) } \mathcal{C}\text{-module category}}$ is a category $\mathcal{M}$ equipped with an action bifunctor $\underline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

module associativity constraint $\quad m_{X,Y,M} : (X \otimes Y) \underline{\otimes} M \xrightarrow{\sim} X \underline{\otimes} (Y \underline{\otimes} M) \qquad X, Y \in \mathcal{C}, \; M \in \mathcal{M}$

and $l_M : \mathbb{1} \underline{\otimes} M \xrightarrow{\sim} M$

such that the following diagrams commute

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \underline{\otimes} M \arrow[dl, "a_{X,Y,Z} \underline{\otimes} I_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \underline{\otimes} M \arrow[d, "m_{X, Y \otimes Z, M}"'] & & (X \otimes Y) \underline{\otimes} (Z \underline{\otimes} M) \arrow[d, "m_{X,Y,Z \underline{\otimes}}"] \\
X \underline{\otimes} ((Y \otimes Z) \underline{\otimes} M) \arrow[rr, "I_X \underline{\otimes} m_{Y,Z,M}"'] & & X \underline{\otimes} (Y \underline{\otimes} (Z \underline{\otimes} M))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \underline{\otimes} M \arrow[rr, "m_{X,\mathbb{1},M}"] \arrow[dr, "r_X \underline{\otimes} I_M"'] & & X \underline{\otimes} (\mathbb{1} \underline{\otimes} M) \arrow[dl, "I_X \underline{\otimes} l_M"] \\
 & (X \underline{\otimes} M) &
\end{tikzcd}$$

Defn: A $\boxed{\text{right } \mathcal{C}\text{-module category}}$ is a left $\mathcal{C}^{op}$-module category.

Let $\mathcal{C}, \mathcal{D}$ be monoidal categories.

<u>Defn</u>: A $\boxed{(\mathcal{C},\mathcal{D})\text{-bimodule category}}$ is a category $\mathcal{M}$ s.t.

(i) $(\mathcal{M}, \underline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}, m_{X,Y,M})$ is a left $\mathcal{C}$-module category.

(ii) $(\mathcal{M}, \bar{\otimes} : \mathcal{M} \times \mathcal{D} \to \mathcal{M}, n_{M,W,Z} : M \bar{\otimes} (W \otimes Z) \xrightarrow{\sim} (M \bar{\otimes} W) \bar{\otimes} Z)$ is a right $\mathcal{D}$-module category.

(iii) We have natural isomorphisms

$$b_{X,M,Z} : (X \underline{\otimes} M) \bar{\otimes} Z \xrightarrow{\sim} X \underline{\otimes} (M \bar{\otimes} Z)$$

middle associativity constraint

such that the following commute:

$$\begin{tikzcd}
& ((X \otimes Y) \underline{\otimes} M) \bar{\otimes} Z \arrow[dl, "m_{X,Y,M} \bar{\otimes} I_Z"'] \arrow[dr, "b_{X\otimes Y, M, Z}"] & \\
(X \underline{\otimes} (Y \underline{\otimes} M)) \bar{\otimes} Z \arrow[d, "b_{X, Y\underline{\otimes} M, Z}"'] & & (X \otimes Y) \underline{\otimes} (M \bar{\otimes} Z) \arrow[d, "m_{X,Y,M\bar{\otimes} Z}"] \\
X \underline{\otimes} ((Y \underline{\otimes} M) \bar{\otimes} Z) \arrow[rr, "id_X \otimes b_{Y,M,Z}"'] & & X \underline{\otimes} (Y \underline{\otimes} (M \bar{\otimes} Z))
\end{tikzcd}$$

$$\begin{tikzcd}
& X \underline{\otimes} (M \bar{\otimes} (W \otimes Z)) \arrow[dl, "I_X \otimes n_{M,W,Z}"'] & \\
X \underline{\otimes} ((M \bar{\otimes} W) \bar{\otimes} Z) & & (X \underline{\otimes} M) \bar{\otimes} (W \otimes Z) \arrow[ul, "b_{X,M,W\otimes Z}"'] \arrow[d, "n_{X\underline{\otimes} M, W, Z}"] \\
(X \underline{\otimes} (M \bar{\otimes} W)) \bar{\otimes} Z \arrow[u, "b_{X, M\bar{\otimes} W, Z}"] & & ((X \underline{\otimes} M) \bar{\otimes} W) \bar{\otimes} Z \arrow[ll]
\end{tikzcd}$$

$\forall\ X, Y \in \mathcal{C},\ Z, W \in \mathcal{D}$, and $M \in \mathcal{M}$

Defn: Let $\mathcal{M}$ and $\mathcal{N}$ be two (left) module categories over $\mathcal{C}$. A $\boxed{\mathcal{C}\text{- module functor}}$ from $\mathcal{M}$ to $\mathcal{N}$ consists of

