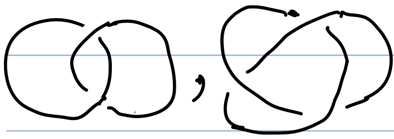
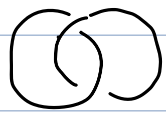
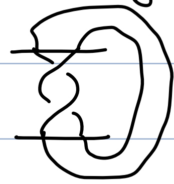
APM seminar talk (3 April)

 distinguishing knots & links is a difficult problem

<u>Alexander</u>: every knot or link can be obtained as closure of a braid

Eg.  = 

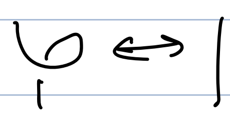
<u>Markov</u>: Two braids give same knot or link iff they are connected by Markov moves

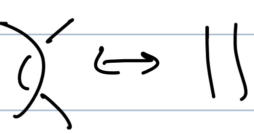
→ Thus to distinguish knots it suffices to
- construct a representation of braid group
- take a trace that is invariant under Markov moves

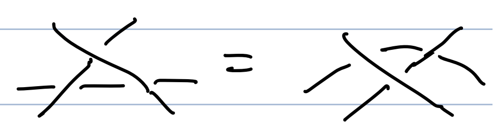
<u>Jones</u> (1982):
- Constructed representation using TL-alg
- took Markov trace

⇝ got the Jones polynomial J_K

Second perspective: Two knots are same ⟺ related by Reidemeister moves

R1 

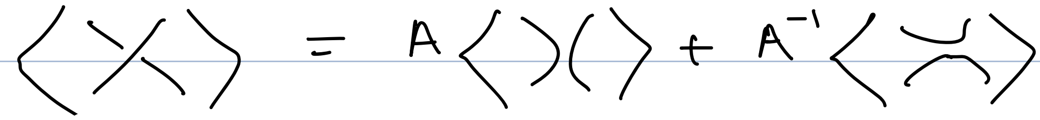
R2 

R3 

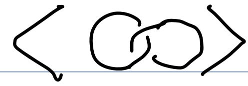
Kauffman (1987)

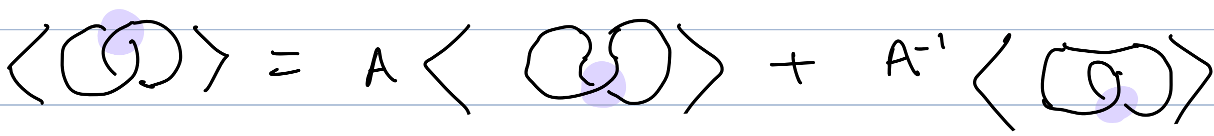
- For a link L he defined a polynomial invariant called the Kauffman bracket $\langle L \rangle$
- $\langle L \rangle$ defined using two relations

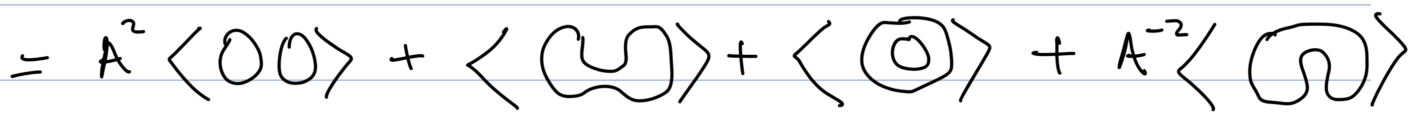
i) $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle$

ii) 

- The relations (i) and (ii) are called skein relations.

Example: What is 





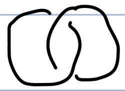
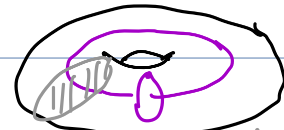
$= (A^2+1)(-A^2+A^{-2})^2 + (1+A^{-2})(-A^2-A^{-2})$

$= (A^2+A^{-2})(A^4+A^{-4})$

Note: • $\langle L \rangle$ obtained this way is not invariant under R1. Have to add a correction factor

• After a change of variables, we get the Jones polynomial.

One can go beyond the Jones polynomial in many ways

$L=$  $\longrightarrow$ put L in a 3 manifold M 

$\langle L \rangle$

$\longrightarrow$ replaced threads in a knot by ribbons (i.e. look at framed knots)

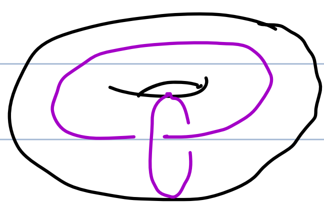
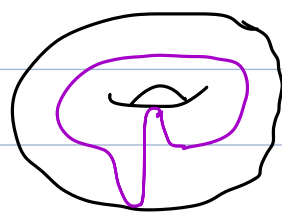
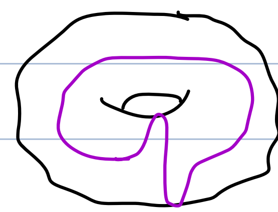
$\longrightarrow$ change the algebraic input

$U_q(sl_2) \rightsquigarrow \mathcal{C} = Rep(U_q(sl_2)) \rightsquigarrow \mathcal{C} =$ ribbon tensor category

