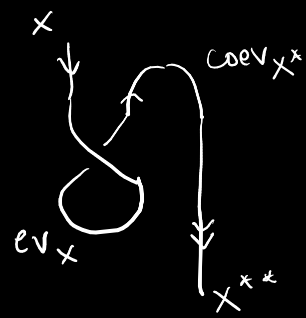
RECAP

$\mathcal{C}$ braided tensor category

→ Using the braiding c, we defined a natural transformation u

$u_X =$ 

$$\{u_X : X \longrightarrow X^{**}\}_{X \in \mathcal{C}}$$

DRINFELD MORPHISM

(Prop 8.10.6 : u is a natural isomorphism)

→ Defined a TWIST θ, which is a natural isomorphism $\{\theta_X : X \longrightarrow X\}_{X \in \mathcal{C}}$ satisfying

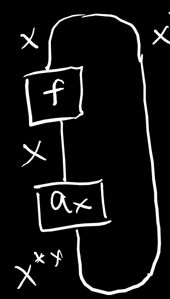
$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \quad \forall\ X, Y \in \mathcal{C}$$

→ A twist θ is called a RIBBON STRUCTURE if $(\theta_X)^* = \theta_{X^*}$ $\forall\ X \in \mathcal{C}$

SOME MORE (NEW) DEFINITIONS

Let $\mathcal{C}$ be a rigid monoidal category

→ A PIVOTAL STRUCTURE on $\mathcal{C}$ is a monoidal natural isomorphism $a : \mathrm{Id}_{\mathcal{C}} \longrightarrow (-)^{**}$

→ Using the pivotal sturucture, we can define the trace of any $f \in \mathrm{End}_{\mathcal{C}}(X)$ as $Tr(f) =$ 

$\in \mathrm{End}(\mathbb{1}) = \Bbbk$

→ Define $\dim(X) = Tr(\mathrm{Id}_X)$

→ A pivotal structure is called SPHERICAL if $\dim(X) = \dim(X^*)$ $\forall$ $X \in \mathcal{C}$

→ $\mathcal{C}$ is called PIVOTAL (SPHERICAL) if it is equipped with a pivotal (spherical) structure a.

→ The following properties of trace are easy to check on a spherical category.

- $\mathrm{Tr}(f \otimes g) = \mathrm{Tr}(f)\,\mathrm{Tr}(g)$
- $\mathrm{Tr}(f \oplus g) = \mathrm{Tr}(f) + \mathrm{Tr}(g)$
- $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f)$ for $f : X \longrightarrow Y$, $g : Y \longrightarrow X$
- $\mathrm{Tr}(\alpha f) = \alpha \,\mathrm{Tr}(f)$ for $\alpha \in \mathbb{k}$
- $\mathrm{Tr}(f) = \mathrm{Tr}(f^*)$

CONNECTION BETWEEN RIBBON TWISTS and PIVOTAL / SPHERICAL STRUCTURE

Now, $\mathcal{C}$ is a braided tensor category.

→ Let $\psi = \{ \psi_X = u_X \theta_X : X \to X^{**} \}_{X \in \mathcal{C}}$. Then,

$$\left\{ \begin{array}{l} \psi \text{ is a pivotal} \\ \text{structure on } \mathcal{C} \end{array} \right\} \Longleftrightarrow \left\{ \theta \text{ is a twist} \right\}$$

→ Suppose $\mathcal{C}$ is further a fusion category. Then,

[Prop 8.10.12] $$\left\{ \begin{array}{l} \psi \text{ is a spherical} \\ \text{structure on } \mathcal{C} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \theta \text{ is a ribbon} \\ \text{structure} \end{array} \right\}$$

§ 8.3 The S-matrix of a Premodular Category

For this talk, $\Bbbk$ = alg. closed field of char. 0

Defn (PRE-MODULAR CATEGORY)

A monoidal category $\mathcal{C}$ is called PRE-MODULAR if

(i) $\mathcal{C}$ is fusion,

(ii) $\mathcal{C}$ is braided, and

(iii) $\mathcal{C}$ has a ribbon structure θ.

