Today, we will use Hopf algebras to define invariants of oriented, closed 3-manifolds.

[In 2-dim, oriented closed manifolds are one of Σ_g =


$g \geq 0$

In 3-dim, problem is much more interesting]

§1 Heegaard diagrams

Every 3-diml closed, orientable 3-dimensional manifold can be described using Heegaard diagrams (uniquely upto some moves).

Defn A Heegaard diagram is a triple

$$D = (\Sigma_g, \{u_i\}, \{l_i\})$$

Σ_g: genus g closed oriented surface

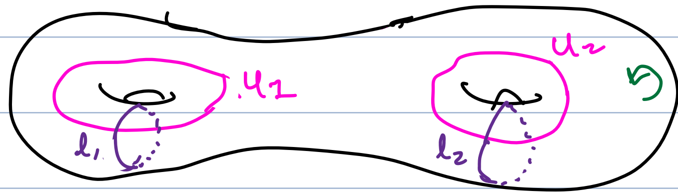
$\{u_i\}$: g 'upper' circles (oriented)

$\{l_i\}$: g 'lower' circles (oriented)

Conditions on circles $\{u_i\}, \{l_i\}$ $1 \leq i \leq n$ for any $n \geq 1$

- → $\{u_i\}, \{l_i\}$ are respectively disjoint
- → $\{u_i\}$ separates Σ_g into planar regions
- → $\{l_i\}$ separates Σ_g into planar regions

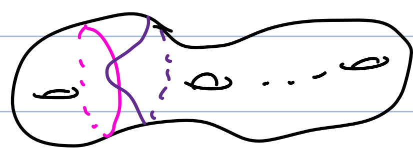
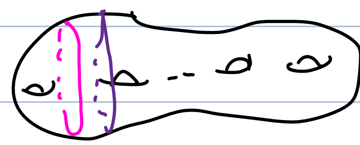
Example:

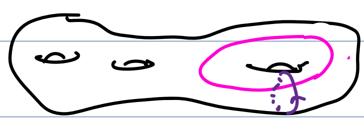
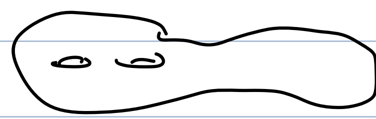


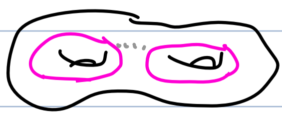
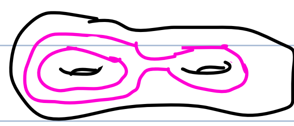
Σ_2

FACTS

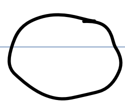
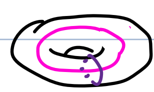
- Given a Heegaard diagram D, one can glue along the circles to get a closed, oriented 3-manifold $M(D)$
- Given a oriented, closed 3-mfld M, $\exists$ a Heegaard diagram D such that $M(D) \simeq M$.
- There can be multiple Heegaard diagrams D representing a 3-mfld M. Two H.D. D_1, D_2 represent the same mfld. if they are related by the following 3 relations

(R1)  $\sim$  (isotopy)

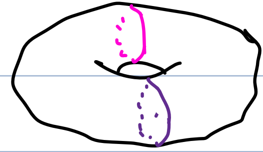
(R2)  $\sim$  (stabilization)

(R3)  $\sim$  (circle slide)

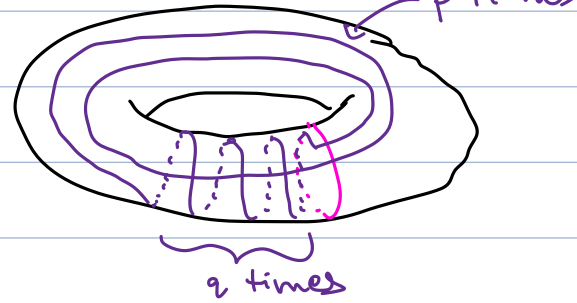
$$D_1 \sim_{R1, R2, R3} D_3 \iff M(D_1) \underset{\text{homeo}}{\simeq} M(D_2)$$

Eg: • $D_{S^3} =$  $\sim$ 

- $D_{S_2 \times S_1}$ =



- Lens space $L(p,q)$ $\quad D_{L(p,q)}$ =



Now, we are ready to define invariants of 3-mflds

<u>Step 1:</u> Assign a number $\theta(D)$ to every Heegard diagram

<u>Step 2:</u> Ensure that if $D_1 \sim D_2$ then
$$\theta(D_1) \sim \theta(D_2)$$

Now for any oriented, closed 3-mfld M, consider its Heegard diagram D_M. Then the assignment ψ
$$\psi : M \longmapsto \theta(D_M)$$
is an invariant of closed, oriented 3-mflds.