(i) a functor $F : \mathcal{M} \longrightarrow \mathcal{N}$, and

(ii) a natural isomorphism

$$s_{X,M} : F(X \underline{\otimes} M) \longrightarrow X \underline{\otimes} F(M) \qquad X \in \mathcal{C},\ M \in \mathcal{M}$$

such that the following diagrams commutes

$$\begin{tikzcd}
& F((X \otimes Y) \underline{\otimes} M) \arrow[dl, "s_{X\otimes Y, M}"'] \arrow[dr, "F(m_{X,Y,M})"] & \\
(X \otimes Y) \underline{\otimes} F(M) \arrow[d, "m_{X,Y,F(M)}"'] & & F(X \underline{\otimes} (Y \underline{\otimes} M)) \arrow[d, "s_{X, Y\underline{\otimes} M}"] \\
X \underline{\otimes} (Y \underline{\otimes} F(M)) & & X \underline{\otimes} F(Y \underline{\otimes} M) \arrow[ll, "I_X \otimes s_{Y,M}"]
\end{tikzcd}$$

$$\begin{tikzcd}
F(\mathbb{1} \underline{\otimes} M) \arrow[rr, "s_{\mathbb{1},M}"] \arrow[dr, "F(l_M)"'] & & \mathbb{1} \underline{\otimes} F(M) \arrow[dl, "l_{F(M)}"] \\
& F(M) &
\end{tikzcd}$$

$\forall\ X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$

Defn: A $\boxed{\text{morphism of } \mathcal{C}\text{-module functors}}$ from (F, s) to (G, t) is a natural transformation $\nu : F \Rightarrow G$ such that the following diagram commutes for any $X \in \mathcal{C}$ and $M \in \mathcal{M}$

$$\begin{tikzcd}
F(X \otimes M) \arrow[r, "s_{X,M}"] \arrow[d, "\nu_{X \otimes M}"'] & X \otimes F(M) \arrow[d, "I_X \otimes \nu_M"] \\
G(X \otimes M) \arrow[r, "t_{X,M}"'] & X \otimes G(M)
\end{tikzcd}$$

- $\mathcal{C}$-module functors between two $\mathcal{C}$-module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ form a category with morphisms as above.
- Compositions of two $\mathcal{C}$-module functors is also a $\mathcal{C}$-module functor.
- Let $\mathcal{C}^*_{\mathcal{M}}$ = category of $\mathcal{C}$-module endofunctors $F : \mathcal{M} \longrightarrow \mathcal{M}$

  Clearly, $\mathcal{C}^*_{\mathcal{M}}$ is a monoidal category under composition

Module Categories over (multi) tensor categories

Let $\mathcal{C}$ be a multi tensor category over k.

Defn: A module category over $\mathcal{C}$ is a cat. $\mathcal{M}$ s.t.

(i) $\mathcal{M}$ is locally finite, abelian

(ii) $\mathcal{M}$ is equipped with $(\underline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M},\ m_{X,Y,M})$ making it a left $\mathcal{C}$-module category s.t.

- $\underline{\otimes}$ is bilinear on morphisms
- $\underline{\otimes}$ is exact in first variable

- In the above setting, the category $\mathcal{C}^*_{\mathcal{M}} = \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ is a finite multitensor category.
- If $\mathcal{M}$ is indecomposable, the $\mathcal{C}^*_{\mathcal{M}}$ is finite tensor cat.

§8.7 Module categories over braided tensor categories

- Let $(\mathcal{C}, c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X)$ be a braided tensor category and $(\mathcal{M}, \underline{\otimes}, m)$ be a module category over $\mathcal{C}$.

- For each $X \in \mathcal{C}$, we get an endofunctor

$$\begin{aligned} X \underline{\otimes} - : \mathcal{M} &\longrightarrow \mathcal{M} \\ M &\longmapsto X \underline{\otimes} M \end{aligned}$$

- $X \underline{\otimes} -$ can be turned into a $\mathcal{C}$-module functor in two ways as follows:

(i) $(H^+(X) = X \underline{\otimes} -, s^+_X)$

$$(s^+_X)_{Y,M} :$$

$$\begin{tikzcd}
H^+(X)(Y \underline{\otimes} M) = X \underline{\otimes} (Y \underline{\otimes} M) \arrow[r, "m^{-1}_{X,Y,M}"] \arrow[dd] & (X \otimes Y) \underline{\otimes} M \arrow[d, "c_{X,Y} \otimes I_M"] \\
 & (Y \otimes X) \underline{\otimes} M \arrow[d, "m_{Y,X,M}"] \\
Y \underline{\otimes} H^+(X)(M) = Y \underline{\otimes} (X \underline{\otimes} M) & Y \underline{\otimes} (X \underline{\otimes} M)
\end{tikzcd}$$

