FROM DOUBLE GROUPOIDS TO WEAK HOPF ALGEBRAS

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Abstract. Double groupoid is a categorical structure that generalizes groupoids. We review known techniques of obtaining semisimple weak Hopf algebras from a double groupoid. Additionally, we review how to obtain a double groupoid using certain data of groupoids. All these constructions are illustrated using the example of a matched pair of groups.

Contents

1. Introduction

The goal of this report is to introduce the reader to double groupoids. Double groupoids are interesting algebraic structures that appear in algebraic topology and symplectic geometry. However, we are interested in them because they can be used to construct semisimple weak Hopf algebra, which in turn yield fusion categories.

$$\text{Double groupoid} \rightsquigarrow \underset{(H,m,u,\Delta,\varepsilon,S)}{\text{Weak Hopf algebra}} \rightsquigarrow \underset{\mathcal{C}=\mathsf{Rep}(H)}{\text{Fusion category}}$$

This is because fusion categories hold significant importance in 21st century mathematics. They offer a powerful framework for studying symmetries in various contexts, generalizing the well-understood concept of finite groups.

While groups (and groupoids) and related constructions have been well utilized to construct fusion categories, the connection to double groupoids has been studied in fewer papers. The main ones are [AN03, AN06, AN09]. Therefore, in this report, we review the main constructions in the aforementioned papers with the example of a matched pair of groups.

We start with a review of matched pairs of groups in Section 2. Section 3 reviews the definition of a double groupoid. Using matched pairs of groups, we construct an example of a double groupoid. Section 4 introduces weak Hopf algebras and reviews the construction of semisimple weak Hopf algebra using double groupoids. Section 5 reviews the construction a double groupoids using groupoids. Focusing on the example of a matched pair of groups, this construction recovers the well-known construction of the Zappa-Szép product of groups. Finally, in section 6, we briefly discuss our attempts to realize the Yang-Lee weak Hopf algebra from [BS96] using double groupoids.

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2. Matched pair of groups

Let G be a group. We will denote its identity element as 1_G. In this section, we introduce a matched pair of groups [Kac68, Mac70, Tak81, Maj90].

Definition 2.1. Let G and F be two finite groups. We say that they form a *matched pair of groups* if we have two group actions:

$$\triangleright : G \times F \to F, \quad \triangleleft : G \times F \to G$$

satisfying the following two conditions:

(a) $g \triangleright ff' = (g \triangleright f)((g \triangleleft f) \triangleright f')$,
(b) $gg' \triangleleft f = (g \triangleleft (g' \triangleright f))(g' \triangleleft f)$

for all $g, g' \in G$ and $f, f' \in F$.

Example 2.2. Let G, F denote the groups $\mathbb{Z}_n, \mathbb{Z}_m$ respectively for integers $m, n > 1$. Additionally, define the functions

$$\begin{array}{ccccccc} \triangleright : & \mathbb{Z}_n & \times & \mathbb{Z}_m & \to & \mathbb{Z}_m \\ & g & \triangleright & f & \mapsto & f \\ \triangleleft : & \mathbb{Z}_n & \times & \mathbb{Z}_m & \to & \mathbb{Z}_n \\ & g & \triangleleft & f & \mapsto & g \end{array}$$

that will form a matched pair of groups.

Proof. We can check that $\triangleleft$ is a group action:

(a) Firstly, we check the identity element: $g \triangleleft 0 = z_n$.
(b) Next, we check compatibility: $g \triangleleft f_1 \triangleleft f_2 = g = g \triangleleft f_1 f_2$.

Thus, $\triangleleft$ is a group action. This is similar proof that $\triangleright$ is a group action also. Let us check that $(G, F, \triangleleft, \triangleright)$ is a matched pair of groups:

(a) Evaluate left-hand part: $g \triangleright ff' = ff'$. Similarly with right-hand part: $(g \triangleright f)((g \triangleleft f) \triangleright f') = (f)(f') = ff'$. Thus, $g \triangleright ff' = (g \triangleright f)((g \triangleleft f) \triangleright f')$.

(b) Evaluate left-hand part: $gg' \triangleleft f = gg'$. Similarly with right-hand part: $(g \triangleleft (g' \triangleright f))(g' \triangleleft f) = (g)(g') = gg'$. Thus, $gg' \triangleleft f = (g \triangleleft (g' \triangleright f))(g' \triangleleft f)$.

So, $(G, F, \triangleleft, \triangleright)$ is a matched pair of groups. $\square$

Next we collect some basic properties of that we will need later.

Lemma 2.3. *Let G, F be a matched pair of groups. Then $1_G \triangleleft f = 1_G$ and $g \triangleright 1_F = 1_F$. Also, $(g \triangleright f)^{-1} = (g \triangleleft f) \triangleright f^{-1}$ and $(g \triangleleft f)^{-1} = g^{-1} \triangleleft (g \triangleright f)$.*

Proof. We prove each claim separately:

- Let us take $g\prime = 1_G$ and look at $(1_G \cdot g\prime) \triangleleft f = (1_G \triangleleft (g\prime \triangleright f))(g\prime \triangleleft f)$, or $1_G = 1_G \triangleleft (g\prime \triangleright f) = 1_G \triangleleft f$. Thus, $1_G = 1_G \triangleleft f$.
- Let us take $f\prime = 1_G$ and look at $g \triangleright (f\prime \times 1_F) = (g \triangleright f\prime)((g \triangleleft f\prime) \triangleright 1_F)$, or $1_F = (g \triangleleft f\prime) \triangleright 1_F$. Thus $1_F = g \triangleright 1_F$.
- $g \triangleright 1_G = (g \triangleright f)((g \triangleleft f) \triangleright f^{-1})$, multiply by $(g \triangleright f)^{-1}$ and get $(g \triangleright f)^{-1} = (g \triangleleft f) \triangleright f^{-1}$.
- $1_G \triangleleft f = (g^{-1} \triangleleft (g \triangleright f))(g \triangleleft f)$, multiply by $(g \triangleleft f)^{-1}$ and get $(g \triangleleft f)^{-1} = g^{-1} \triangleleft (g \triangleright f)$. $\square$

Lemma 2.4. *Let $\overline{g} \in G, \overline{f} \in F$ be two elements. Then there are unique elements $g \in G, f \in F$ such that $g \triangleright f = \overline{f}$ and $g \triangleleft f = \overline{g}$.*

Proof. To prove the claim, it suffices to write a formula for g, f using $\overline{g}, \overline{f}$.

Define $g = \overline{g} \triangleleft (\overline{g}^{-1} \triangleright \overline{f}^{-1})$ and $f = (\overline{g}^{-1} \triangleleft \overline{f}^{-1}) \triangleright \overline{f}$. Now, observe that

$$\begin{aligned} g \triangleleft f &= [\overline{g} \triangleleft (\overline{g}^{-1} \triangleright \overline{f}^{-1})] \triangleleft [(\overline{g}^{-1} \triangleleft \overline{f}^{-1}) \triangleright \overline{f}] \\ &= \overline{g} \triangleleft [(\overline{g}^{-1} \triangleright \overline{f}^{-1})((\overline{g}^{-1} \triangleleft \overline{f}^{-1}) \triangleright \overline{f})] \\ &\overset{(a)}{=} \overline{g} \triangleleft [\overline{g}^{-1} \triangleright \overline{f}^{-1}\overline{f}] = \overline{g} \triangleleft [\overline{g}^{-1} \triangleright 1_F] = \overline{g} \triangleleft 1_F = \overline{g}. \end{aligned}$$

In the second line of the above, we have used the group action property $g \triangleleft (a \triangleleft b) = g \triangleleft ab$. Similarly for $g \triangleright f = \overline{f}$.

$$\begin{aligned} g \triangleright f &= [\overline{g} \triangleleft (\overline{g}^{-1} \triangleright \overline{f}^{-1})] \triangleright [(\overline{g}^{-1} \triangleleft \overline{f}^{-1}) \triangleright \overline{f}] \\ &= [(\overline{g} \triangleleft (\overline{g}^{-1} \triangleright \overline{f}^{-1}))(\overline{g}^{-1} \triangleleft \overline{f}^{-1})] \triangleright \overline{f} \\ &\overset{(b)}{=} [\overline{g}^{-1}\overline{g} \triangleleft \overline{f}^{-1}] \triangleright \overline{f} = [1_G \triangleleft \overline{f}^{-1}] \triangleright \overline{f} = 1_G \triangleright \overline{f} = \overline{f}. \end{aligned}$$

$\square$

Next, we show that a matched pair of groups allows us to give a non-trivial group structure to the set $G \times F$. This construction is known as the *Zappa-Szép product* or *bicrossed product* in the literature.

Proposition 2.5. *Let $(G, F, \triangleleft, \triangleright)$ be a matched pair of groups. Then $G \bowtie F = (G \times F, m, 1_{G \times F})$ is a group where:*

(a) $G \times F$ is the set of elements of the group.

