TALK 1 :

→ $\boxed{\text{MONOIDAL CATEGORY}}$: It is a category $\mathcal{C}$ with

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- an object $\mathbb{1} \in \mathcal{C}$ called unit object
- natural isomorphisms

$$\alpha = \{\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C\}_{A,B,C \in \mathcal{C}}$$
$$l = \{l_A : \mathbb{1} \otimes A \longrightarrow A\}_{A \in \mathcal{C}}$$
$$r = \{r_A : A \otimes \mathbb{1} \longrightarrow A\}_{A \in \mathcal{C}}$$

such that α, l, r satisfy certain coherence conditions called the <u>pentagon</u> & <u>triangle</u> axioms.

→ A monoidal category is $\boxed{\text{BRAIDED}}$ if $\exists$ a natural isom. $\{c = c_{A,B} : A \otimes B \to B \otimes A\}_{A,B \in \mathcal{C}}$ satisfying some coherence conditions

→ A braided monoidal category is called $\boxed{\text{SYMMETRIC}}$ if c satisfies $c_{A,B} \circ c_{B,A} = Id_{B \otimes A}$.

→ A $\boxed{\text{MONOIDAL FUNCTOR}}$ between monoidal categories $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \bar{\otimes}, \mathbb{1}_{\mathcal{D}})$ is a functor $F : \mathcal{C} \to \mathcal{D}$ equipped with natural trans.

$$F_{A,B} : \{F(A) \bar{\otimes} F(B) \longrightarrow F(A \otimes B)\}_{A,B \in \mathcal{C}}$$

and a morphism

$$F_0 : \mathbb{1}_{\mathcal{D}} \longrightarrow F(\mathbb{1}_{\mathcal{C}})$$

satisfying certain coherence conditions.

→ A $\boxed{\text{BRAIDED MONOIDAL FUNCTOR}}$ is a monoidal functor between braided categories $(\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}}, c)$ and $(\mathcal{D}, \bar{\otimes}, \mathbb{1}_{\mathcal{D}}, \bar{c})$ satisfying

$$F(c_{A,B}) \circ F_{A,B} = F_{B,A} \circ \bar{c}_{F(A),F(B)}$$

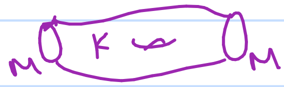
→ A $\boxed{\text{SYMMETRIC MONOIDAL FUNCTOR}}$ is a braided functor b/w symmetric monoidal categories.

→ For more precise definitions, go to "math.ucr.edu/home/baez/qg-winter2001/definitions.pdf"

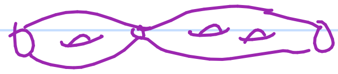
→ KEY EXAMPLES OF SYMMETRIC MONOIDAL CATEGORIES

• Bord_d^{d-1} =

- objects: $d-1$ mflds (eg. $M = S^1 = \bigcirc$ when $d=2$)
- morphisms: a morphism b/w objects M and N is a d-diml mfld K with its bdry ∂K diffeomorphic to $\overline{M} \amalg N$.



- composition: gluing



- monoidal structure:
  - $\otimes : M, N \longrightarrow M \amalg N$ disjoint union
  - monoidal unit is ϕ_{d-1} empty $(d-1)$ diml manifold

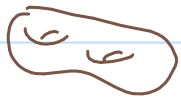
• Vect =

- Objects: $\Bbbk$-vector spaces
- morphisms: $\Bbbk$-linear maps
- composition: comp.
- monoidal structure:
  - $\otimes$: tensor product of vect. spaces
  - unit : $\Bbbk$

→ TOPOLOGICAL FIELD THEORY (TFT)

A TFT_{d-1}^{d} is a symmetric monoidal functor $Z : \mathrm{Bord}_{d-1}^{d} \longrightarrow \mathrm{Vect}$

NOTE: Any closed d-manifold K can be seen as a morphism from ϕ_{d-1} to ϕ_{d-1}. Since Z maps ϕ_{d-1} to $\mathbb{C}$, under Z

Z : K =  $\longmapsto Z(K) : \mathbb{C} \to \mathbb{C}$

$\therefore Z(K) \in \mathbb{C}$

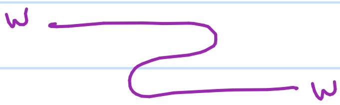
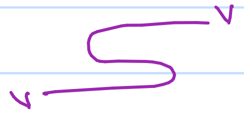
Hence TQFTs yield invariants of manifolds.