By Prop 8.10.12, (iii) is equivalent to

(iii)′ $\mathcal{C}$ has a spherical structure ψ.

NOTATION:

- $\mathcal{O}(\mathcal{C})$ = isomorphism classes of simple objects in $\mathcal{C}$
- For $X, Y, Z \in \mathcal{O}(\mathcal{C})$
  $N_{X,Y}^{Z}$ = multiplicity of Z in $X \otimes Y$

Defn:

→ Let $\mathcal{C}$ be a pre-modular category. The **S-MATRIX** of $\mathcal{C}$ is defined by
$$S := (s_{X,Y})_{X,Y \in \mathcal{O}(\mathcal{C})} \quad \text{where} \quad s_{XY} = \mathrm{Tr}(c_{Y,X} c_{X,Y})$$

→ A pre-modular category is said to be **MODULAR** if the S-matrix is non-degenerate.

SOME OBSERVATIONS :

→ S-matrix is symmetric

$$s_{x,y} = \mathrm{Tr}(c_{y,x} c_{x,y}) = \mathrm{Tr}(c_{x,y} c_{y,x}) = s_{y,x}$$

→ $s_{x,\mathbb{1}} = \mathrm{Tr}(c_{\mathbb{1},x} \circ c_{x,\mathbb{1}})$ but $c_{x,\mathbb{1}} = c_{\mathbb{1},x} = id_x$

$$= \mathrm{Tr}(id_x$$

$$= \dim(X)$$

→ $s_{x^*,y^*} = \mathrm{Tr}(c_{y^*,x^*} \circ c_{x^*,y^*}) = \mathrm{Tr}(c_{y,x}^* \circ c_{x,y}^*)$

$$= \mathrm{Tr}((c_{x,y} \circ c_{y,x})^*)$$

$$= \mathrm{Tr}(c_{x,y} \circ c_{y,x})$$

$$= s_{x,y}$$

EXAMPLE 1

G finite abelian group

$q : G \longrightarrow \Bbbk^\times$ quadratic form on G

$b : G \times G \longrightarrow \Bbbk^\times$ associated symm. bilinear form

→ Consider the pointed braided fusion category $\mathcal{C}(G,q)$.
defined in Sean's talk.

[recall $q \longleftrightarrow (\omega, c)$ then $\mathcal{C}(G,q) = (\mathrm{Vec}_G^\omega, c)$
and $q = c \circ \Delta$]

→ These categories have a canonical ribbon structure

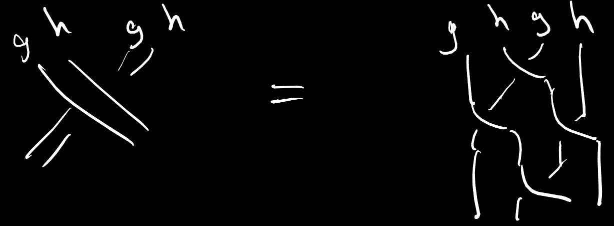
$$\theta_{\delta_g} = q(g)\, \mathrm{Id}_{\delta_h}$$

w.r.t. this ribbon structure,

$$\dim(\delta_g) = 1$$

(easy exercise)

Now,



$$\Rightarrow \quad c(gh, gh) = c(g,h)\, c(g,g)\, c(h,h)\, c(h,g)$$

$$\Rightarrow \quad q(gh) = q(g)\, q(h)\, c(g,h)\, c(h,g)$$

$$\therefore \quad c(g,h)\, c(h,g) = \frac{q(gh)}{q(g)\, q(h)} = b(g,h)$$

$$\therefore \quad S_{\delta_g, \delta_h} = Tr\left(c_{\delta_h, \delta_g} \circ c_{\delta_g, \delta_h} \right) = Tr\left(b(g,h)\, Id_{\delta_{gh}} \right)$$

$$= b(g,h)\, Tr\left(Id_{\delta_{gh}} \right)$$

$$= b(g,h)\, \dim(\delta_{gh})$$

$$= b(g,h)$$

The last example can be generalized.