That is, $\quad M_1 \simeq M_2 \quad \Rightarrow \quad \psi(M_1) = \psi(M_2)$

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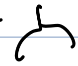
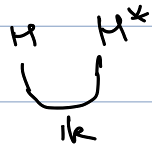
§2 Invariants constructed using involutory Hopf algebras

(Involutory Hopf algebras and 3-manifold invariants, Kuperberg)

- Let $\boxed{H}$ be a finite dimensional Hopf algebra. satisfying $S^2 = id_H$
- Basis $= \{h_i\}$, dual basis $= \{h^i\}$

NOTATIONS:

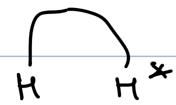
→ $m =$  , $u =$  , $\Delta =$  , $\varepsilon =$  , $S =$ 

→ evaluation map 

h, f

I

$f(h)$

coevaluation map 

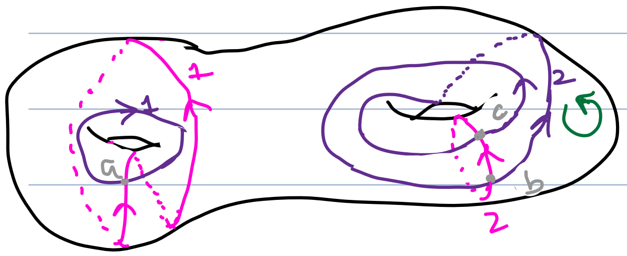
1

I

$h_i \otimes h^i$

→

Consider a Heegaard diagram:

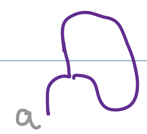


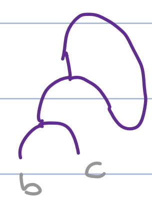
upper

lower

① arbitrarily orient all circles

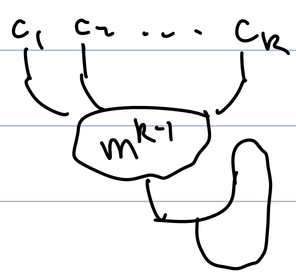
② to each lower circle ℓ, assign the maps

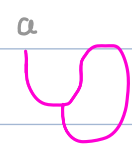
1 : 

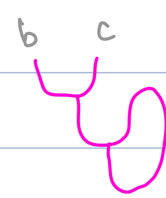
2 : 

here the indices $c_1, c_2, \ldots, c_n$ correspond to crossings on ℓ in the order that they are encountered when travelling along ℓ in positively oriented direction.

③ to each upper circle assign

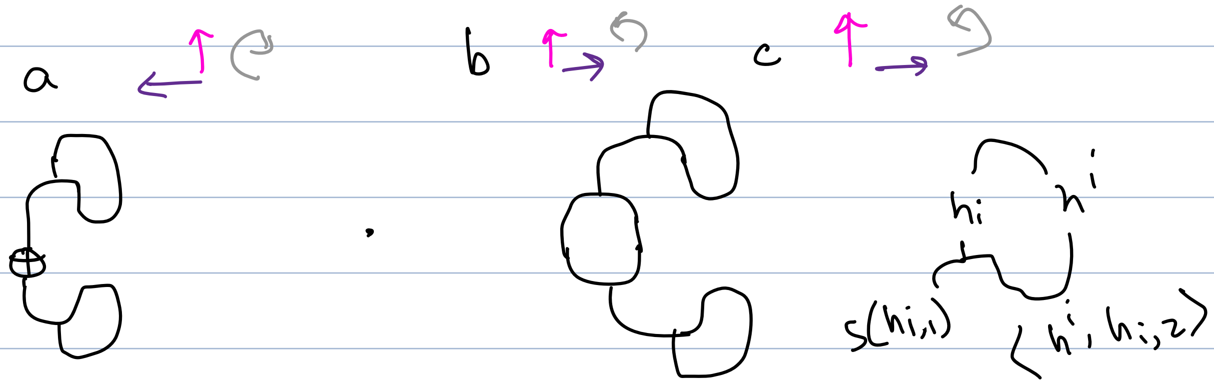


1 : 

2 : 

④ if at crossing c, the tangent vectors of the lower and upper circle, in that order, form a positively oriented basis of $T\Sigma_g$ at c, compose the maps in steps ②, ③.

If not, interpose S in between before composing

$Z(H) =$ 

The invariant is
$$\psi(D, H) = Z(H)\,(\dim H)^{-g}$$

→ Next we need to check that $\psi(D, H)$ is preserved by the moves R1, R2, R3.

- When $H = \Bbbk G$, then
$$\psi(D_M, H) = |\mathrm{Hom}(\pi(M), G)|$$

- With $\mathrm{char}(\Bbbk) = 0$, H is involutory $\Leftrightarrow$ H is semisimple

There is a non-semisimple generalization of above result as well.