(b) Group multiplication is $\begin{array}{ccccc} G \times F & \cdot & G \times F & \to & G \times F \\ (g_1, f_1) & \cdot & (g_2, f_2) & \mapsto & ((g_2 \triangleleft f_1)g_1, f_2(g_2 \triangleright f_1)) \end{array}.$

(c) The inverse of (g, f) is equal to $(g^{-1} \triangleleft f^{-1}, g^{-1} \triangleright f^{-1})$.

(d) *The identity element is* $1_{G\times F} = (1_G, 1_F)$.

Proof. (a) We start with a proof of associativity. First, we observe that

$$\begin{aligned}
[(g_1,f_1)(g_2,f_2)](g_3,f_3) &= ((g_2 \lhd f_1)g_1, f_2(g_2 \rhd f_1))(g_3,f_3)\\
&= ([g_3 \lhd (f_2(g_2 \rhd f_1))](g_2 \lhd f_1)g_1, f_3[g_3 \rhd (f_2(g_2 \rhd f_1))])\\
&= ([(g_3 \lhd f_2) \lhd (g_2 \rhd f_1)](g_2 \lhd f_1)g_1, f_3[g_3 \rhd (f_2(g_2 \rhd f_1))])\\
&= ([((g_3 \lhd f_2)g_2) \lhd f_1]g_1, f_3[g_3 \rhd (f_2(g_2 \rhd f_1))]).
\end{aligned}$$

On the other hand, we observe that

$$\begin{aligned}
(g_1,f_1)[(g_2,f_2)(g_3,f_3)] &= (g_1,f_1)((g_3 \lhd f_2)g_2, f_3(g_3 \rhd f_2))\\
&=([((g_3 \lhd f_2)g_2) \lhd f_1]g_1, f_3(g_3 \rhd f_2)[((g_3 \lhd f_2)g_2) \rhd f_1])\\
&=([((g_3 \lhd f_2)g_2) \lhd f_1]g_1, f_3(g_3 \rhd f_2)[(g_3 \lhd f_2) \rhd (g_2 \rhd f_1)])\\
&=([((g_3 \lhd f_2)g_2) \lhd f_1]g_1, f_3[g_3 \rhd (f_2(g_2 \rhd f_1))]).
\end{aligned}$$

This proves that $[(g_1,f_1)(g_2,f_2)](g_3,f_3) = (g_1,f_1)[(g_2,f_2)(g_3,f_3)]$. Thus, the defined multiplication is associative.

(b) Next, we prove that element $1_{G\times F} = (1_G, 1_F)$ is an identity element. We check the following multiplication:
$(g,f)(1_G,1_F) = ((1_G \lhd f)g, 1_F(1_G \rhd f)) = (1_G * g, 1_F * f) = (g,f)$,
$(1_G,1_F)(g,f) = ((g \lhd 1_F)1_G, f(g \rhd 1_F)) = (g * 1_G, f * 1_F) = (g,f)$.
This show that $(1_G,1_F)$ is identity element of group $G \bowtie F$.

(c) Lastly, we check that $(g^{-1} \lhd f^{-1}, g^{-1} \rhd f^{-1})$ is the inverse for every $(g,f) \in G \bowtie F$:

$$\begin{aligned}
(g,f)(g^{-1} \lhd f^{-1}, g^{-1} \rhd f^{-1}) &= ([(g^{-1} \lhd f^{-1}) \lhd f]g, (g^{-1} \rhd f^{-1})[(g^{-1} \lhd f^{-1}) \rhd f])\\
&= ([g^{-1} \lhd 1_F]g, g^{-1} \rhd (f^{-1}f) = (1_G, 1_F) = 1_{G\times F},\\
(g^{-1} \lhd f^{-1}, g^{-1} \rhd f^{-1})(g,f) &= ([g \lhd (g^{-1} \rhd f^{-1})](g^{-1} \lhd f^{-1}), f[g \rhd (g^{-1} \rhd f^{-1})])\\
&= ((gg^{-1}) \rhd f^{-1}, f[gg^{-1}) \rhd f^{-1}]) = (1_G, 1_F) = 1_{G\times F}.
\end{aligned}$$

Thus, we have proved that every element has an inverse element in $G \bowtie F$. □

Remark 2.6. Later in Section 5 we will see a natural generalization of bicrossed product to groupoids.

3. Double groupoid

This section introduces double groupoids, a higher-order structure generalizing groupoids. They feature a key distinction: two compatible, groupoid-like multiplications acting on the same objects. We'll formally define these operations and explore their interplay. A subsequent example, the matched pair of groups, will illustrate this duality.

3.1. **Definition of a double groupoid.** Let us define a category and a groupoid first.

Definition 3.1. A *category* $\mathcal{C}$ is a collection (A, O, s, t, id, m), where A("arrows" or morphisms) and O("objects"), $s, t : A \to O$("source" and "target" respectively), $\mathsf{id} : O \to A$("identity") and $m : A_e \times_s A \to A$("composition") are in collection. They have to satisfy the associativity and identity axioms for id and m maps.

A category is called a *groupoid* if every morphism is an isomorphism.

Definition 3.2. A *double category* T consists of the following data:

- Four non-empty sets: $\mathcal{B}$ (boxes), $\mathcal{H}$ (horizontal edges), $\mathcal{V}$ (vertical edges), and $\mathcal{P}$ points
- eight boundary functions: $t, b : \mathcal{B} \to \mathcal{H}$; $r, l : \mathcal{B} \to \mathcal{V}$; $r, l : \mathcal{H} \to \mathcal{P}$; $t, b : \mathcal{V} \to \mathcal{P}$;
- four identity functions: $\mathsf{id} : \mathcal{B} \to \mathcal{H}$; $\mathsf{id} : \mathcal{B} \to \mathcal{V}$; $\mathsf{id} : \mathcal{H} \to \mathcal{P}$; $\mathsf{id} : \mathcal{V} \to \mathcal{P}$;
- four identity functions, all denoted by m:
  $\mathcal{B}_b \times \mathcal{B}_t \to \mathcal{B}$ (horizontal composition), $\mathcal{B}_r \times \mathcal{B}_l \to \mathcal{B}$ (vertical composition), $\mathcal{H}_r \times \mathcal{H}_l \to \mathcal{H}$, $\mathcal{V}_b \times \mathcal{V}_t \to \mathcal{V}$.

This data has to satisfy the following axioms:

(i) Boxes with vertical and horizontal composition form categories.
(ii) The horizontal edges and vertical edges with their respective compositions from categories.
(iii) $tr = rt$, $tl = lt$, $br = rb$, $bl = lb$. Where, t, b, r, and l mean, respectively, 'top', 'bottom', 'right', and 'left' of a box.
(iv) Compatibility of the compositions with the boundaries.
(v) Interchange law between horizontal and vertical compositions.
(vi) Horizontal and vertical identities.
(vii) Horizontal and vertical identities of the identities of the points.
(viii) Compatibility of the identity with the composition of arrows.

We refer the reader to [AN03, Definition 1.1] for the detailed definition.

Definition 3.3. A *double groupoid* is a double category in which all categories involved are groupoids.

For later use, we introduce the following notation. Let T be a double groupoid, then for each box $\boxed{t} \in T$, there exist a box $\boxed{t}^h$, such that $\boxed{t}|\boxed{t}^h = \mathsf{id}(l(\boxed{t}))$. Also for each box $\boxed{t} \in T$, there exists a box $\boxed{t}^v$, such that $\dfrac{\boxed{t}}{\boxed{t}^v} = \mathsf{id}(t(\boxed{t}))$. Lastly, we will denote $(\boxed{t}^h)^v = (\boxed{t}^v)^h$ as $\boxed{t}^{-1}$.

We can categorize double groupoids as vacant, slim, or filled (satisfying the filling condition).

- *Vacant* double groupoid
  We call a double groupoid *vacant* if for a fixed pair of a horizontal morphism x and a vertical morphism g there only exists one box β with edges consisting of g, x.
- *Slim* double groupoid
  We call a double groupoid *slim* if for every four morphisms there is *at most* one box β with edges consisting of g, x, g', x'.
- *The filling condition*
  We call a double groupoid satisfying *the filling condition* for fixed pair of a horizontal morphism x and a vertical morphism g (a corner pair (g, x)) there is at least one box β that has that corner.