PROP 8.13.8: We have

$$s_{XY} = \theta_X^{-1} \theta_Y^{-1} \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} \theta_Z \dim(Z) \quad \forall\, X, Y \in \mathcal{O}(\mathcal{C})$$

PROOF: Recall the twist θ satisfies

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$$

Apply Tr to both sides to get

$$\left(X \otimes Y = \bigoplus_{Z \in \mathcal{O}(\mathcal{C})} Z^{\oplus N_{X,Y}^{Z}} \right.$$

and $\theta_{X \otimes Y} = \bigoplus_{Z \in \mathcal{O}(\mathcal{C})} \theta_Z^{\oplus N_{XY}^{Z}}$

and Tr is additive)

$$LHS = Tr(\theta_{X \otimes Y})$$

$$= \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z}\, Tr(\theta_Z)$$

(But Z is simple $\Rightarrow \theta_Z = \theta_Z \, Id_Z$ where $\theta_Z \in \mathbb{k}$)

$$= \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} \theta_Z \, Tr(Id_Z) = \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} \theta_Z \dim(Z)$$

$$\text{RHS} = \operatorname{Tr}\left((\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}\right)$$

Notice that since X is simple, $\theta_X = \theta_X \, id_X$

$\theta_Y = \theta_Y \, id_Y$

$\theta_X, \theta_Y \in k$

$$\therefore \text{RHS} = \theta_X \theta_Y \operatorname{Tr}\left(c_{Y,X} \circ c_{X,Y}\right)$$

$$= \theta_X \theta_Y \, s_{X,Y}$$

Comparing with LHS, we get

$$s_{X,Y} = \frac{1}{\theta_X \theta_Y} \sum_{Z \in \mathcal{O}(\mathcal{C})} N_{XY}^{Z} \, \theta_Z \dim(Z)$$

Remark: The above proof holds even if $\mathcal{C}$ is not finite.

For $\mathcal{C} = \mathcal{C}(G, q)$

$$\theta_{\delta_g} = q(g) \quad ,$$

$$\dim(\delta_g) = 1 \quad \forall\ g \in G$$

$$N_{\delta_g \delta_h}^{\delta_{gh}} = 1 \quad \text{and} \quad N_{a,b}^{c} = 0 \ \text{ else}$$

By Prop 8.13.8,

$$\delta_{\delta_g, \delta_h} = \frac{1}{q(g)\,q(h)} N_{\delta_g \delta_h}^{\delta_{gh}}\, q(gh)$$

$$= \frac{q(gh)}{q(g)\,q(h)}$$

$$= b(g, h)$$

GROTHENDIECK RINGS

Let $\mathcal{C}$ be an abelian category.

Defn: An object $X \in \mathcal{C}$ is said to be of finite length if $\exists$ a filtration, that is, a sequence of morphisms

$$0 = X_0 \xhookrightarrow{f_1} X_1 \xhookrightarrow{f_2} X_2 \cdots\cdots \hookrightarrow X_{n-1} \xhookrightarrow{f_n} X_n = X$$

such that each f_i is a monomorphism and $X_i / X_{i-1} = \operatorname{coker} f_i$ is simple in $\mathcal{C}$.

→ Such a filtration is called a Jordan-Hölder series of X. A simple object Y has multiplicity m in X if we can m different values of i such that $X_i / X_{i-1} \cong Y$.

JORDAN-HÖLDER THEOREM :

Suppose that $X \in \mathcal{C}$ has finite length. Then any filtration can be extended to a Jordan-Hölder series, and any two Jordan-Hölder series of X contain any simple object with the same multiplicity, so in particular have the same length.

→ As a consequence, for any $X, Y \in \mathcal{C}$ with Y simple, we can define

$[X:Y]$ = multiplicity of Y in the Jordan-Hölder series of X

→ Length of X is the length of its Jordan-Hölder series.

GROTHENDIECK GROUP:

Denoted $Gr(\mathcal{C})$, this is the free abelian group generated by isomorphism classes $X_i, i \in I$ of simple objects in $\mathcal{C}$.

→ To every object $X \in \mathcal{C}$, we can canonically associate its class $[X] \in Gr(\mathcal{C})$ given by the formula

$$[X] = \sum_i [X : X_i] \, X_i$$

→ For any exact sequence $0 \to X \to Y \to Z \to 0$ we have that $[Y] = [X] + [Z]$ (How?)

