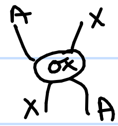
([TV, Chapter 5] , [EGNO, Sections 8.9, 8.20])

Drinfeld-Joyal-Street center of a monoidal category

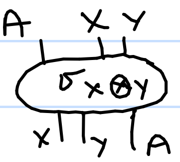
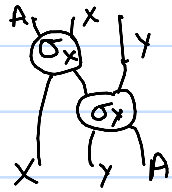
Our setting is that of a strict monoidal cat. $\mathcal{C}$.

(i) Half-braidings

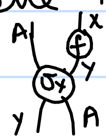
it is a pair (A, σ) where $A \in Ob(\mathcal{C})$ and $\sigma = \{\sigma_X : A \otimes X \xrightarrow{\sim} X \otimes A\}_{X \in Ob(\mathcal{C})}$



is a natural isomorphism between the functors $A \otimes -$ and $- \otimes A$ which is $\otimes$-multiplicative, i.e. $\forall\ X, Y \in Ob(\mathcal{C})$,

 $= \sigma_{X \otimes Y} = (id_X \otimes \sigma_Y)(\sigma_X \otimes id_Y) =$  $\circledast$

The naturality of σ means that

 $= \sigma_Y(id_A \otimes f) = (f \otimes id_A)\sigma_X =$ 

for all morphisms $f : X \to Y$ in $\mathcal{C}$

(Taking $X = Y = \mathbb{1}$ in $\circledast$, we get

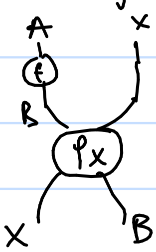
$$\sigma_{\mathbb{1}} = (id_X \otimes \sigma_{\mathbb{1}})(\sigma_{\mathbb{1}} \otimes id_{\mathbb{1}}) = \sigma_{\mathbb{1}}^2$$

But $\sigma_{\mathbb{1}} : A = A \otimes \mathbb{1} \to \mathbb{1} \otimes A = A$ is an iso

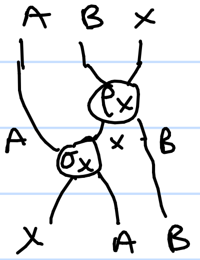
$\Rightarrow \sigma_{\mathbb{1}} = Id_A$)

(ii) The center: $Z(\mathcal{C})$, it is a category with

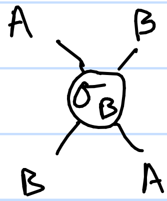
- Objects : Half-braiding (A, σ)
- Morphisms : $(A, \sigma) \longrightarrow (B, \rho)$ in $Z(\mathcal{C})$ are those $f : A \to B$ in $\mathcal{C}$ which satisfy

 $=$ 

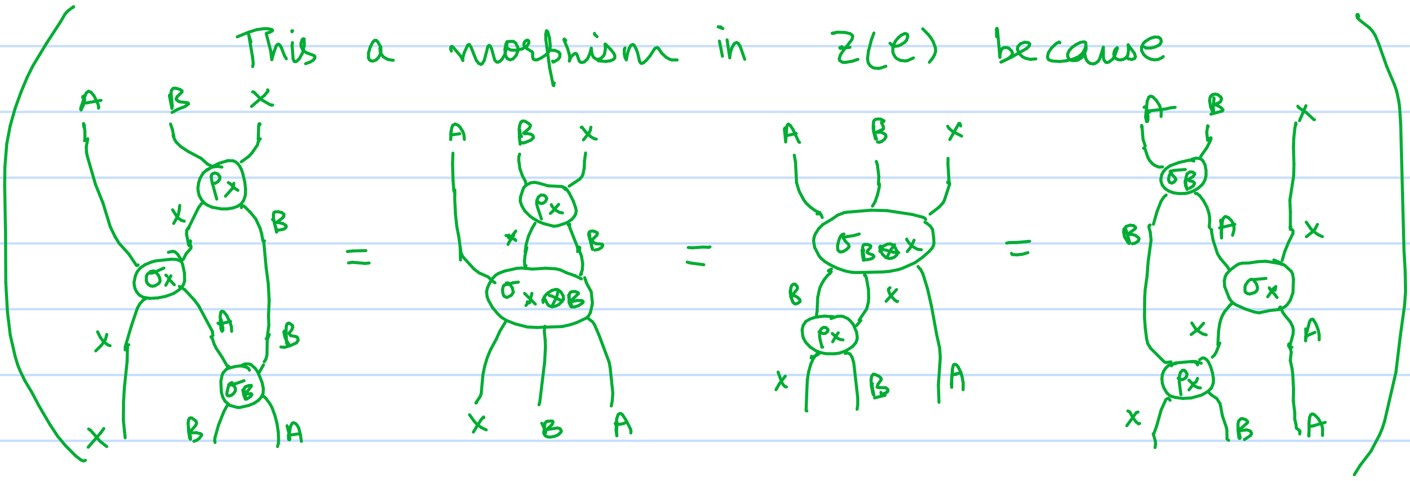
- Unit object : $\mathbb{1}_{Z(\mathcal{C})} = \left(\mathbb{1}_{\mathcal{C}}, \{\mathrm{Id}_X\}_{X \in \mathrm{Ob}(\mathcal{C})}\right)$