3.2. **Example: matched pair of groups.**

Theorem 3.4. *Suppose we have a matched pair of groups* $(G, F, \triangleleft, \triangleright)$*. Then we can form a double category, which we will denote as* $\mathbb{A}(G, F, \triangleleft, \triangleright)$*.*

Proof. We start by defining the data of the double category:

$$\begin{array}{ccc}
\text{objects}(\mathcal{P}) & = & \{*\} \\
\text{horizontal morphisms}(\mathcal{H}) & = & \{g | g \in G\} \\
\text{vertical morphisms}(\mathcal{V}) & = & \{f | f \in F\} \\
\text{boxes}(\mathcal{B}) & = & \left\{ (g,f) = \begin{tikzcd} * \arrow[r, "g \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(g{,}f)"] & * \arrow[d, "g \triangleright f"] \\ * \arrow[r, "g"'] & * \end{tikzcd} \middle| g \in G, f \in F \right\}
\end{array}$$

This picture tells us what the horizontal and vertical morphisms of a box are:

- $t(g, f) = g \triangleleft f$, for $(g, f) \in \mathcal{B}$;
- $b(g, f) = g$, for $(g, f) \in \mathcal{B}$;
- $l(g, f) = f$, for $(g, f) \in \mathcal{B}$;
- $r(g, f) = g \triangleright f$, for $(g, f) \in \mathcal{B}$.

We can notice that each box is determined by the bottom and left labels. So, sometimes in place of a box, we just write (g, f) to denote the box corresponding to the bottom label g and left label f. Given an object $p \in \mathcal{P}$, we denote the identity box on that point as Ω_p. Since we only have one object $*$, we use the notation $\Omega = \Omega_* = (1_G, 1_F)$.

Next, we check that this data satisfies the conditions for being a double category:

(i) G is a group, so $\mathbf{B}G$ is category, and $\mathcal{H} = \mathbf{B}G$. So horizontal morphisms form a category.
(ii) F is a group, so $\mathbf{B}F$ is category, and $\mathcal{V} = \mathbf{B}F$. So vertical morphisms form a category.
(iii) Boxes with horizontal composition from a category with composition given by:

$$\begin{tikzcd} * \arrow[r, "g \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(g{,}f)"] & * \arrow[d, "g \triangleright f"] \\ * \arrow[r, "g"'] & * \end{tikzcd} \Bigg| \begin{tikzcd} * \arrow[r, "x \triangleleft (g \triangleright f)"] \arrow[d, "g \triangleright f"'] \arrow[dr, phantom, "(x{,} g \triangleright f)"] & * \arrow[d, "x \triangleright (g \triangleright f)"] \\ * \arrow[r, "x"'] & * \end{tikzcd} = \begin{tikzcd} * \arrow[r, "(x \triangleleft (g \triangleright f))(g \triangleleft f)"] \arrow[d, "f"'] \arrow[dr, phantom, "?"] & * \arrow[d, "x \triangleright (g \triangleright f)"] \\ * \arrow[r, "xg"'] & * \end{tikzcd} = \begin{tikzcd} * \arrow[r, "xg \triangleleft f"] \arrow[d, "f"'] \arrow[dr, phantom, "(xg{,}f)"] & * \arrow[d, "xg \triangleright f"] \\ * \arrow[r, "xg"'] & * \end{tikzcd}.$$

So, the general rule of the horizontal composition is: $(g, f)|(x, y) = \begin{cases} (xg, f) & , \quad \text{if } y = g \triangleright f \\ 0 & , \quad \text{otherwise} \end{cases}$.

(iv) The identity box for the horizontal composition is $\mathsf{id}(f) = f \underset{1_G}{\overset{1_G}{\boxed{}}} f$ or $(1_G, f)$. Then, we can check that $(1_G, f)|(g, f) = (g, f)$.

(v) Next, we explain the category of boxes with vertical composition:

$$\frac{\begin{array}{ccc} * & \xrightarrow{x\triangleleft y} & * \\ {\scriptstyle y}\downarrow & (x,y) & \downarrow{\scriptstyle x\triangleright y} \\ * & \xrightarrow[x]{} & * \end{array}}{\begin{array}{ccc} * & \xrightarrow{g\triangleleft f} & * \\ {\scriptstyle f}\downarrow & (g,f) & \downarrow{\scriptstyle g\triangleright f} \\ * & \xrightarrow[g]{} & * \end{array}} = \begin{array}{ccc} * & \xrightarrow{(g\triangleleft f)\triangleleft y} & * \\ {\scriptstyle fy}\downarrow & ? & \downarrow{\scriptstyle (g\triangleright f)((g\triangleleft f)\triangleright y)} \\ * & \xrightarrow[g]{} & * \end{array} = \begin{array}{ccc} * & \xrightarrow{g\triangleleft fy} & * \\ {\scriptstyle fy}\downarrow & (g,fy) & \downarrow{\scriptstyle g\triangleright fy} \\ * & \xrightarrow[g]{} & * \end{array}.$$

So, the general rule of the vertical composition is: $\dfrac{(x,y)}{(g,f)} = \begin{cases} (g,fy) & , \text{ if } x = g \triangleleft f \\ 0 & , \text{ otherwise} \end{cases}.$

(vi) The identity box for the vertical composition is $\mathsf{id}(g) = 1_F \overset{g}{\underset{g}{\boxed{}}} 1_F$ or $(g, 1_F)$. Then, we can observe that $\dfrac{(g,f)}{(g,1_F)} = (g,f)$.

(vii) Next, we check the interchange law. Assume that $\frac{(a,b)}{(e,f)}$, $\frac{(c,d)}{(g,h)}$, $(a,b)|(c,d)$, $(e,f)|(g,h)$ are composable:

(a) Firstly, $\dfrac{(a,b)|(c,d)}{(e,f)|(g,h)} = \dfrac{(ca,b)}{(ge,f)}$. If $\frac{(a,b)}{(e,f)}$, $\frac{(c,d)}{(g,h)}$ and $(e,f)|(g,h)$ are composable, then $a = e \triangleleft f$, $c = g \triangleleft h$ and $h = e \triangleright f$, so

$$\begin{aligned} ca &= (g \triangleleft h)(e \triangleleft f) \\ &= (g \triangleleft (e \triangleright f))(e \triangleleft f) \\ &= ge \triangleleft f. \end{aligned}$$

Thus, $\dfrac{(a,b)|(c,d)}{(e,f)|(g,h)} = \dfrac{(ca,b)}{(ge,f)} = (ge, fb)$.

(b) Secondly, $\left.\dfrac{(a,b)}{(e,f)}\right|\dfrac{(c,d)}{(g,h)} = (e,fb)|(g,hd)$. If $(a,b)|(c,d)$, $(e,f)|(g,h)$ and $\frac{(a,b)}{(e,f)}$ are composable, then $d = a \triangleright b$, $h = e \triangleright f$ and $a = e \triangleleft f$, so

$$\begin{aligned} hd &= (e \triangleright f)(a \triangleright b) \\ &= (e \triangleright f)((e \triangleleft f) \triangleright b) \\ &= e \triangleright fb. \end{aligned}$$

Thus, $\left.\dfrac{(a,b)}{(e,f)}\right|\dfrac{(c,d)}{(g,h)} = (e,fb)|(g,hd) = (ge,fb)$.

So, interchange law holds.

(viii) Next, we want to check that we can from Ω box both from horizontal identity map and vertical identity map: $\mathsf{id}(1_F) = (1_G, 1_F) = \Omega$, $\mathsf{id}(1_G) = (1_G, 1_F) = \Omega$.

(ix) Given $f_1, f_2 \in F$, we want to check that $\dfrac{(1_G, f_1)}{(1_G, f_2)} = (1_G, f_1 f_2)$.
$1_G = 1_G \lhd f_2$, so $\dfrac{(1_G, f_1)}{(1_G, f_2)} = (1_G, f_2 f_1)$.

(x) Given $g_1, g_2 \in F$, we want to check that $(g_1, 1_F)|(g_2, 1_F) = (g_2 g_1, 1_F)$.
$1_F = g_1 \rhd 1_F$, so $(g_1, 1_F)|(g_2, 1_F) = (g_2 g_1, 1_F)$. □

With this, we can prove the main result of this section.

Theorem 3.5. *The double category $\mathbb{A}(G, F, \lhd, \rhd)$ is a double groupoid.*

Proof. We check the conditions for being a double groupoid:

(i) G is a group, so $\mathbf{B}G$ is groupoid, and $\mathcal{H} = \mathbf{B}G$.
(ii) F is a group, so $\mathbf{B}F$ is groupoid, and $\mathcal{V} = \mathbf{B}F$.
(iii) Let (g, f) be a cell, then there exist $(g^{-1}, g \rhd f)$, such $(g, f)|(g^{-1}, g \rhd f) = (g^{-1}g, f) = (1_G, f) = \mathsf{id}(f)$, so horizontal composition of cells is invertible.
(iv) Let (g, f) be a cell, then there exist $(g \lhd f, f^{-1})$, such $\dfrac{(g \lhd f, f^{-1})}{(g, f)} = (g, ff^{-1}) = (g, 1_F) = \mathsf{id}(g)$, so the vertical composition of cells is invertible. □

Next we show certain properties of the groupoid $\mathbb{A}(G, F, \lhd, \rhd)$.