→ $\mathcal{C}$ monoidal category. An object $W \in \mathcal{C}$ is **RIGHT DUAL** to V if $\exists$ morphisms

$\mathrm{coev} : \mathbb{1}_{\mathcal{C}} \longrightarrow V \otimes W$ and $\mathrm{ev} : W \otimes V \longrightarrow \mathbb{1}_{\mathcal{C}}$

(denoted as ) (denoted )

Satisfying

 $= \mathrm{Id}_W$ &  $= \mathrm{Id}_V$

RESULT 1:

$$\{\mathrm{TFT}_0^1\} \longleftrightarrow \left\{\begin{array}{c}\text{finite dimensional} \\ \text{vector spaces}\end{array}\right\}$$

$$\left(\begin{array}{ccc} Z & \longmapsto & Z(\bullet^+) \\ Z : \mathrm{Bord}_0^1 \to \mathrm{Vect} & \longleftarrow\!\shortmid & V \\ \bullet^+ \mapsto V & & \\ \bullet^- \mapsto V^* & & \end{array}\right)$$

RESULT 2:

$$\{\mathrm{TFT}_1^2\} \longleftrightarrow \{\text{commutative Frobenius algebras}\}$$

$$Z \longmapsto Z(S^1)$$

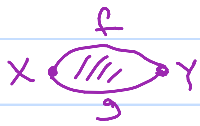
→ A **FROBENIUS ALGEBRA** in a $\mathbb{k}$-vector space equipped with maps $m : A \otimes A \to A$, $u : \mathbb{k} \to A$, $\Delta : A \to A \otimes A$, $\varepsilon : A \to \mathbb{k}$ such that

- (A, m, u) is a $\mathbb{k}$-algebra
- (A, Δ, ε) is a $\mathbb{k}$-coalgebra
- The following diagram commutes

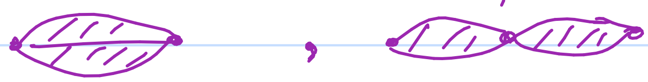
$$\begin{tikzcd}
A \otimes A \arrow[rr, "I \otimes \Delta"] \arrow[dr, "m"] \arrow[dd, "\Delta \otimes I"'] & & A \otimes A \otimes A \arrow[dd, "m \otimes I"] \\
& A \arrow[dr, "\Delta"] & \\
A \otimes A \otimes A \arrow[rr, "I \otimes m"'] & & A \otimes A
\end{tikzcd}$$

(Frobenius law)

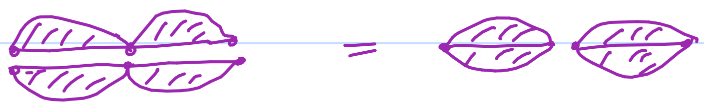
TALK-2:

→ A **BICATEGORY** consists of
- objects (denoted as $\overset{X}{\bullet}$)
- 1-morphism (denoted $X \xrightarrow{f} Y$)
- 2-morphisms ($\begin{tikzcd} X \arrow[r, bend left, "f"{name=F}] \arrow[r, bend right, "g"'{name=G}] \arrow[from=F, to=G, Rightarrow, "\alpha"] & Y \end{tikzcd}$ or 

- have usual composition of morphisms
- 2-morphisms can be composed in 2-ways



and these 2-ways satisfy the condition



+ few more coherence conditions

Example: The bicategory of CATEGORIES $= \begin{cases} \text{Obj}: & \text{Categories} \\ \text{1-Mor}: & \text{Functors} \\ \text{2-Mor}: & \text{Natural transformations} \end{cases}$

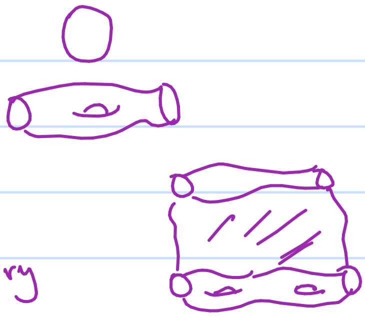
→ A **MONOIDAL BICATEGORY** is a bicategory in which we can "multiply objects".