(ii) $(H^-(X) = X \underline{\otimes} -, s^-_X)$

$$
(s^-_X)_{Y,M} : \quad
\begin{tikzcd}
H^-(X)(Y\underline{\otimes}M) = X\underline{\otimes}(Y\underline{\otimes}M) \arrow[r, "m^{-1}_{X,Y,M}"] \arrow[d] & (X\otimes Y)\underline{\otimes}M \arrow[d, "c^{-1}_{Y,X}\otimes I_M"] \\
Y\underline{\otimes}H^-(X)(M) = Y\underline{\otimes}(X\underline{\otimes}M) & (Y\otimes X)\underline{\otimes}M \arrow[l, "m_{Y,X,M}"]
\end{tikzcd}
$$

$\forall\, X, Y \in \mathcal{C},\ M \in \mathcal{M}.$

Let's check that $(X\otimes -, s^+_X)$ is indeed a $\mathcal{C}$-module endofunctor. Recall that we need the following diagram to commute

$$
\begin{tikzcd}[column sep=small]
 & F((X\otimes Y)\underline{\otimes}M) \ (1) \arrow[dl, "s_{X\otimes Y,M}"'] \arrow[dr, "F(m_{X,Y,M})"] & \\
(2)\ (X\otimes Y)\underline{\otimes}F(M) \arrow[d, "n_{X,Y,F(M)}"'] & & F(X\underline{\otimes}(Y\underline{\otimes}M)) \ (5) \arrow[d, "s_{X,Y\otimes M}"] \\
(3)\ X\underline{\otimes}(Y\underline{\otimes}F(M)) & & X\underline{\otimes}F(Y\underline{\otimes}M) \ (4) \arrow[ll, "I_X\otimes s_{Y,M}"]
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}[column sep=small]
 & X\underline{\otimes}((Y\otimes Z)\underline{\otimes}M) \arrow[dl, "(s^+_X)_{Y\otimes Z,M}"'] \arrow[dr, "I_X\otimes m_{Y,Z,M}"] & \\
(Y\otimes Z)\underline{\otimes}(X\underline{\otimes}M) \arrow[d, "m_{Y,Z,X\otimes M}"'] & & X\underline{\otimes}(Y\underline{\otimes}(Z\underline{\otimes}M)) \arrow[d, "(s^+_X)_{Y,Z\otimes M}"] \\
Y\underline{\otimes}Z\underline{\otimes}(X\underline{\otimes}M) & & Y\underline{\otimes}(X\underline{\otimes}(Z\underline{\otimes}M)) \arrow[ll, "I_Y\otimes (s^+_X)_{Z,M}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\text{1}\quad X\otimes[(Y\otimes Z)\otimes M] \arrow[rr, "I_X\otimes m_{Y,Z,M}"] \arrow[d, "m^{-1}_{X,Y\otimes Z,M}"'] & & X\otimes[Y\otimes(Z\otimes M)] \arrow[rr, "x{,}y{,}z"] & & (X\otimes Y)\otimes(Z\otimes M) \arrow[d, "c_{X,Y}\otimes I_{Z\otimes M}"] \\
{[X\otimes(Y\otimes Z)]\otimes M} \arrow[d, "c_{X,Y\otimes Z}"'] & & {[(X\otimes Y)\otimes Z]\otimes M} \arrow[ll, "a_{X,Y,Z}\otimes I_M"'] \arrow[urr, "m_{X\otimes Y,Z,M}"] \arrow[r, "c_{X,Y}"] & {[(Y\otimes X)\otimes Z]\otimes M} \arrow[r, "m_{Y\otimes X,Z,M}"] \arrow[d, "a_{Y,X,Z}"] & (Y\otimes X)\otimes(Z\otimes M) \arrow[d, "m_{Y,X,Z\otimes M}"] \\
{[(Y\otimes Z)\otimes X]\otimes M} \arrow[rr, "a_{Y,Z,X}"] \arrow[d, "m_{Y\otimes Z,X,M}"'] & & {[Y\otimes(Z\otimes X)]\otimes M}\ \text{5} \arrow[dd, "m_{Y,Z\otimes X,M}"] & {[Y\otimes(X\otimes Z)]\otimes M} \arrow[l, "c_{X,Z}"'] \arrow[r, "m_{Y,X\otimes Z,M}"] & Y\otimes(X\otimes(Z\otimes M))\quad\text{4} \arrow[d, "I_Y\otimes m^{-1}_{X,Z,M}"] \\
\text{2}\quad (Y\otimes Z)\otimes[X\otimes M] \arrow[d, "m_{Y,Z,X\otimes M}"'] & & & & Y\otimes((X\otimes Z)\otimes M) \arrow[dll, "I_Y\otimes c_{X,Z}\otimes I_M"] \\
\text{3}\quad Y\otimes(Z\otimes(X\otimes M)) & & Y\otimes((Z\otimes X)\otimes M) \arrow[ll, "I_Y\otimes m_{Z,X,M}"'] & &
\end{tikzcd}
$$

pentagon axiom for m

braid relation 1

naturality of m

Prop 8.7.1, EGNO The assignments

$$H^+ : \mathcal{C} \longrightarrow \mathcal{C}^*_{\mathcal{M}} \qquad \text{and} \qquad H^- : \mathcal{C}^{op} \longrightarrow \mathcal{C}^*_{\mathcal{M}}.$$