Lemma 3.6. *The double category $\mathbb{A}(G, F, \lhd, \rhd)$ is vacant.*

Proof. By Lemma 2.4, for each pair (g, f), such $g \in G$ and $f \in F$, there exist exactly one $\overline{g}, \overline{f}$, such that there is a box $\begin{tikzcd} * \arrow[r, "g"] \arrow[d, "\overline{f}"'] & * \arrow[d, "f"] \\ * \arrow[r, "g"'] & * \end{tikzcd}$ .Thus, $\mathbb{A}(G, F, \lhd, \rhd)$ is vacant. □

Lemma 3.7. *The double category $\mathbb{A}(G, F, \lhd, \rhd)$ is slim.*

Proof. There exists exactly one cell: $\begin{tikzcd} * \arrow[r] \arrow[d, "f"'] & * \arrow[d] \\ * \arrow[r, "g"'] & * \end{tikzcd}$, so there exists at most one cell such as: $\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "f"'] & * \arrow[d, "y"] \\ * \arrow[r, "g"'] & * \end{tikzcd}$. □

Lemma 3.8. *The double category $\mathbb{A}(G, F, \lhd, \rhd)$ is filled.*

Proof. By Lemma 3.6, the double category $\mathbb{A}(G, F, \lhd, \rhd)$ is vacant, so for every pair (g, f), we have at least 1 box with that corner. □

4. Weak Hopf algebra from double groupoid

In this section we introduce weak Hopf algebra and explain how to construct a semisimple Hopf algebra using a finite double groupoid that satisfies the filling condition. To illustrate this procedure, we discuss the example of a matched pair of groups in detail.

4.1. **Definition of a weak Hopf algebra.** First, we introduce the definition of a weak Hopf algebra.

Definition 4.1. *A weak Hopf algebra* is an algebraic structure that consists of:

(i) a vector space H,

(ii) a map $m : H \otimes H \to H$ and a map $u : \mathbb{C} \to H$,
(iii) a map $\Delta : H \to H \otimes H$ and a map $\varepsilon : H \to \mathbb{C}$,
(iv) a map $S : H \to H$ called antipode,

such that, the following conditions are satisfied:

(a) The triple (H, m, u) is an algebra.
  (a) Multiplication map m is associative. That is, $a(bc) = (ab)c$ for all $a, b, c \in A$. This is represented by following commutative diagram.

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "m\otimes\mathsf{id}"] \arrow[d, "\mathsf{id}\otimes m"'] & H \otimes H \arrow[d, "m"] \\
H \otimes H \arrow[r, "m"] & A
\end{tikzcd}$$

  (b) The unit condition is satisfied. That is, the element $1_H = u(1)$ satisfies $a \cdot 1_H = 1_H \cdot a = a$.

(b) The triple (H, Δ, ε) is a coalgebra. This mean that it satisfies the following conditions:

$$\begin{tikzcd}
 & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[dr, "\mathsf{id}\otimes\Delta"'] & & C \otimes C, \arrow[dl, "\Delta\otimes\mathsf{id}"] \\
 & C \otimes C \otimes C \arrow[d, leftrightsquigarrow] & \\
 & (\mathsf{id} \otimes \Delta)\Delta = (\Delta \otimes \mathsf{id})\Delta &
\end{tikzcd}
\quad
\begin{tikzcd}
 & C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[dr, "\mathsf{id}\otimes\varepsilon"'] & & C \otimes C \arrow[dl, "\varepsilon\otimes\mathsf{id}"] \\
 & C \arrow[d, leftrightsquigarrow] & \\
 & (\mathsf{id} \otimes \varepsilon)\Delta = \mathsf{id} = (\varepsilon \otimes \mathsf{id})\Delta &
\end{tikzcd}.$$

We will use the Sweedler notation to denote that comultiplication Δ as $\Delta(h) = h_1 \otimes h_2$.

(c) The unit and counit satisfy the following conditions:
- $\Delta(ab) = \Delta(a)\Delta(b)$;
- $(\mathsf{id} \otimes \Delta)\Delta(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H)$;
- $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$.

(d) Antipode S satisfies the following conditions:
- $m(\mathsf{id} \otimes S)\Delta(h) = (\varepsilon \otimes \mathsf{id})[\Delta(1_H)(h \otimes 1_H)]$;
- $m(S \otimes \mathsf{id})\Delta(h) = (\mathsf{id} \otimes \varepsilon)[(1_H \otimes h)\Delta(1_H)]$;
- $[m(m \otimes \mathsf{id})](S \otimes \mathsf{id} \otimes S)[(\Delta \otimes \mathsf{id})\Delta(h)] = S(h)$.

To clarify the definition, we provide an example.

Example 4.2. Let us define weak Hopf algebra using a group G.

(i) The underlying vector space is $H = \mathbb{C}G$. Its basis is $\{\delta_g\}_{g \in G}$;
(ii) $m : \delta_g \otimes \delta_h \mapsto \delta_{gh}$, $u : 1 \mapsto 1\delta_e$;
(iii) $\Delta : \delta_g \mapsto \delta_g \otimes \delta_g$, $\varepsilon : \delta_g \mapsto 1$;
(iv) $S : \delta_g \mapsto \delta_{g^{-1}}$.

Proof. Let us show that $(H, m, u, \Delta, \varepsilon, S)$ is a Hopf algebra by showing that it satisfies the following conditions:

(a) $A = (H, m, u)$ is an algebra: straigthforward.

(b) (H, Δ, ε) is a coalgebra: It will be true because both coassociativity and counitality are satisfied as we show below.

- *Coassociativity:* $(\mathsf{id} \otimes \Delta)\Delta = (\Delta \otimes \mathsf{id})\Delta$
  For element δ_g, from the left side $\delta_g \xrightarrow{\Delta} \delta_g \otimes \delta_g \xrightarrow{\mathsf{id}\otimes\Delta} (\delta_g \otimes \delta_g) \otimes \delta_g = \delta_g \otimes \delta_g \otimes \delta_g$
  And from the other side: $\delta_g \xrightarrow{\Delta} \delta_g \otimes \delta_g \xrightarrow{\Delta\otimes\mathsf{id}} \delta_g \otimes (\delta_g \otimes \delta_g) = \delta_g \otimes \delta_g \otimes \delta_g$
  Thus, we get that $(\mathsf{id} \otimes \Delta)\Delta = (\Delta \otimes \mathsf{id})\Delta$
- *Counitality:* $(\mathsf{id} \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathsf{id})\Delta$
  For element δ_g, the left side is $\delta_g \xrightarrow{\Delta} \delta_g \otimes \delta_g \xrightarrow{\varepsilon\otimes\mathsf{id}} 1 \otimes \delta_g = \delta_g = \mathsf{id}$
  And the right side: $\delta_g \xrightarrow{\Delta} \delta_g \otimes \delta_g \xrightarrow{\mathsf{id}\otimes\varepsilon} \delta_g \otimes 1 = \delta_g = \mathsf{id}$
  Thus, we get $(\mathsf{id} \otimes \varepsilon)\Delta = \mathsf{id} = (\varepsilon \otimes \mathsf{id})\Delta$

(c) Three following conditions are satisfied:
For any $\delta_g, \delta_{g'}, \delta_{g''}$ in A,
- $\Delta(\delta_g \delta_{g'}) = \Delta(\delta_g)\Delta(\delta_{g'})$:
  $\Delta(\delta_g \delta_{g'}) = \delta_g \delta_{g'} \otimes \delta_g \delta_{g'} = (\delta_g \otimes \delta_g)(\delta_{g'} \otimes \delta_{g'}) = \Delta(\delta_g)\Delta(\delta_{g'})$
- $(\mathsf{id} \otimes \Delta)\Delta(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H)$:
  $(\mathsf{id} \otimes \Delta)\Delta(1_H) = (\mathsf{id} \otimes \Delta)(1_H \otimes 1_H) = 1_H \otimes 1_H \otimes 1_H$
  $(\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = (1_H \otimes 1_H \otimes 1_H)(1_H \otimes 1_H \otimes 1_H) = 1_H \otimes 1_H \otimes 1_H$
  $(1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H) = (1_H \otimes 1_H \otimes 1_H)(1_H \otimes 1_H \otimes 1_H) = 1_H \otimes 1_H \otimes 1_H$
- $\varepsilon(\delta_g \delta_{g'} \delta_{g''}) = \varepsilon(\delta_g \delta_{g'_1})\varepsilon(\delta_{g'_2} \delta_{g''}) = \varepsilon(\delta_g \delta_{g'_2})\varepsilon(\delta_{g'_1} \delta_{g''})$: this is true because all three terms are equal to 1.