- Monoidal product : $(A,\sigma) \otimes (B,\rho) = \Big(A \otimes B,$  $\Big)$

- Braiding τ : $\tau_{(A,\sigma),(B,\rho)} : (A,\sigma) \otimes (B,\rho) \to (B,\rho) \otimes (A,\sigma)$



This a morphism in $Z(\mathcal{C})$ because



We have the forgetful functor

$$F : Z(\mathcal{C}) \longrightarrow \mathcal{C}$$
$$\left\{ \begin{array}{ccc} (A,\sigma) & \longmapsto & A \\ f & \longmapsto & f \end{array} \right\}$$

F is strict monoidal & reflects isomorphisms
(i.e. $F(f)$ is an iso $\Rightarrow$ f is an iso.)

JUST SAY

Remark: The above defined half-braidings are sometimes called left half braidings & $Z(\mathcal{C})$ the left center. We also have right with obj's (A,σ) where $\{\sigma : X \otimes A \to A \otimes X\}$ is nat. iso
(right center) $\cong$ mirror of left center
$(A,\sigma) \longmapsto (A,\sigma^{-1})$

Example(s):

① Take G a monoid. Consider the mon. category $\underline{G}$

Objects : Only one $*$

Morphisms : $* \xrightarrow{g} *$ $\quad g \in G$

Monoid str. : $* \otimes * := *$

Then $Z(\underline{G})$ has

Objects : $(*, \{\sigma : * \otimes X \to X \otimes *\}_{X \in \underline{G}})$

$\sigma : * \to *$, thus $\sigma = g \in G$

σ is $\otimes$ multiplicative

$$\Rightarrow \begin{tikzcd} * = * \otimes * \arrow[r, "Id_* \otimes h = h"] \arrow[d, "\sigma = g"'] & * \otimes * = * \arrow[d, "\sigma = g"] \\ * \otimes * \arrow[r, "h \otimes Id_* = h"'] & * \otimes * = * \end{tikzcd}$$

commutes $\forall h \in G$

$\Rightarrow gh = hg \quad \Rightarrow g \in Z(G)$

We further require g to be invertible in G

$\therefore Z(\underline{G})$ has objects $\{(*, g)\}_{\substack{g \in Z(G) \\ g \text{ invertible}}}$

Morphisms from $(*, g) \longrightarrow (*, h)$

are those $k \in G$ s.t.

$$\begin{tikzcd} * \otimes * \arrow[r, "k"] \arrow[d, "g"'] & * \otimes * \arrow[d, "h"] \\ * \otimes * \arrow[r, "k"'] & * \otimes * \end{tikzcd}$$

i.e. $hk = kg$

$\Rightarrow h = kgk^{-1}$

② For G a monoid, consider the category $\underline{G}$

with obj : $\{g\}_{g \in G}$

mor : $mor(g, h) = \Bbbk \delta_{g,h}$

Then $Z(\underline{G})$ has

objects : (a, σ) $\quad a \in Z(G)$

$\sigma : G \to \Bbbk^*$ monoid homomorphism

Thus, $Z(\underline{G})$ is (slightly) bigger than the category $\underline{Z(G)}$.