→ A **SYMMETRIC MONOIDAL BICATEGORY** is a monoidal bicategory with symmetric monoidal structure

→ Similarly definite **(symmetric) monoidal bifunctors**.

Key examples

① $\mathrm{Bord}_1^3 = \begin{cases} \text{Objects}: & \text{1-mflds} \\ \text{1-morph}: & \text{2-mfld} \\ \text{2-morphism}: & \text{3-mfld}\,/\,\substack{\text{diffeo} \\ \text{rel bdry}} \end{cases}$



→ A category is called LINEAR if $\mathrm{Hom}(X,Y)$ $\forall\, X, Y \in \mathcal{C}$ is a $\mathbb{k}$-vector space & composition is bilinear.

② $\mathrm{LinCat} = \begin{cases} \text{Obj}: & \text{Linear categories} \\ \text{1-mor}: & \text{Linear functors} \\ \text{2-mor}: & \text{Natural transformations} \end{cases}$

→ An EXTENDED TFT (TFT_1^3) is a symm. monoidal bifunctor
$$Z : \mathrm{Bord}_1^3 \longrightarrow \mathrm{LinCat}$$

• We choose Lincat instead of any other symm. monoidal bicategory because this appropriately generalizes the defn of TFT_2^3. Upon restriction to a single object $\emptyset_1$, we get a TFT_2^3.

→ An object $X \in \mathcal{C}$ is called SIMPLE if it has no proper subobjects.

→ A category $\mathcal{C}$ is called SEMISIMPLE if every object can be written as a direct sum of finitely many simple objects.

→ As a step towards classifying Extended TFT_1^3, we study what is $Z(S^1)$ for a TFT_1^3 Z.

$(S^1 = \bigcirc)$

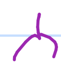
→ Realize that the linear category $\mathcal{C} = Z(S^1)$ is a monoidal linear category. So, we try to construct examples of these.

→ For any $\mathbb{k}$-alg A, the category A-mod = category of left A-modules is a linear category.

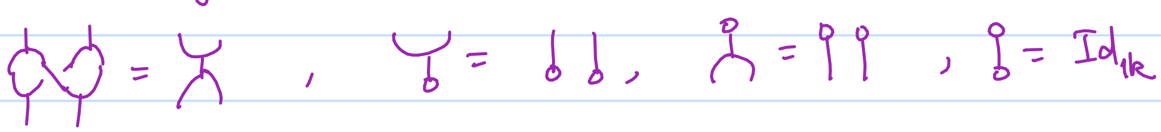
RESULT 1: For a $\mathbb{k}$-algebra A, if A is a bialgebra then the category A-mod is monoidal

→ A **BIALGEBRA** is a $\mathbb{k}$-vector space equipped with maps $m: A \otimes A \to A$, $u: \mathbb{k} \to A$, $\Delta: A \to A \otimes A$, $\varepsilon: A \to \mathbb{k}$ such that:

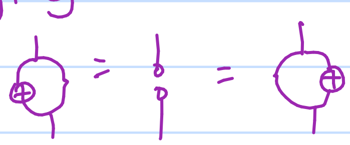
- (A, m, u) is a $\mathbb{k}$-algebra
- (A, Δ, ε) is a $\mathbb{k}$-coalgebra
- $m, u, \Delta, \varepsilon$ satisfy bialgebra axioms

denote $m =$ , $u =$ , $\Delta =$ , $\varepsilon =$ 

then the bialg. axioms are



→ A **HOPF ALGEBRA** is a bialgebra equipped with an antipode, i.e. a map $S: H \longrightarrow H$ ($\oplus$) satisfying:



RESULT 2: If H is a Hopf algebra, the category Mod-H is monoidal and has duals.