$$X \longmapsto (X \otimes -, s^+_X) \qquad\qquad X \longmapsto (X \otimes -, s^-_X)$$

define tensor functors

PROOF: To give H^+ a monoidal structure, we need a natural isomorphism

$$J^+_{X,Y} : H^+(X)H^+(Y) \Longrightarrow H^+(X \otimes Y)$$

this functor takes $M \mapsto X \otimes (Y \otimes M)$

this functor takes $M \mapsto (X \otimes Y) \otimes M$

$J^+_0 : I_{\mathcal{M}} \Longrightarrow H^+(\mathbb{1}_{\mathcal{C}})$

$(J^+_0)_M = l_M$

the components of $J^+_{X,Y}$ are

$$(J^+_{X,Y})_M : X \otimes (Y \otimes M) \xrightarrow{m^{-1}_{X,Y,M}} (X \otimes Y) \otimes M$$

to check if $J_{X,Y}$ is indeed a morphism of $\mathcal{C}$-module functors $H^+(X)H^+(Y)$ & $H^+(X \otimes Y)$, we need to check if the following commutes

$$\begin{tikzcd}
{[H^+(X)H^+(Y)](Z \otimes M)} \arrow[r, "s^+_X s^+_Y"] \arrow[d, "(J_{X,Y})_{Z \otimes M}"'] & Z \otimes ([H^+(X)H^+(Y)](M)) \arrow[d, "I_Z \otimes (J_{X,Y})_M"] \\
{[H^+(X \otimes Y)](Z \otimes M)} \arrow[r, "s^+_{X \otimes Y}"] & Z \otimes H^+(X \otimes Y)(M)
\end{tikzcd}$$

Expanding out, this diagram is :

$$
\begin{tikzcd}
X \otimes (Y \underline{\otimes} (Z \underline{\otimes} M)) \arrow[rr, "I_X \otimes m^{-1}_{Y,Z,M}"'] & & X \underline{\otimes} ((Y \otimes Z) \underline{\otimes} M) \arrow[rr, "c_{Y,Z}"] & & X \underline{\otimes} ((Z \otimes Y) \underline{\otimes} M) \arrow[d, "I_X \otimes m_{Z,Y,M}"] \\
(X \otimes Y) \underline{\otimes} (Z \underline{\otimes} M) \arrow[u, "m_{X,Y,Z \otimes M}"] \arrow[d, "m^{-1}_{X \otimes Y,Z,M}"'] & & (X \otimes (Y \otimes Z)) \underline{\otimes} M \arrow[u, "m_{X,Y \otimes Z,M}"] \arrow[r, "c_{Y,Z}"] & (X \otimes (Z \otimes Y)) \underline{\otimes} M \arrow[ur] & X \underline{\otimes} (Z \underline{\otimes} (Y \underline{\otimes} M)) \arrow[d, "m^{-1}_{X,Z,Y \otimes M}"] \\
[(X \otimes Y) \otimes Z] \underline{\otimes} M \arrow[urr, "a_{X,Y,Z} \otimes I_M"'] \arrow[d, "c_{X \otimes Y,Z} \otimes I_M"'] & & & [(X \otimes Z) \otimes Y] \underline{\otimes} M \arrow[u, "a"] \arrow[r, "m"] \arrow[d, "c_{X,Z}"'] & (X \otimes Z) \underline{\otimes} (Y \underline{\otimes} M) \arrow[d, "c_{X,Z}"] \\
[Z \otimes (X \otimes Y)] \underline{\otimes} M \arrow[drr, "m_{Z,X \otimes Y,M}"'] & & & [(Z \otimes X) \otimes Y] \underline{\otimes} M \arrow[lll, "a_{Z,X,Y} \otimes I_M"'] \arrow[r, "m_{Z \otimes X,Y,M}"'] & (Z \otimes X) \underline{\otimes} (Y \underline{\otimes} M) \arrow[d, "m_{Z,X,Y \otimes M}"] \\
& & Z \underline{\otimes} [(X \otimes Y) \underline{\otimes} M] \arrow[rr, "I_Z \otimes m_{X,Y,M}"'] & & Z \underline{\otimes} (X \underline{\otimes} (Y \underline{\otimes} M))
\end{tikzcd}
$$

$\wr$ module associators

$\wr$ braid condition

$\wr$ by naturality of m

Similarly for H^-, we define the natural transformation

$$J^-_{X,Y} : H^-(X)H^-(Y) \Longrightarrow H^-(X \otimes^{op} Y) = H^-(Y \otimes X)$$

the component maps of J^- are

$$(J^-_{X,Y})_M : X \underline{\otimes} (Y \underline{\otimes} M) \xrightarrow{m^{-1}_{X,Y,M}} (X \otimes Y) \underline{\otimes} M \xrightarrow{c_{X,Y} \underline{\otimes} I_M} (Y \otimes X) \underline{\otimes} M$$