(d) And for antipode S the following is true:
- $m(\mathsf{id} \otimes S)\Delta(\delta_h) = (\varepsilon \otimes \mathsf{id})[\Delta(1)(\delta_h \otimes 1)]$:
  $m(\mathsf{id} \otimes S)\Delta(\delta_h) = m(\mathsf{id} \otimes S)(\delta_h \otimes \delta_h) = 1_H$
  $(\varepsilon\otimes\mathsf{id})[\Delta(1)(\delta_h\otimes 1_H)] = (\varepsilon\otimes\mathsf{id})[(1_H\otimes 1_H)(\delta_h\otimes 1_H)] = (\varepsilon\otimes\mathsf{id})(\delta_h\otimes 1)H) = 1\otimes 1_H = 1_H$
- $m(S \otimes \mathsf{id})\Delta(\delta_h) = (\mathsf{id} \otimes \varepsilon)[(1_H \otimes \delta_h)\Delta(1_H)]$:
  $m(S \otimes \mathsf{id})\Delta(\delta_h) = m(S \otimes \mathsf{id})(\delta_h \otimes \delta_h) = 1_H$
  $(\mathsf{id} \otimes \varepsilon)[(1_H \otimes \delta_h)\Delta(1)] = (\mathsf{id} \otimes \varepsilon)(1_H \otimes 1_H) = 1$
- $[m(m \otimes \mathsf{id})](S \otimes \mathsf{id} \otimes S)[(\Delta \otimes \mathsf{id})\Delta(\delta_h)] = S(\delta_h)$: this follows because

$$\begin{aligned}
[m(m \otimes \mathsf{id})](S \otimes \mathsf{id} \otimes S)[(\Delta \otimes \mathsf{id})\Delta(\delta_h)] &= [m(m \otimes \mathsf{id})](S \otimes \mathsf{id} \otimes S)[(\Delta \otimes \mathsf{id})(\delta_h \otimes \delta_h)] \\
&= [m(m \otimes \mathsf{id})](S \otimes \mathsf{id} \otimes S)[\delta_h \otimes \delta_h \otimes \delta_h] \\
&= [m(m \otimes \mathsf{id})](\delta_{h^{-1}} \otimes \delta_h \otimes \delta_{h^{-1}}) = \delta_{h^{-1}} = S(\delta_h)
\end{aligned}$$

Thus, we have checked all the conditions and the proof is finished. □

4.2. **Weak Hopf algebra from double groupoid.** In this section, we explain how to construct a weak Hopf algebra $(H, m, u, \Delta, \varepsilon, S)$ using a double groupoid, defined in [AN06, §2.2]. For constructing it we use different properties of the double groupoids.

- To construct the vector space H we use the boxes in the double groupoid.
- To get a multiplication map m we use the vertical composition of boxes.
- For the u unit map we use identity boxes in the double groupoid.
- In constructing a co-multiplication map Δ we use the horizontal composition of boxes.
- For constructing the co-unital map ε we use certain boxes.

- Lastly, for antipode map S we use the inverse of arrows under horizontal and vertical compositions.

To start, consider the following definition.

Definition 4.3. Define 4 maps $\llcorner : \mathcal{B} \to \mathbb{C}, \lrcorner : \mathcal{B} \to \mathbb{C}, \ulcorner : \mathcal{B} \to \mathbb{C}, \urcorner : \mathcal{B} \to \mathbb{C}$

$$\llcorner(\boxed{b_1}) = \#\{\boxed{b_2} \in \mathcal{B} | b(\boxed{b_1}) = b(\boxed{b_2}), l(\boxed{b_1}) = l(\boxed{b_2})\}$$
$$\lrcorner(\boxed{b_1}) = \#\{\boxed{b_2} \in \mathcal{B} | b(\boxed{b_1}) = b(\boxed{b_2}), r(\boxed{b_1}) = r(\boxed{b_2})\}$$
$$\urcorner(\boxed{b_1}) = \#\{\boxed{b_2} \in \mathcal{B} | t(\boxed{b_1}) = t(\boxed{b_2}), r(\boxed{b_1}) = r(\boxed{b_2})\}$$
$$\ulcorner(\boxed{b_1}) = \#\{\boxed{b_2} \in \mathcal{B} | t(\boxed{b_1}) = t(\boxed{b_2}), l(\boxed{b_1}) = l(\boxed{b_2})\}$$

The following is the main result of [AN06].

Theorem 4.4. *Let T be a double groupoid. Then we get a weak Hopf algebra with the following data:*

(i) *as a vector space $A = \mathbb{C}\mathcal{B}$;*

(ii) *the multiplication map is $m : A \otimes A \to A$*

$$m(\boxed{b_1} \otimes \boxed{b_2}) = \begin{cases} \begin{array}{c} \boxed{b_1} \\ \hline \boxed{b_2} \end{array} & \textit{if } b_1 \textit{ and } b_2 \textit{ are vertically composable} \\ 0 & \textit{otherwise} \end{cases};$$

(iii) *the unit map is $u : \mathbb{C} \to A$*

$$u(1) = \sum_{x \in \mathcal{H}} \mathsf{id} \underset{x}{\overset{x}{\boxed{}}} \mathsf{id};$$

(iv) *the comultiplication map is $\Delta : A \to A \otimes A$*

$$\Delta(\boxed{b_1}) = \sum_{\boxed{b_2} | \boxed{b_3} = \boxed{b_1}} \frac{1}{\urcorner(\boxed{b_3})} \boxed{b_2} \otimes \boxed{b_3};$$

(v) *the counit map is $\varepsilon : H \to \mathbb{C}$*

$$\varepsilon(\boxed{b_1}) = \begin{cases} \urcorner(\boxed{b_1}) & \textit{if } t(\boxed{b_1}) = b(\boxed{b_1}) = \mathsf{id} \\ 0 & \textit{otherwise} \end{cases}$$

(vi) *the antipode map is $S : H \to H$*

$$S(\boxed{b_1}) = \frac{\ulcorner(\boxed{b_1})}{\llcorner(\boxed{b_1})} \boxed{b_1}^{-1} = \frac{\ulcorner(\boxed{b_1})}{\llcorner(\boxed{b_1})} \left(\boxed{b_1}^h\right)^v = \frac{\ulcorner(\boxed{b_1})}{\llcorner(\boxed{b_1})} \left(\boxed{b_1}^v\right)^h$$

4.3. **Example: matched pair of groups.** Let us form a weak Hopf algebra from the double groupoid $T(\mathcal{B}, \mathcal{V}, \mathcal{H}, \mathcal{P})$ formed by a matched pair of groups $(G, F, \triangleleft, \triangleright)$. Let D denote the set $G \times F = \{(g, f) | \ g \in G, f \in F\}$.

Theorem 4.5. *Define the following maps:*

(i)
$$\begin{array}{ccccccc} m: & \mathbb{C}D & \otimes & \mathbb{C}D & \to & \mathbb{C}D \\ & (g,f) & \otimes & (x,y) & \mapsto & \begin{cases} (x, yf) & , \quad \textit{if } g = x \triangleleft y\,; \\ 0 & , \quad \textit{otherwise} \end{cases} \end{array}$$

(ii)
$$\begin{array}{cccc} u: & \mathbb{C} & \to & \mathbb{C}D \\ & 1 & \mapsto & \sum_{g \in G} (g, 1_F) \end{array};$$

(iii)
$$\begin{array}{cccccc} \Delta: & \mathbb{C}D & \to & \mathbb{C}D & \otimes & \mathbb{C}D \\ & (g,f) & \mapsto & \sum_{g' \in G} \ (g', f) & \otimes & (gg'^{-1}, g' \triangleright f) \end{array};$$

(iv) $\begin{array}{cccc} \varepsilon: & \mathbb{C}D & \to & \mathbb{C} \\ & (g,f) & \mapsto & \begin{cases} 1 & , \textit{ if } g = 1_G; \\ 0 & , \textit{ otherwise} \end{cases} \end{array}$

(v) $\begin{array}{cccc} S: & \mathbb{C}D & \to & \mathbb{C}D \\ & (g,f) & \mapsto & ((g \triangleleft f)^{-1}, (g \triangleright f)^{-1}) \end{array}.$

Then $(\mathbb{C}D, m, u, \Delta, \varepsilon, S)$ *is a weak Hopf algebra.*

There are two ways of proving this Theorem. The first is by using Theorem 4.4 and the second is by directly checking the axioms of a weak Hopf algebra. We present both proofs below.

First proof: using [AN06]. We will check that each map is formed from Theorem 4.4.

(a) A multiplication map m, formed by way described in Theorem 4.4.

$$\text{Then } m(\boxed{b_1} \otimes \boxed{b_2}) = m((g,f) \otimes (x,y)) = \begin{cases} \dfrac{\boxed{b_1}}{\boxed{b_2}} = \frac{(g,f)}{(x,y)} = (x, yf) & , \text{if } g = x \triangleleft y \\ 0 & , \text{otherwise} \end{cases}.$$

(b) A map u, contain all elements with id vertical morphisms.