Example: For the category $\mathcal{C} = Vec_G$,

$$G(\mathcal{C}) = \mathbb{Z}G$$

• Let $\mathcal{C}$ be a tensor category
(Note that rigidity and $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) = \mathbb{k}$ are not required)

The tensor product on $\mathcal{C}$ induces a natural multiplication on $\mathrm{Gr}(\mathcal{C})$ given by

$$(*) \qquad X_i X_j = [X_i \otimes X_j] = \sum_{k \in I} [X_i \otimes X_j : X_k] X_k \qquad i, j \in I$$

$(*)$ is called the fusion rule of $\mathcal{C}$

<u>Lemma 4.5.1</u>: The above multiplication on $\mathrm{Gr}(\mathcal{C})$ is associative. The resulting ring is denoted $K_0(\mathcal{C})$

<u>Defn</u>: $K_0(\mathcal{C})$ is called the Grothendieck ring of $\mathcal{C}$.

FROBENIUS-PERRON THEOREM (trimmed version of THM 3.2.1, EGNO)

Let B be a square matrix with non-negative real entries. Then:

→ B has a largest non-negative eigenvalue $\lambda(B)$.

→ $\exists$ an eigenvector v of B with eigenvalue $\lambda(B)$ with all entries non-negative.

FROBENIUS-PERRON DIMENSION

Given a finite tensor category $\mathcal{C}$ and a simple $X_i \in \mathcal{C}$ with consider the $n \times n$ matrix N_i (where $n = |I|$)

$$(N_i)_j^k = [X_i \otimes X_j : X_k] \in \mathbb{Z}_+ \qquad j, k \in I$$

Then $\quad \mathrm{FPdim}(X_i) := \lambda(N_i)$

→ Extending additively, we get a ring homomorphism

$$\mathrm{FPdim} : \mathrm{Gr}(\mathcal{C}) \longrightarrow \mathbb{C}$$

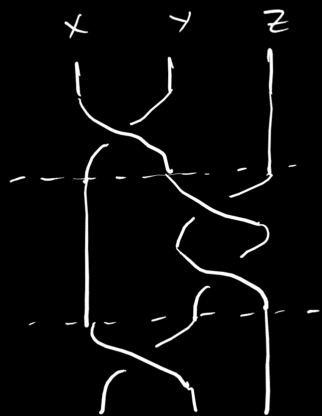
Some more properties of $s_{X,Y}$

Prop 8.13.10: $s_{X,Y}$ satisfy the relation

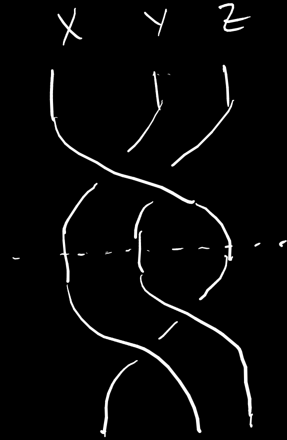
$$s_{XY}\, s_{XZ} = \dim(X) \sum_{W \in \mathcal{O}(\mathcal{C})} N_{YZ}^{W}\, s_{XW} \qquad X, Y, Z \in \mathcal{O}(\mathcal{C})$$

PROOF: Consider

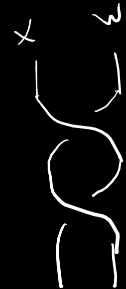
$\star$



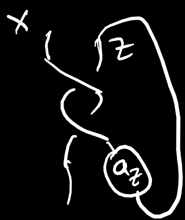
$=$



$= \bigoplus_{W \in \mathcal{O}(\mathcal{C})} N_{YZ}^{W}$

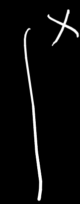


Notice that

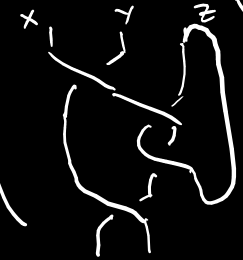


$\in \mathrm{End}(X) \cong \Bbbk \Rightarrow$

$= \alpha$



Taking trace $\Rightarrow$ $s_{X,Z} = \alpha \dim(X) \Rightarrow \alpha = \dfrac{s_{X,Z}}{\dim(X)}$