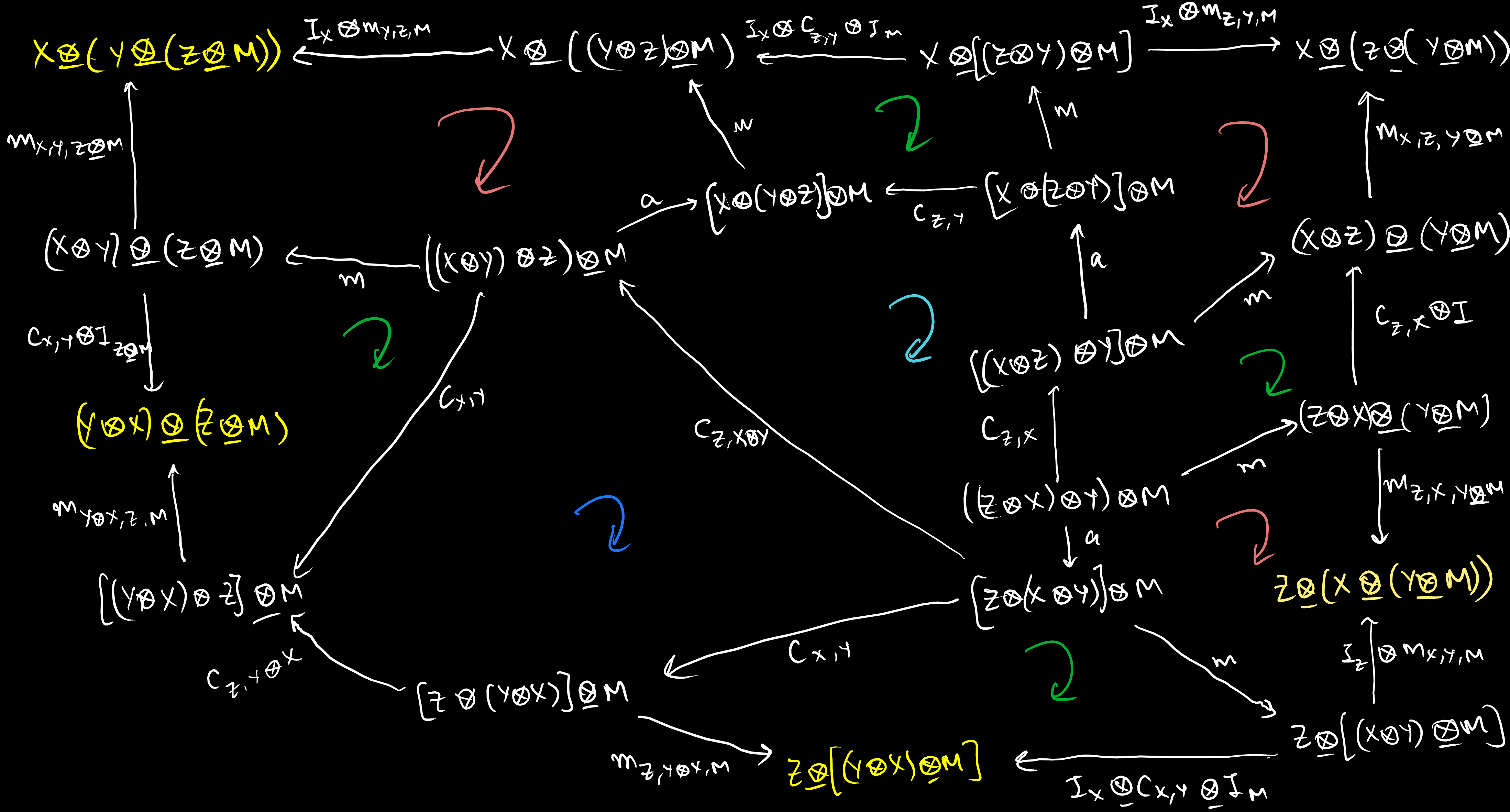
and we also have $J^- : I_M \Longrightarrow H^-(\mathbb{1})$

$$(J^-)_M = l_M$$

then $J^-_{X,Y}$ is a morphism of $\mathcal{C}$-module functors because the following diagram commutes

$$\begin{tikzcd}
{[H^-(X)H^-(Y)](Z \underline{\otimes} M)} \arrow[r, "s^-_X s^-_Y"] \arrow[d, "(J^-_{X,Y})_{Z\underline{\otimes} M}"'] & Z \underline{\otimes} (H^-(X)H^-(Y)(M)) \arrow[d, "I_Z \otimes (J^-_{X,Y})_M"] \\
H^-(X \otimes^{op} Y)(Z \underline{\otimes} M) \arrow[r, "s^-_{X\otimes Y}"'] & Z \otimes (H^-(X \otimes^{op} Y)(M))
\end{tikzcd}$$