(c) A map Δ, should split boxes, by horizontal composition. In this example, $\urcorner$ map equals 1 for all corners. By reforming $\boxed{b_2} \| \boxed{b_3} = \boxed{b_1}$, we can get that $\boxed{b_3} = \boxed{b_2}^h \boxed{b_1}$. If use that $\boxed{b_1} = (g,f)$, $\boxed{b_2} = (g', f)$, we will get the same result.

(d) A map ε, should be non-zero for all maps where horizontal morphisms are id. In this example, $\urcorner$ map equals 1 for all corners. Thus, suppose that $\boxed{b_1} = (g,f)$ than, if $g = 1_G$, both horizontal morphisms are id, and $\varepsilon(g,f) = 1$. Otherwise, it equals 0.

(e) An antipode map S, should be equal to $\dfrac{\ulcorner(\boxed{b_1})}{\llcorner(\boxed{b_1})}\boxed{b_1}^{-1}$. In this example, $\ulcorner$, $\llcorner$ maps equal 1 for all corners. Thus, $S(g,f) = (g,f)^{-1} = ((g,f)^h)^v = (g^{-1}, g \triangleright f)^v = (g^{-1} \triangleleft (g \triangleright f), (g \triangleright f)^{-1}) = ((g \triangleleft f)^{-1}, (g \triangleright f)^{-1})$.

This finished the proof. □

Next, we provide a direct check of all the conditions for being a weak Hopf algebra.

Second proof: direct check. Let us look at the conditions for A to be a weak Hopf algebra. The associativity of multiplication and coassociativity of comultiplication is easy to check. So is checking that multiplication is unital and comultiplication is countial. Therefore, we skip these proofs.

1) We check the condition $\Delta(ab) = \Delta(a)\Delta(b)$. Let $a = (g,f)$, $b = (x,y)$. Then there are two cases:

(i) $\underline{g \neq x \triangleleft y}$: In this case, $ab = 0$ and $\Delta(ab) = 0$. $\Delta(a)\Delta(b) = \sum_{g',x' \in G} (g',f) \cdot (x',y) \otimes (gg'^{-1}, g' \triangleright f) \cdot (xx'^{-1}, x' \triangleright y)$. This sum is non-zero, if only if $(g',f) \cdot (x',y) \neq 0$ and $(gg'^{-1}, g' \triangleright f) \cdot (xx'^{-1}, x' \triangleright y) \neq 0$, for some $g', x' \in G$. It can be if and only if $g' = x' \triangleleft y$ and $gg'^{-1} = (xx'^{-1}) \triangleleft (x' \triangleright y)$. But if both equations hold, then $(gg'^{-1})(g') = ((xx'^{-1}) \triangleleft (x' \triangleright y))(x' \triangleleft y) = (xx'^{-1}x) \triangleleft y = x \triangleleft y$. It is a contradiction, so if $g \neq x \triangleleft y$ then $\Delta(ab) = \Delta(a)\Delta(b) = 0$.

(ii) $\underline{g = x \triangleleft y}$: Then, $\Delta(ab) = \Delta(x, yf) = \sum_{x' \in G} (x', yf) \otimes (xx'^{-1}, x' \triangleright yf)$. At the right-side we have $\Delta(a)\Delta(b) = \sum_{g',x' \in G} (g',f) \cdot (x',y) \otimes (gg'^{-1}, g' \triangleright f) \cdot (xx'^{-1}, x' \triangleright y)$. Suppose $(g',f) \cdot (x',y) \neq 0$, this mean that $g' = x' \triangleleft y$. By notice that $gg'^{-1} = (x \triangleleft y)(x' \triangleleft y)^{-1} =$

$((xx')x'^{-1} \triangleleft y)(x' \triangleleft y)^{-1} = (xx' \triangleleft (x' \triangleright y))(x' \triangleleft y)(x' \triangleleft y)^{-1} = (xx' \triangleleft (x' \triangleright y))$, we get that $(gg'^{-1}, g' \triangleright f) \cdot (xx'^{-1}, x' \triangleright y) = (xx'^{-1}, (x' \triangleright y)(g' \triangleright f)) = (xx'^{-1}, (x' \triangleright y)((x' \triangleleft y) \triangleright f)) = (xx'^{-1}, x' \triangleright yf)$. Other part is equal $(g', f) \cdot (x', y) = (x', yf)$. Thus $\Delta(ab) = \Delta(a)\Delta(b)$.

2) Next we check the condition

$$(\mathsf{id} \otimes \Delta)\Delta(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H). \tag{4.1}$$

Note that $\Delta(1_H) = \sum_{g \in G} \Delta(g, 1_F) = \sum_{a,b \in G} (a, 1_F) \otimes (b, 1_F)$.

(i) the first term of (4.1) is:

$$(\mathsf{id} \otimes \Delta)\Delta(1) = \sum_{a,b \in G} (id \otimes \Delta)((a, 1_F) \otimes (b, 1_F)) = \sum_{a,b,c \in G} (a, 1_F) \otimes (b, 1_F) \otimes (c, 1_F).$$

(ii) If we simplify the second term of (4.1), we can get

$$(\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = \sum_{a,b,c,d \in G} ((a, 1_F) \otimes (b, 1_F) \otimes 1_H)(1_H \otimes (c, 1_F) \otimes (d, 1_F).)$$

However, if $(b, 1_F) \cdot (c, 1_F) \neq 0$, then $b = c$. Thus, $(\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H))$ equals

$$\sum_{a,b,c \in G} ((a, 1_F) \otimes (b, 1_F) \otimes 1_H)(1_H \otimes (b, 1_F) \otimes (c, 1_F)) = \sum_{a,b,c \in G} (a, 1_F) \otimes (b, 1_F) \otimes (c, 1_F).$$

(iii) If we simplify the third term of (4.1), we can get

$$(1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H) = \sum_{a,b,c,d \in G} (1_H \otimes (a, 1_F) \otimes (b, 1_F))((c, 1_F) \otimes (d, 1_F) \otimes 1_H).$$

If $(a, 1_F) \cdot (d, 1_F) \neq 0$, then $a = d$. Thus, $(\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H))$ equals

$$\sum_{a,b,c \in G} (1_H \otimes (a, 1_F) \otimes (b, 1_F))((c, 1_F) \otimes (a, 1_F) \otimes 1_H) = \sum_{a,b,c \in G} (a, 1_F) \otimes (b, 1_F) \otimes (c, 1_F).$$

In this way, we see that (4.1) holds.

3) Next we check the condition $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$.
Suppose that $a = (g_1, f_1), b = (g_2, f_2), c = (g_3, f_3)$. There are three cases:

(i) If $\varepsilon(abc) = 1$, then $\varepsilon(g_3, f_3f_2, f_1) = 1$, thus $g_3 = 1_G$, $g_2 = 1_G$ and $g_1 = 1_G$. If $g_{2,1} \neq 1_G$, then $ab_1 = 0$ and $b_1c = 0$, otherwise $\varepsilon((1_G, f_1) \cdot (1_G, f_2)) = 1$ and $\varepsilon((1_G, f_2) \cdot (1_G, f_3)) = 1$. Thus, $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$, if $\varepsilon(abc) = 1$.

(ii) If $abc \neq 0$, but $\varepsilon(abc) = 0$, then $\varepsilon(g_3, f_3f_2, f_1) = 0$, thus $g_3 \neq 1_G$. In this case, $\varepsilon(b_1c)$ and $\varepsilon(b_2c)$, both equal zero. If b_xc (for $x = 1$ or 2) is composable, then $b(b_xc) = b(c) \neq 1_G$ and $\varepsilon(b_1c) = 0$.

(iii) Otherwise, $abc = 0$. If $g_3 \neq 1_G$, every $\varepsilon(b_xc) = 0$. So $g_3 = 1_G$. Suppose $g_1 \neq 1_G$, then $\varepsilon(ab_x) = 0$ (for $x = 1$ or 2) because: if $ab_x \neq 0$, then $t(b_x) = b(a) \neq 1_G$, so $\varepsilon(ab_x) = 0$. Thus $g_1, g_3 = 1_G$, and $g_2 \neq 1_G$. There is no b_1, b_2, such $b(b_1) = 1_G$ and $b(b_2) = 1_G$, and otherwise $\varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c) = 0$.

Thus, this condition holds.

4) Lastly we check the conditions for the antipode map. Suppose that $h = (g, f)$.

(i) $m(\mathsf{id} \otimes S)\Delta(h) = (\varepsilon \otimes \mathsf{id})[\Delta(1_H)(h \otimes 1_H)]$.