③ For H Hopf algebra,
$\mathcal{Z}(\mathrm{Rep}(H)) \simeq \mathrm{Rep}(D(H))$ as braided mon. cats

where $D(H)$ = Drinfeld double of the Hopf algebra H.

④ For a Hopf algebra H,
$\mathcal{Z}(\mathrm{Corep}(H)) \simeq {}^{H}_{H}\mathcal{YD}$ (category of left-left Yetter-Drinfeld modules)

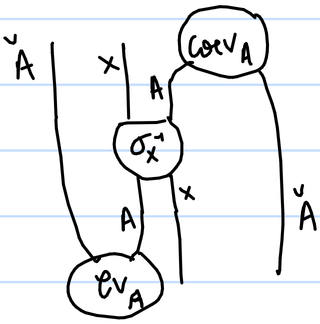
Centers of rigid and pivotal categories:

(i) Rigid categories:

$\mathcal{C}$ - rigid category

have a left duality $\{({}^\vee X, ev_X)\}_{X \in Ob(\mathcal{C})}$ and a right duality $\{(X^\vee, \widetilde{ev}_X)\}_{X \in Ob(\mathcal{C})}$. Denote by $coev_X$ & $\widetilde{coev}_X$ the inverses

Consider a half braiding (A, σ) of $\mathcal{C}$

Set $({}^+\sigma)_X =$  & $({}^+\sigma)_X^{-1} =$ 

CLAIM: ${}^+\sigma = \{ ({}^+\sigma)_X : {}^\vee A \otimes X \to X \otimes {}^\vee A \}_{X \in Ob(\mathcal{C})}$ is a half braiding with inverse given by ${}^+\sigma^{-1}$ defined above

Similarly can define σ^+

$\rightsquigarrow$ $({}^\vee A, {}^+\sigma)$ & $(A^\vee, \sigma^+)$ are half braidings of $\mathcal{C}$. Also $ev_A, \widetilde{ev} \cdots$ are morphism in $\mathcal{Z}(\mathcal{C})$

UPSHOT: $\mathcal{C}$ rigid $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ is also rigid.

(ii) Pivotal categories

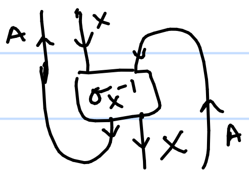
Recall pivotal means $\forall\ A \in Ob(\mathcal{C})\ \exists\ A^* \in Ob(\mathcal{C})$ which plays role of both left & right dual.

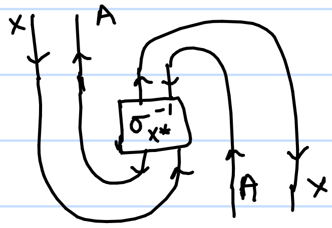
For a half braiding (A, σ) in $\mathcal{C}$
Considering $(A^*, ev_A, coev_A)$ as left dual we get
$(A^*, {}^{+}\sigma)$ a half braiding in $\mathcal{Z}(\mathcal{C})$
Similarly get half braiding (A^*, σ^{+})

Lemma: ${}^{+}\sigma = \sigma^{+}$

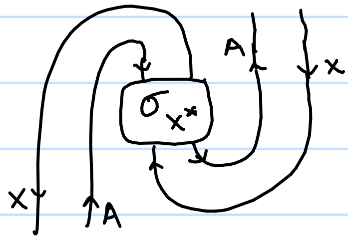
Proof:

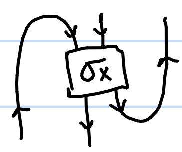
A^* is left dual
arrows on A go to left

$({}^{+}\sigma)_X =$ 

$({}^{+}\sigma)_X^{-1} =$ 

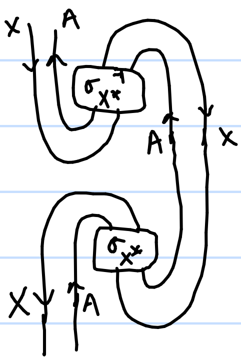
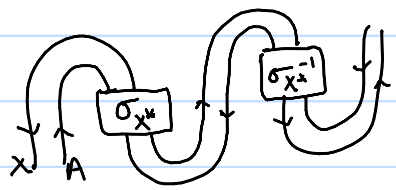
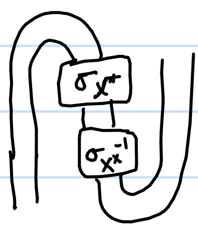
A^* is right dual. So arrows to right

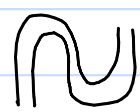
$(\sigma^{+})_X =$ 

$(\sigma^{+})_X^{-1} =$ 

These are mirror images of the top row

Then

$({}^{+}\sigma)_X^{-1} \circ (\sigma^{+})_X =$  $=$  $=$ 

$=$ 

$=$ 

Since the inverse is unique,

$$(\sigma^{+})_X = ({}^{+}\sigma)_X$$

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UPSHOT: The family

$$\{(A,\sigma)^* = (A^*, \sigma^{+}),\ ev_{(A,\sigma)} = ev_A,\ \widetilde{ev}_{(A,\sigma)} = \widetilde{ev}_A\}_{(A,\sigma) \in Ob(\mathcal{C})}$$

is a pivotal duality

$\mathcal{C}$ pivotal $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ is pivotal

Further the forgetful functor

$$F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$$

is strictly pivotal (i.e. $F^{\ell} = F^{r}$)

($F^{\ell}: F({}^{\vee}X) \to {}^{\vee}F(X)$, $F^{r}: F(X^{\vee}) \to F(X)^{\vee}$)

Consequently

$\mathcal{C}$ is spherical $\Rightarrow$ $\mathcal{Z}(\mathcal{C})$ is spherical

(left & right trace coincide)

(iii) (Pre)-Fusion categories:

Recall that a ribbon cat. is a braided pivotal category $\mathcal{C}$ whose left twist θ^{ℓ} & right twist θ^{r} are equal. We had seen ribbon $\Rightarrow$ spherical. Following lemma tells us that the center construction provides a way to go in other direction.

Lemma: The center of a spherical pre-Fusion $\mathbb{k}$-category is ribbon.

Example:

Fundamental theorems of Müger

Thm 1: Let $\mathbb{k}$ be algebraically closed & let $\mathcal{C}$ be an additive pivotal fusion $\mathbb{k}$-cat with $\dim(\mathcal{C}) \neq 0$. Then $\mathcal{Z}(\mathcal{C})$ is an additive pivotal braided fusion category.

Thm 2: If under the assumptions of the above theorem, the category $\mathcal{C}$ is spherical, then $\mathcal{Z}(\mathcal{C})$ is an additive anomaly free modular $\mathbb{k}$-category with $\Delta_{+} = \Delta_{-} = \dim(\mathcal{C})$.

Eg $\quad \mathrm{Rep}(G) \xrightarrow{\mathcal{Z}} \mathrm{Rep}(D(G))$

G- finite

	Starting category $\mathcal{C}$	$\mathcal{Z}(\mathcal{C})$
from defn	monoidal	braided monoidal
easy to see	$\Bbbk$-linear	$\Bbbk$-linear
will discuss	rigid pivotal spherical	rigid pivotal spherical
easy lemma just say	spherical pre-fusion	ribbon $(\theta^{l} = \theta^{r})$
[Müger] ↑ big theorems	additive pivotal fusion $\Bbbk$-cat $\Bbbk$-alg closed, $\dim(\mathcal{C}) \neq 0$	additive pivotal braided fusion $\Bbbk$-cat $\dim \mathcal{Z}(\mathcal{C}) = \dim(\mathcal{C})^2$
↓ [Müger]	$\Bbbk$-alg. closed, $\dim(\mathcal{C}) \neq$ $\mathcal{C}$ spherical additive fusion $\Bbbk$-cat	additive anomaly free modular $\Bbbk$-category $\Delta_+ = \Delta_- = \dim(\mathcal{C})$