we check this on next page



$$
\begin{tikzcd}
{[H^{-1}(X)H^{-1}(Y)](Z\otimes M)} \arrow[r, "\delta_X^{-1}\delta_Y^{-1}"] \arrow[d, "(J^{-1}_{X,Y})_{Z\otimes M}"'] & Z\otimes(H^{-1}(X)H^{-1}(Y)(M)) \arrow[d, "I_Z\otimes(J^{-1}_{X,Y})"] \\
H^{-1}(X\otimes Y)(Z\otimes M) \arrow[r, "\delta^{-1}_{X\otimes Y}"'] & Z\otimes(H^{-1}(X\otimes Y)(M))
\end{tikzcd}
$$

naturality

braid eqn

module pentagon

naturality

⇝ Using the tensor functor $H^+ : \mathcal{C} \to \mathcal{C}^*_{\mathcal{M}}$, $\mathcal{M}$ becomes a left $\mathcal{C}$-module category with

$$\mathcal{C} \times \mathcal{M} \to \mathcal{M}$$
$$X, M \mapsto X \underline{\otimes} M$$

and module constraint $= m$

⇝ Using the tensor functor $H^- : \mathcal{C}^{op} \to \mathcal{C}^*_{\mathcal{M}}$, $\mathcal{M}$ becomes a right $\mathcal{C}$-module category with

$$\mathcal{M} \times \mathcal{C} \to \mathcal{M}$$
$$(M, X) \mapsto \underbrace{X \underline{\otimes} M}_{=: \, M \bar{\otimes} X}$$

and module associativity constraint

$$m'_{X,Y,M} = \left[(Y \otimes X) \underline{\otimes} M \xrightarrow{c_{X,Y} \otimes I_M} (X \otimes Y) \underline{\otimes} M \xrightarrow{m_{X,Y,M}} X \underline{\otimes} (Y \underline{\otimes} M) \right]$$

⇝ We also have natural iso

$$b_{X,Y,M} : \underset{Y \underline{\otimes} (X \underline{\otimes} M)}{\overset{\shortparallel}{(X \underline{\otimes} M) \bar{\otimes} Y}} \longrightarrow \underset{X \underline{\otimes} (Y \underline{\otimes} M)}{\overset{\shortparallel}{X \underline{\otimes} (M \bar{\otimes} Y)}}$$

given by $b_{X,Y,M} = (b^-_Y)_{X,M}$

⇝ Together $(\mathcal{M}, m, m', b)$ becomes a $\mathcal{C}$-bimodule category.

Commutative algebras and Central functors

Let $\mathcal{C}$ be a braided tensor category

Defn: An <u>algebra</u> in $\mathcal{C}$ is a triple $(A, m : A\otimes A \to A, u : \mathbb{1} \to A)$ such that the following diagrams commute

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m\otimes I_A"'] & & A\otimes(A\otimes A) \arrow[d, "I_A\otimes m"] \\
A\otimes A \arrow[r, "m"'] & A & A\otimes A \arrow[l, "m"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "I_A\otimes u"] \arrow[dr, "\rho_A"'] & A\otimes A \arrow[d, "m"] & \mathbb{1}\otimes A \arrow[l, "u\otimes I_A"'] \arrow[dl, "\lambda_A"] \\
& A &
\end{tikzcd}$$

Defn: An algebra (A, m, u) in $\mathcal{C}$ is called <u>commutative</u> if

$$\begin{tikzcd}
A\otimes A \arrow[rr, "c_{A,A}"] \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}$$

commutes.

Examples ① Let (H,R) be a quasitriangular Hopf algebra. Here $R = R_i \otimes R^i$ is the R-matrix.

Recall, the category $\mathrm{Rep}(H)$ is braided with braiding

$$c_{M,N} : M \otimes N \longrightarrow N \otimes M$$
$$m \otimes n \longmapsto (R^i n \otimes R_i \cdot m)$$
$$= R^{21}(n \otimes m)$$
$$\forall\, M, N \in \mathrm{Rep}(H)$$

An algebra $(A, m, u) \in \mathrm{Rep}(H)$ is braided

$$\Updownarrow$$

$$\begin{tikzcd} A \otimes A \arrow[r, "c_{A,A}"] \arrow[dr, "m"'] & A \otimes A \arrow[d, "m"] \\ & A \end{tikzcd}$$

commutes i.e. $\forall\, a, b \in A$

$$ab = m(c_{A,A}(a \otimes b))$$
$$= m(R^{21}(b \otimes a))$$
$$\Rightarrow ab = (R^i \cdot b)(R_i \cdot a)$$

Example continued ....

For $H = \Bbbk \mathbb{Z}_2 = \langle u \rangle$

and $R = R_u = \frac{1}{2}(1\otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u)$

$\rightarrow$ $\mathrm{Rep}(H, R) = s\mathrm{Vec}$

$\rightarrow$ For any vector space V, the exterior power

$$\wedge V \in \mathrm{Rep}(\Bbbk \mathbb{Z}_2, R_u)$$

and further it is commutative.