Trace of LHS of ⊛ $= \mathrm{Tr}\Big($  $\Big) = \dfrac{S_{XZ}}{\dim(X)} \mathrm{Tr}\Big($  $\Big)$

$$= \frac{S_{XZ}\, S_{XY}}{\dim(X)}$$

Trace of RHS $= \sum_{W \in \mathcal{O}(\mathcal{C})} N_{YZ}^{W} S_{X,W}$

Compairing these we get the desired relation.

<u>Remark</u>: • This proof remains valid when $\mathcal{C}$ is not finite.

• When $\dim(X) = 0$, then putting $Y = Z$ yields

$$S_{XY}^{2} = 0$$

$$\Rightarrow S_{X,Y} = 0 \quad \forall \quad Y \in \mathcal{O}(\mathcal{C})$$

PROP 8.13.11:

(i) For any fixed $X \in \mathcal{O}(\mathcal{C})$ the map

$$h_X : Y \longmapsto \frac{s_{XY}}{\dim(X)}, \quad Y \in \mathcal{O}(\mathcal{C})$$

defines a homomorphism $K_0(\mathcal{C}) \longrightarrow \Bbbk$

(ii) The numbers $\dfrac{s_{XY}}{\dim(X)}$ are algebraic integers.

PROOF: (i)

$$h_X(YZ) = h_X\left(\sum N_{YZ}^{W} W\right) = \sum N_{YZ}^{W} h_X(W)$$

$$= \sum N_{YZ}^{W} \frac{s_{XW}}{\dim(X)}$$

$$h_X(Y)\, h_X(Z) = \frac{s_{XY}}{\dim(X)} \frac{s_{XZ}}{\dim(X)}$$

$$= \frac{\dim(X) \sum N_{YZ}^{W} s_{XW}}{\dim(X)^2} \quad \text{(by Prop 8.13.10)}$$

$$= h_X(YZ)$$

(ii) Let $\vec{s}_X$ be the $1 \times n$ matrix with $(\vec{s}_X)_W = s_{X,W}$ for $W \in \mathcal{O}(\mathcal{C})$

Let $(\vec{N}_Y)$ be the $n \times n$ matrix with $(\vec{N}_Y)_Z^W = N_{Y,Z}^W$

then

$$\boxed{s_{XY}\, s_{XZ} = \dim(X) \sum_{W \in \mathcal{O}(\mathcal{C})} N_{YZ}^W\, s_{XW}} \implies \frac{s_{XY}}{\dim(X)} \vec{s}_X = \vec{N}_Y \vec{s}_X$$

$\implies h_X(Y)$ is an eigenvalue

$\therefore$ root of characteristic poly.

$\implies$ it's an algebraic integer

Remarks:

$\rightarrow$ If $\dim(X) \neq 0$ then Prop 8.10.11 (i) holds for X with same proof even if $\mathcal{C}$ is not finite.

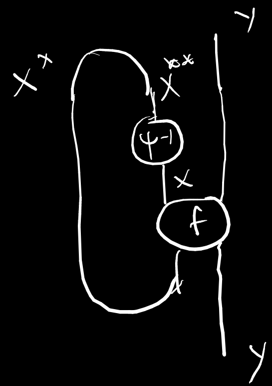
$\rightarrow$ When $\dim(X) \neq 0$, part (ii) holds when $\mathcal{C}$ is finite

TRACE OF FACTORS OF MOPHISMS BETWEEN TENSOR PRODUCTS

Consider $f : X \otimes Y \to X \otimes Y$

Define trace on X $\quad id_X \otimes Tr(f) :=$

Define trace on Y $\quad Tr \otimes id_Y(f) :=$



T – Matrix :

It is an $n \times n$ matrix $(n = |O(\mathcal{C})|)$
where $T = \operatorname{diag}(\theta_i)$

→ The pair of matrices $\{S, T\}$ for any modular category is called its MODULAR DATA.

→ A lot of information about the modular category can be obtained from the modular data.