- $\boxed{\Lambda^r}$ = right integral of H $\qquad \Lambda^r h = \varepsilon(h)\Lambda^r$
- $\boxed{\lambda^r}$ = right cointegral of H $\qquad \langle \lambda^r, h_1\rangle h_2 = \langle \lambda^r, h\rangle 1_H$

these satisfy $\langle \lambda^r, \Lambda^r\rangle = 1$

- $\boxed{g}$ = distinguished grouplike elt. of H
  $g := \Lambda_1 \langle \lambda, \Lambda_2\rangle$
- $\boxed{\alpha}$ = distinguished character of H
  $\langle \alpha, h\rangle := \langle \lambda, h\Lambda\rangle$

$\Lambda_1 \langle \alpha, \Lambda_2\rangle$
$= \Lambda_1 \langle \lambda, \Lambda_2 \Lambda\rangle$

- For $n \in \mathbb{Z}$, define $\boxed{\Lambda_{n-\frac{1}{2}}}$, $\boxed{\lambda_{n-\frac{1}{2}}}$

$\Lambda_{n-\frac{1}{2}} := \Lambda_1 \langle \alpha, \Lambda_2 \Lambda_3 \cdots \Lambda_{n+1}\rangle$ $\qquad$ e.g. $\Lambda_{-\frac{1}{2}} = \langle \alpha, \Lambda_1\rangle \Lambda_2$
$(n=1)$ $\qquad = \Lambda$

$\langle \lambda_{n-\frac{1}{2}}, h\rangle := \langle \lambda, h g^n\rangle$ $\qquad$ $\lambda_{-\frac{1}{2}} = \lambda$

- Set $\boxed{q} = \langle \alpha, g\rangle$ then q is a root of unity

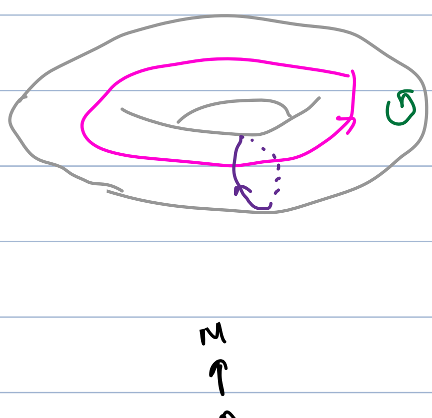
REFERENCE: On two invariants of three manifolds from Hopf algebras. (Chang, Cui)

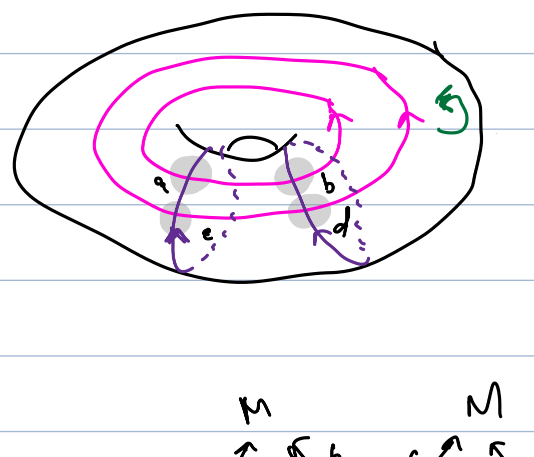
Notation	My notes	Chang-Cui
	Λ	e^R
	λ	μ^R
	g	a
	α	α

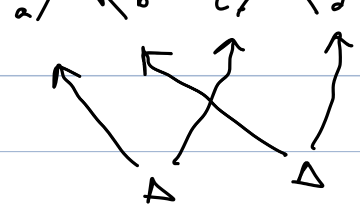
- $\boxed{S}$ antipode of H

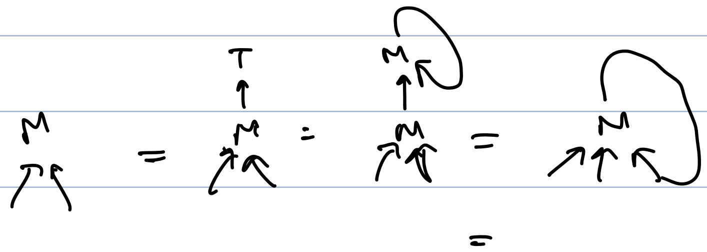
- $\boxed{T}$: $T(h) := \langle \alpha, S^{-2}(h)_1 \rangle \, S^{-2}(h)_2 \, \langle \alpha^{-1}, S^{-2}(h)_3 \rangle$

$$= \langle \alpha, S^{-2}(h_1) \rangle \, S^{-2}(h_2) \, \langle \alpha^{-1}, S^{-2}(h_3) \rangle$$





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