The left-hand side can be simplified to

$$\sum_{ba=g}(a,f)\cdot S(b,a\triangleright f)=\sum_{ba=g}(a,f)\cdot([b\triangleleft(a\triangleright f)]^{-1},(b\triangleright(a\triangleright f))^{-1}).$$

So for non-zero components $a=[b\triangleleft(a\triangleright f)]^{-1}\triangleleft[b\triangleright(a\triangleright f)]^{-1}=b^{-1}\triangleleft(b\triangleright(a\triangleright f)]\triangleleft[b\triangleright(a\triangleright f)]^{-1}=b^{-1}$. In general if $g=1_G$, then the left-hand side is equal to $\sum_{g\in G}(g,1_F)$, otherwise it is equal to zero.

On the other hand, the right-hand side can be simplified to

$$\sum_{a,b\in G}(\varepsilon\otimes\mathsf{id})((a,1_F)(g,f)\otimes(b,1_F))=\sum_{b\in G}(\varepsilon\otimes\mathsf{id})((g,f)\otimes(b,1_F))=\sum_{b\in G}\varepsilon(h)\otimes(b,1_F).$$

So, in general, if $g=1_G$, then the right-hand side is equal to $\sum_{g\in G}(g,1_F)$, otherwise it is equal to zero.

Thus, this condition holds.

(ii) $m(S\otimes\mathsf{id})\Delta(h)=(\mathsf{id}\otimes\varepsilon)(1_H\otimes h)[\Delta(1_H)]$.

The left-hand side can be simplified to

$$\sum_{ba=g}S(a,f)\cdot(b,a\triangleright f)=\sum_{ba=g}((a\triangleleft f)^{-1},(a\triangleright f)^{-1}\cdot(b,a\triangleright f).$$

For non-zero components, $(a\triangleleft f)^{-1}=b\triangleleft(a\triangleright f)$, so $a^{-1}\triangleleft(a\triangleright f)$, and $a=b^{-1}$. In general if $g=1_G$, then the left-hand side is equal to $\sum_{g\in G}(g,1_F)$, otherwise it is equal to zero.

On the other hand, the right-hand side can be simplified to

$$\sum a,b\in G(\mathsf{id}\otimes\varepsilon)((a,1_F)\otimes(g,f)(b,1_F))=\sum a\in G(\mathsf{id}\otimes\varepsilon)((a,1_F)\otimes(g,f)).$$

So, in general, if $g=1_G$, then the right-hand side is equal to $\sum_{g\in G}(g,1_F)$, otherwise it is equal to zero.

Thus, this condition holds.

(iii) $[m(m\otimes\mathsf{id})](S\otimes\mathsf{id}\otimes S)[(\Delta\otimes\mathsf{id})\Delta(h)]=S(h)$. Suppose $h=(g,f)$.

The left-hand side can be simplifies to

$$\sum_{a,b\in G}S(a,f)\cdot(b,a\triangleright f)\cdot S(ga^{-1}b^{-1},ba\triangleright f).$$

To start, not that $S(a,f)\cdot(b,a\triangleright f)=((a\triangleleft f)^{-1},(a\triangleright f)^{-1})\cdot(b,a\triangleright f)$. It is non-zero if and only if $(a\triangleleft f)^{-1}=b\triangleleft(a\triangleright f)$, or $a^{-1}\triangleleft(a\triangleright f)=b\triangleleft(a\triangleright f)$. Thus $b=a^{-1}$. Consequently, one can check that $((a\triangleleft f)^{-1},(a\triangleright f)^{-1})\cdot(a^{-1},a\triangleright f)=(a^{-1},1_F)$. So the left-hand side is equal to

$$\sum_{a\in G}(a^{-1},1_F)\cdot S(g,f)=\sum_{a\in G}(a^{-1},1_F)\cdot((g\triangleleft f)^{-1},(g\triangleright f)^{-1}).$$

If $a=[(g\triangleleft f)^{-1}\triangleleft(g\triangleright f)^{-1}]^{-1}$, then it equals $((g\triangleleft f)^{-1},(g\triangleright f)^{-1})$, otherwise it is equal to zero. In general left-hand side is equal to $((g\triangleleft f)^{-1},(g\triangleright f)^{-1})$.

The right-hand side is equal to $((g\triangleleft f)^{-1},(g\triangleright f)^{-1})$, thus, this condition holds. □

Remark 4.6. The weak Hopf algebra constructed using a matched pair of groups is actually a Hopf algebra. So one does not need to check all the axioms of a weak Hopf algebra. We however provide all the details for instructive purposes.

4.4. **Weak Hopf algebra to fusion category.** A fusion category is a $\Bbbk$-linear, abelian, semisimple, finite, rigid monoidal category with simple unit object and bilinear tensor product.

How fusion categories generalize groups:

$$G \rightsquigarrow \underset{\text{Weak Hopf algebra}}{\mathbb{C}G} \rightsquigarrow \underset{\text{Fusion category}}{\mathsf{Rep}(G)}$$

Using a semisimple weak Hopf algebra $(H, m, u, \Delta, \varepsilon, S)$ we construct a category $\mathcal{C} = \mathsf{Rep}(H)$ which is in fact a fusion category. It consists of the following:

- Objects: a pair (V, ρ) consisting of a vector space V and a map $\rho : H \times V \to V$ satisfying
  (a) $\rho(h, \rho(h', v)) = \rho(hh', v)$;
  (b) $\rho(1_H, v) = v$.
- Morphisms: Given two objects (V, ρ), (W, σ) then a morphism: $f : (V, \rho) \to (W, \sigma)$ is a map $f : V \to W$ which satisfies: $\sigma(h, f(w)) = f(\rho(h, v))$.
- For constructing objects and morphisms we use H, m, u from weak Hopf algebra.
- Fusion category is monoidal, thus, there is tensor map $\otimes : (V, \rho) \otimes (W, \sigma) \to (U, \tau)$. For its construction, we use Δ, ε from weak Hopf algebra.
- Fusion category is rigid, so there are *ev*, *coev*, X^* maps that follow some condition. For their construction, we use a S map from weak Hopf algebra.

So, in general, we use horizontal and vertical composition from double groupoids to construct weak Hopf algebra, which we use to construct the fusion category. This generalizes double groupoids, making it easier to research it using fusion categories.

5. Constructing double groupoids using groupoids

In this section, we use the definition of the diagram, and the way of transforming the diagram to a double groupoid from [AN09, §2.1].

5.1. **Building diagram for double groupoid formed by matched pair of groups.**

Definition 5.1. A *diagram* $(\mathcal{D}, j, i)$ over $\mathcal{H}$ and $\mathcal{V}$, is a groupoid $\mathcal{D}$ over $\mathcal{P}$, with two maps: $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$.

Definition 5.2. Each diagram $(\mathcal{D}, j, i)$ has an associated double groupoids $\square(\mathcal{D}, j, i)$ defined as follows. Boxes in $\square(\mathcal{D}, j, i)$ are of the form

$$A = h \underset{y}{\overset{x}{\square}} g \in \square(\mathcal{V}, \mathcal{H}),$$

with $x, y \in \mathcal{H}$, $g, h \in \mathcal{V}$, such that $i(x)j(g) = j(h)i(y)$.

The set of boxes $\square(\mathcal{D}, j, i)$ is stable under vertical and horizontal compositions in $\square(\mathcal{V}, \mathcal{H})$, so it is a double groupoid.

5.2. **Example: matched pair of groups.**

Theorem 5.3. *Let $\mathcal{D} = \mathbb{B}(G \bowtie F)$ be a groupoid, define following groupoids:* $\begin{array}{cccc} i: & G & \to & \mathcal{D} \\ & g & \mapsto & (g, 1_F) \end{array}$ *and* $\begin{array}{cccc} j: & F & \to & \mathcal{D} \\ & f & \mapsto & (1_G, f) \end{array}$, *then* $\square(\mathcal{D}, j, i) = \mathbb{A}(G, F, \triangleleft, \triangleright)$

Proof. (a) They have same $\mathcal{P}, \mathcal{V}, \mathcal{H}$ by definition.

(b) Condition for box $f\,\overset{x}{\underset{g}{\square}}\,y$ in $\mathbb{A}(G, F, \lhd, \rhd)$, is that $x = g \lhd f$ and $y = g \rhd f$. While in $\square(D, j, i)$ condition is

$$i(x)j(y) = j(f)i(g), i(x)j(y) = (x, 1_F)(1_G, y) = (x, y),$$
$$j(f)i(g) = (1_G, f)(g, 1_F) = ([g \lhd f]1_G, 1_F[g \rhd f]) = (g \lhd f, g \rhd f).$$

But note that $i(x)j(y) = j(f)i(g) \Leftrightarrow (x, y) = (g \lhd f, g \rhd f) \Leftrightarrow (x = g \lhd f) \wedge (y = g \rhd f)$. So both conditions are the same, so $\mathcal{B}$ is the same.