(Notice, as an algebra in Vec, $\wedge V$ is not commutative)

→ Let A be a comm. alg. in a BTC $\mathcal{C}$

→ Let $\mathrm{Mod}_{\mathcal{C}}(A)$ = category of <u>right A-modules</u> in $\mathcal{C}$

[i.e. pairs $(M, p : M \otimes A \to M)$ s.t. following commul

$$\begin{tikzcd}
(M\otimes A)\otimes A \arrow[rr, "a"] \arrow[d, "p\otimes I_A"'] & & M\otimes (A\otimes A) \arrow[d, "I_M\otimes m"] \\
M\otimes A \arrow[dr, "p"'] & & M\otimes A \arrow[dl, "p"] \\
& M &
\end{tikzcd}$$

]

→ Every $M \in \mathrm{Mod}_{\mathcal{C}}(A)$ becomes a left A-module in two ways

(i) $q^{+} : \quad A\otimes M \xrightarrow{c_{A,M}} M\otimes A \xrightarrow{p} M$

(ii) $q^{-} : \quad A\otimes M \xrightarrow{c^{-1}_{M,A}} M\otimes A \xrightarrow{p} M$

Further, p is compatible with $q^{\pm}$ and we get that

$M_{+} = (M, p, q^{+})$ and $M_{-} = (M, p, q^{-})$

are A-bimodules in $\mathcal{C}$.

Thus, we get full and faithful functors (& injective on objects)

$$F_{\pm} : \mathrm{Mod}_A(\mathcal{C}) \longrightarrow \mathrm{Bimod}_A(\mathcal{C})$$
$$(M, p) \longmapsto M_{\pm}$$

→ When A is exact (i.e. $\mathrm{Mod}_{\mathcal{C}}(A)$ is exact module cat.) $\mathrm{Bimod}_A(\mathcal{C})$ is a multitensor category.

→ The tensor product of A-bimodules M', N' (written $M' \otimes_A N'$) is defined as a coker of a morphism from

$$M' \otimes A \otimes N' \longrightarrow M' \otimes N'$$

(see defn 7.8.21 EGNO)

→ Since $F_{\pm}$ are full, faithful, the image of $\mathrm{Mod}_{\mathcal{C}}(A)$ under $F_{\pm}$ is abelian.

$\Rightarrow$ For $M, N \in \mathrm{Mod}_{\mathcal{C}}(A)$, $F(M) \otimes_A F(N) \in F_{\pm}(\mathrm{Mod}_{\mathcal{C}}(A))$

So, we can define $M \otimes_A N = F(M) \otimes_A F(N)$ making $\mathrm{Mod}_{\mathcal{C}}(A)$ a multitensor category.

Remark: The two tensor structures on $\mathrm{Mod}_{\mathcal{C}}(A)$ obtained using F_+ and F_- are opposite (in terms of tensor product) of each other.

Remark: Let $\mathcal{M} = \mathrm{Mod}_A(\mathcal{C})$ viewed as a $\mathcal{C}$-module category.

Then $\mathcal{C}^*_{\mathcal{M}}$ is identified with $\mathrm{Bimod}_{\mathcal{C}}(A)^{op}$

The functors $H^+ : \mathcal{C} \to \mathcal{C}^*_{\mathcal{M}}$ and $H^- : \mathcal{C} \to \mathcal{C}^*_{\mathcal{M}}$ are the same as compositions

$$\mathcal{C} \xrightarrow{F_A} \mathrm{Mod}_{\mathcal{C}}(A) \xrightarrow{F_+} \mathrm{Bimod}_{\mathcal{C}}(A)$$

and

$$\mathcal{C} \xrightarrow{F_A} \mathrm{Mod}_{\mathcal{C}}(A) \xrightarrow{F_-} \mathrm{Bimod}_{\mathcal{C}}(A)$$

respectively.

$$F_A(X) := X \otimes A$$

Let $\mathcal{C}$ be a B.T.C. and let $\mathcal{A}$ be a finite tensor cat.

<u>Defn</u>: Let $F : \mathcal{C} \to \mathcal{A}$ be a tensor functor. A structure of of a $\boxed{\text{central functor}}$ on F is a braided tensor functor $F' : \mathcal{C} \longrightarrow \mathcal{Z}(\mathcal{A})$ together with an isomorphism $F \cong (\text{Forget}) \circ F'$
where $\text{Forget} : \mathcal{Z}(\mathcal{A}) \longrightarrow \mathcal{A}$ is the forgetful functor.