$V =$ std 2D rep $\qquad V \in obj\ \mathcal{C}$

For a 3-mfld M (due to Przytcki, Turaev)

$$Sk_{SL_2}(M, A) := \frac{\mathbb{Q}(A) \text{ vector space spanned by all framed links in } M}{\text{isotopy equivalence} + \text{Kauffman relations}}$$

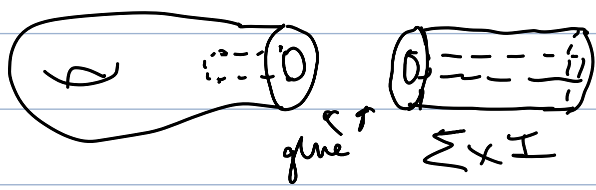
Eg:  $= A$  $+ A^{-1}$ 

Observations:

(i) $Sk_{SL_2}(\Sigma \times I, A)$ is an algebra.
(by stacking)

$\Sigma =$ 

(ii) let M be a 3-mfld by bdry Σ.
Then $Sk_{SL_2}(M, A)$ is a $Sk_{SL_2}(\Sigma \times I, A)$-module.

$M =$ solid torus 

$\Sigma =$ 

(G-)Character variety of a surface

$X =$ compact manifold, $G =$ (algebraic) group

character variety
$Ch_G(X) = \{\rho : \pi_1(X) \longrightarrow G\}^G$

$=$ set of group homomorphism for the fundamental group of S to G upto G-conjugation

(this can be though of as data of a principal G-bundle E, $\nabla_\gamma : E_x \to E_y$, $\eta =$ triv of fibers: $E_x \cong G$)

→ Choosing a set of m generators with n relations identifies

$\{\rho : \pi_1(X) \to G\} \leftrightsquigarrow$ closed subvariety of G^m defined by the n relations

→ $Ch_G(X)$ is then obtained by taking the GIT quotient by the G-action by conjugation

⇝ this admite the structure of stack.
$\underline{Ch}_G(X) =$ stack version of $Ch_G(X)$
↳ this is better for TQFT statements

for $G = SL_2$ and $X = \Sigma$ a surface we have that

$$Ch_{SL_2}(\Sigma) \simeq \mathrm{Spec}(Sk_{SL_2}(\Sigma))$$

For a surface Σ,

$Sk_{SL_2}(\Sigma) = Sk_{SL}(\Sigma \times I, A = -1)$

$=$ closed curves drawn on Σ

$\mathbb{Q}$ vector space spanned by all

isotopy +

$\times =)(+ \asymp$

$\bigcirc = 2 \cdot \varnothing$

Thus, the skien algebra $SL_{SL_2}(\Sigma \times I, A)$ is a deformation of (the coordinate ring) of SL_2-character variety of Σ.

Witten conjectured

<u>Theorem</u>: The skein module $Sk_{SL_2}(M, A)$ of any oriented 3-mfld M is finite dimensional.

Proved recently by Gunningham–Jordan–Safronov.

- use Heegaard splitting $M = H_g \cup_{\Sigma_g} H_g$
- get $Sk_{SL_2}(M, A) \simeq \left(Sk^{int}_{SL_2}(H_g, A) \otimes_{SL^{int}_{SL_2}(\Sigma \times I, A)} Sk^{int}_{SL}(H_g, A)\right)^{inv}$

$= N_1 \otimes_A N_2$

- show $-A =$ defor. quant of Poisson variety G^{2g}
  - N_1, N_2 also def. quant. of Lagrangian subvar. $G^g \hookrightarrow G^{2g}$
- using def. quant mod theory of Kashiwara–Schapira get f.d.

What about $A=-1$ specialization of $Sk_{SL_2}(M,A)$?

→ It recovers the algebra of functions on the character variety of M.

(but this is rarely a nice deformation because

- Poisson bracket does not lift
- $Sk_{SL_2}(M,A)$ dose not have a natural algebra structure.

What if A is another root of unity?

→ ongoing research.

(Higher) categorical perspective

$$\underset{\text{Skein modules}}{Sk_{\mathcal{E}}(\Sigma\times I, A)} \cong \mathrm{End}_{\mathcal{D}}(\mathbb{1}_{\mathcal{D}})$$

where $\mathcal{D} = Skcat_{\mathcal{E}}(\Sigma)$

$=$ skein module category

$=$ factorization homology $= \int_{\Sigma} \mathcal{E}$

This construction is in fact a 3-2 TFT

$$\begin{aligned} Bord^3_2 &\longrightarrow Cat \\ \Sigma &\longmapsto \int_{\Sigma} \mathcal{E} \end{aligned}$$

Q What are the resulting categories $\mathcal{M} = \int_{\Sigma} \mathcal{C}$?

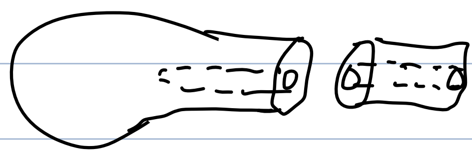
→ They are braided $\mathcal{C}$-module categories.
(this is one of the results of 1606.04769)

$$\boxed{\int_{\text{Annulus}} \mathcal{C} \simeq \mathcal{C}}$$

$$\boxed{\int_{S^\circ} \mathcal{C} = \text{braided } \mathcal{C}\text{-modules}}$$

S° = surface with circle boundary

action given by gluing annulus



algebraically this looks like a relative tensor product.