(c) Both groupoids have the same vertical composition:

Note that $\dfrac{v_1\,\overset{h_1}{\underset{h_2}{\square}}\,v_3}{v_2\,\overset{h_3}{\underset{h_4}{\square}}\,v_4} = v_2v_1\,\overset{h_1}{\underset{h_4}{\square}}\,v_4v_3$ if and only if $h_2 = h_3$, otherwise equal 0, for all $h_1, h_2, h_3, h_3 \in \mathcal{H}$, $v_1, v_2, v_3, v_3 \in \mathcal{V}$.

(d) Both groupoids have the same horizontal composition:

Note that $v_1\,\overset{h_1}{\underset{h_2}{\square}}\,v_3 \,\Big|\, v_2\,\overset{h_3}{\underset{h_4}{\square}}\,v_4 = v1\,\overset{h_3h_1}{\underset{h_4h_2}{\square}}\,v4$ if and only if $v_2 = v_3$, otherwise equal 0, for all $h_1, h_2, h_3, h_3 \in \mathcal{H}$, $v_1, v_2, v_3, v_3 \in \mathcal{V}$. □

6. Our findings

In this section, we will present our findings about transforming double groupoid into weak Hopf algebra of direct sum of matrix algebras. The motivation for this section is the fact that forming weak Hopf algebra of direct sum of matrices algebras is complicated, and by observing double groupoids, we can try to find an easier path. We will use the algebra $A_{L.Y.} = M_2 \oplus M_3$ formed in the paper [BS96, §5]. In this section, M_2 and M_3 will denote the matrix algebra formed from 2×2 and 3×3 matrices, respectively.

6.1. **Weak Hopf algebra of direct sum of matrix algebras.** Given $A_{L.Y.} = M_2 \oplus M_3$. We will fix matrix units e_0^{ij} in M_2 and e_1^{ij} in M_3. Let $z = \sqrt{(\sqrt{5} - 1)/2}$. Let us define a weak Hopf algebra for this algebra. The coproduct is given by:

- $\Delta(e_0^{11}) = e_0^{11} \otimes e_0^{11} + e_1^{11} \otimes e_1^{33}$
- $\Delta(e_0^{12}) = e_0^{12} \otimes e_0^{12} + z^2 e_1^{13} \otimes e_1^{31} + z e_1^{12} \otimes e_1^{32}$
- $\Delta(e_0^{22}) = e_0^{22} \otimes e_0^{22} + z^4 e_1^{33} \otimes e_1^{11} + z^3 e_1^{32} \otimes e_1^{12} + z^3 e_1^{23} \otimes e_1^{21} + z^2 e_1^{22} \otimes e_1^{22}$
- $\Delta(e_1^{11}) = e_0^{11} \otimes e_1^{11} + e_1^{11} \otimes e_0^{22} + e_1^{11} \otimes e_1^{22}$
- $\Delta(e_1^{12}) = e_0^{12} \otimes e_1^{12} + e_1^{12} \otimes e_0^{22} + z e_1^{13} \otimes e_1^{21} - z^2 e_1^{12} \otimes e_1^{22}$
- $\Delta(e_1^{13}) = e_0^{12} \otimes e_1^{13} + e_1^{13} \otimes e_0^{21} + e_1^{12} \otimes e_1^{23}$
- $\Delta(e_1^{22}) = e_0^{22} \otimes e_1^{22} + e_1^{22} \otimes e_0^{22} + z^2 e_1^{33} \otimes e_1^{11} - z^3 e_1^{32} \otimes e_1^{12} - z^3 e_1^{23} \otimes e_1^{21} + z^4 e_1^{22} \otimes e_1^{22}$
- $\Delta(e_1^{23}) = e_0^{22} \otimes e_1^{23} + e_1^{23} \otimes e_0^{21} + z e_1^{32} \otimes e_1^{13} - z^2 e_1^{22} \otimes e_1^{23}$
- $\Delta(e_1^{33}) = e_0^{22} \otimes e_1^{33} + e_1^{33} \otimes e_0^{11} + e_1^{22} \otimes e_1^{33}$

and $\Delta(e_x^{ij}) = \Delta(e_x^{ij})$. The counit and antipode are as follows:

$$\begin{array}{llll} \varepsilon(e_0^{ij}) = 1 & i,j=1,2 & \varepsilon(e_1^{ij}) = 0 & i,j=1,2,3 \\ S(e_0^{ij}) = e_0^{ji} & i,j=1,2 & S(e_1^{ij}) = z^{i-j} e_1^{\bar{j}\bar{i}} & i,j=1,2,3 \end{array}$$

where we denote $\bar{1} = 3$, $\bar{2} = 2$ and $\bar{3} = 1$.

6.2. **Attempt to construct using double groupoids.** Suppose there exists a double groupoid $T(\mathcal{P}, \mathcal{H}, \mathcal{V}, \mathcal{B})$ that could be used to define this weak Hopf algebra.

By observing the definition we can see that the set of boxes $\mathcal{B}$ must contain e_0^{ij} for $i,j = 1,2$ and e_1^{ij} for $i,j = 1,2,3$. This is because the proposed elements are used in the definition of S, Δ, ε and m.

Proposition 6.1. *There are at most* 2 *points in* $\mathcal{P}$.

Proof. By observing the algebras M_2, M_3, and map m we can observe following:

- $m(\boxed{e_0^{11}} \otimes \boxed{e_0^{12}}) = \boxed{e_0^{12}}$, so $\boxed{e_0^{11}}$ did not change result, thus $l(\boxed{e_0^{11}}) = r(\boxed{e_0^{11}}) = \mathsf{id}$.
- $m(\boxed{e_0^{12}} \otimes \boxed{e_0^{22}}) = \boxed{e_0^{12}}$, so $\boxed{e_0^{22}}$ did not change result, thus $l(\boxed{e_0^{22}}) = r(\boxed{e_0^{22}}) = \mathsf{id}$.
- $m(\boxed{e_0^{12}} \otimes \boxed{e_0^{21}}) = \boxed{e_0^{11}}$, thus they are inverse elements, so $l(\boxed{e_0^{12}}) = l(\boxed{e_0^{21}})^{-1}$ and $r(\boxed{e_0^{12}}) = r(\boxed{e_0^{21}})^{-1}$.
- $m(\boxed{e_0^{11}} \otimes \boxed{e_0^{22}}) = 0$, so $\boxed{e_0^{22}}$ and $\boxed{e_0^{11}}$ are not composable.

We can observe from Δ map, that for each element $\boxed{e_1^{ij}}$, there exist element $\boxed{e_0^{ij}}$, such $\boxed{e_0^{ij}} \| \boxed{e_1^{ij}} \neq 0$, and there exist element $\boxed{e_0^{ij}}$, such $\boxed{e_1^{ij}} \| \boxed{e_0^{ij}} \neq 0$. So in general there are 3 or 4 vertical morphisms in $\mathcal{V}$.

Thus, in $\mathcal{V}$ there are two id morphisms, so there are 1 or 2 objects in $\mathcal{P}$. $\square$

Remark 6.2. The way of constructing described here, cannot build this weak Hopf algebra with irrational coefficients. If we use generalized corner functions described in [AN06, §3.2] we still cannot construct this weak Hopf algebra:

- There are at most 2 unique points, so should be at most 2 coefficients, but there are at least 7 unique coefficients. This is a contradiction.
- If we construct weak Hopf algebra, coefficients in Δ correspond to the corner function from the right box, but in the defined decomposition of $\Delta(e_1^{12})$ there is term $ze_1^{13} \otimes e_1^{21}$, while in the decomposition of $\Delta(e_1^{22})$ there is term $-z^3 e_1^{23} \otimes e_1^{21}$. This is also a contradiction.

Remark 6.3. Other proof that this weak Hopf algebra cannot be constructed from a double groupoid, can be obtained from the ε function: In $\Delta(\boxed{e_1^{22}})$ there is term $z^4 e_1^{22} \otimes e_1^{22}$, thus $\boxed{e_1^{22}} \| \boxed{e_1^{22}} = \boxed{e_1^{22}}$, so $t(\boxed{e_1^{22}}) = b(\boxed{e_1^{22}}) = \mathsf{id}$, and this give contradicts with $\varepsilon(\boxed{e_1^{22}}) = 0$.

Corollary 6.4. *A weak Hopf algebra built from the direct sum of matrix algebras cannot be constructed from a double groupoid.* $\square$

7. Miscellaneous

In this section, we collect information on natural questions that arise after the preceding discussion.

Let T denote a double groupoid and $H(T)$ the corresponding weak Hopf algebra.

7.1. **Braidings of fusion categories coming from double groupoids.** It is natural to wonder if there is a classification of braidings on the fusion category $\mathsf{Rep}(H(T))$. This problem has been studied for T being the double groupoid corresponding to a matched pair of groups in [LYZ00, LYZ01] (see [Tak03] for a nice survey). These results were extended to the double groupoid coming from a matched pair of double groupoids in [AA05].

7.2. **Drinfeld double.** Done for matched pair of groupoids in [AA05] extending the prior work on matched pair of groups in [BGM96].

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