<u>Examples</u>: (i) The forgetful functor $\text{Forget} : \mathcal{Z}(\mathcal{A}) \longrightarrow \mathcal{A}$ is a central functor with $F' = \mathrm{Id}_{\mathcal{Z}(\mathcal{A})}$

(ii) Any braided tensor functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is central with F' as the composition

$$\mathcal{C}_1 \xrightarrow{F} \mathcal{C}_2 \xhookrightarrow{i} \mathcal{Z}(\mathcal{C}_2)$$

then $(\text{Forget}) \circ F' = \text{Forget} \circ i \circ F = F$

Proof 8.8.8: Let $F : \mathcal{C} \to \mathcal{A}$ be a central functor. Let $I : \mathcal{A} \to \mathcal{C}$ be the right adjoint of F. Then the object $A = I(\mathbb{1})$ has a canonical structure of a commutative algebra in $\mathcal{C}$.

PROOF: → Using F, $\mathcal{A}$ becomes a $\mathcal{C}$-module category

$$\begin{cases} \mathcal{C} \times \mathcal{A} \longrightarrow \mathcal{A} \\ (X, Y) \longmapsto F(X) \otimes Y \end{cases} \quad \forall X \in \mathcal{C},\ Y \in \mathcal{A}$$

→ Further $\mathrm{Hom}_{\mathcal{C}}(X, I(Y)) \overset{\text{adjoint}}{=} \mathrm{Hom}_{\mathcal{A}}(F(X), Y) = \mathrm{Hom}_{\mathcal{A}}(F(X) \otimes \mathbb{1}, Y)$

Recall defn. of internal homs

$$\mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M_1, M_2)) = \mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2)$$

$$\Rightarrow \quad I(Y) = \underline{\mathrm{Hom}}(\mathbb{1}, Y)$$

$\therefore A = I(\mathbb{1}) = \underline{\mathrm{Hom}}(\mathbb{1}, \mathbb{1})$

Hence, $I(\mathbb{1})$ is an algebra in $\mathcal{C}$.

$\rightarrow$ $$\mathrm{Hom}_{\mathcal{A}}(F(A \otimes A), \mathbb{1}) \underset{\text{adjoint}}{\cong} \mathrm{Hom}_{\mathcal{C}}(A \otimes A, \overset{I(\mathbb{1})}{\overset{\shortparallel}{A}}) \quad \text{— } \circledast$$

$\rightarrow$ Let $\tilde{m}$ be the preimage of multiplication m of A under the above iso.

$\rightarrow$ Let c = braiding of $\mathcal{C}$, $\tilde{c}$ = braiding of $\mathcal{Z}(\mathcal{C})$

$\rightarrow$ Under the iso. $\circledast$,

$m \circ c_{A,A}$ corresponds to $\tilde{m} \circ F(c_{A,A}) \in \mathrm{Hom}_{\mathcal{A}}(F(A \otimes A), \mathbb{1})$

$\therefore$ To show $m = m \circ c_{A,A}$ it suffices to show that $\tilde{m} = \tilde{m} \circ F(c_{A,A})$ in $\mathcal{A}$

$\Updownarrow$

$\tilde{m} = \tilde{m} \circ F'(c_{A,A})$ in $\mathcal{Z}(\mathcal{A})$

<u>Example</u>: ① Let $\mathcal{C} = \mathrm{Rep}(G)$

$F : \mathcal{C} \longrightarrow \mathrm{Vec}$ be the forgetful Braided tensor functor

the adjoint functor is $I : \mathrm{Vec} \longrightarrow \mathcal{C}$

$$V \longmapsto \mathrm{Fun}(G, V)$$

$\therefore$ $I(\mathbb{1}) = \mathrm{Fun}(G, \Bbbk)$ is a commutative alg in $\mathrm{Rep}(G)$

(also called regular algebra of $\mathrm{Rep}(G)$)

We saw that $\left.\begin{array}{l} F: \mathcal{C} \to A \text{ central} \\ \mathcal{C} \leftarrow A : I \text{ adjoint} \end{array}\right\} \Rightarrow A = I(\mathbb{1})$ is comm. alg in $\mathcal{C}$

Conversely suppose that A is comm. exact algebra in $\mathcal{C}$. Further $\mathrm{Hom}(\mathbb{1}, A) \cong \mathbb{k}$, then $\mathrm{Mod}_{\mathcal{C}}(A)$ is indecomposable. also $\mathrm{Bimod}_{\mathcal{C}}(A)$ is finite tensor

Thus, via the functor $F_-: \mathrm{Mod}_{\mathcal{C}}(A) \to \mathrm{Bimod}_{\mathcal{C}}(A)$, $\mathrm{Mod}_{\mathcal{C}}(A)$ becomes a finite tensor cat.

Prop 8.8.10: The functor $F_A: \mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}(A)$ is central.

$$X \mapsto X \otimes A$$

Proof: We show that $\forall\, X \in \mathcal{C}$ and $Y \in \mathrm{Mod}_{\mathcal{C}}(A)$, show that $F_A(X) \otimes_A Y \cong Y \otimes_A F_A(X)$ via braiding c

$\therefore F_A$ lifts to a BTF

$$F_A' : \mathcal{C} \to Z(\mathrm{Mod}_{\mathcal{C}}(A))$$