

# LET'S PLAY HEX: OPEN PROBLEMS

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computing science    ualberta    supported by NSERC

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## THANK YOU

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Gao, Martin Müller, Bjarne Toft, Bedir Tapkan,  
Peter Selinger, Eric Demer

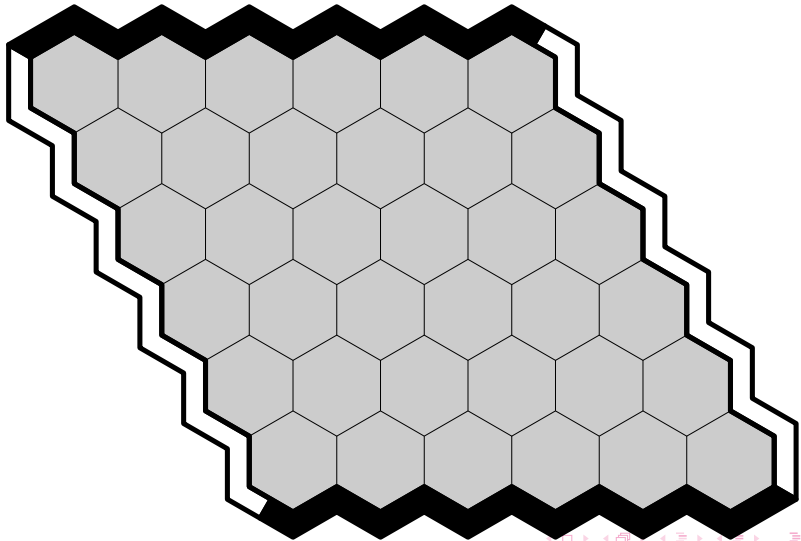
## LET'S PLAY HEX: OPEN PROBLEMS

- *Hex, A Playful Intro*

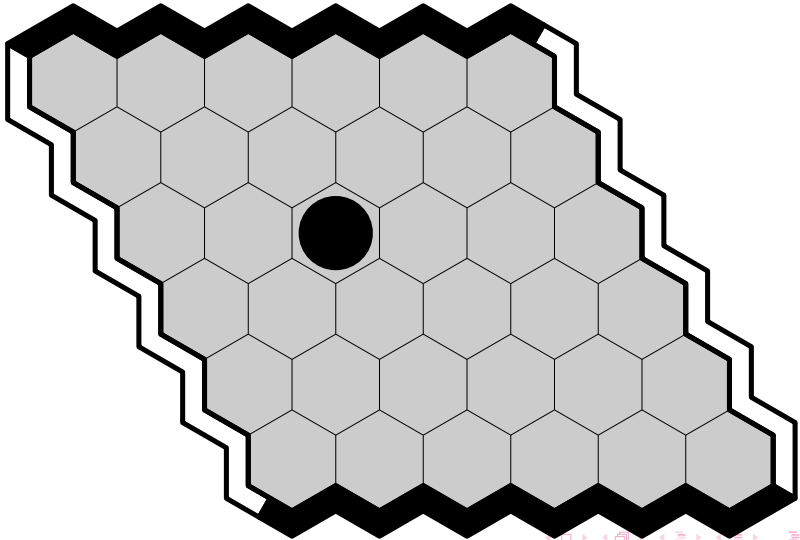
MAA 2021 (soon)

- problems: classic hex
- problems: random hex
- problems: kriegspiel hex

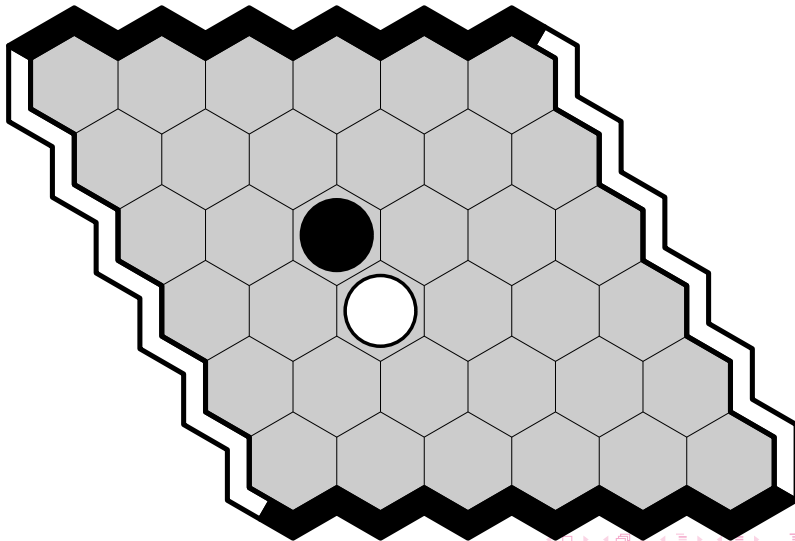
# RULES OF CLASSIC HEX



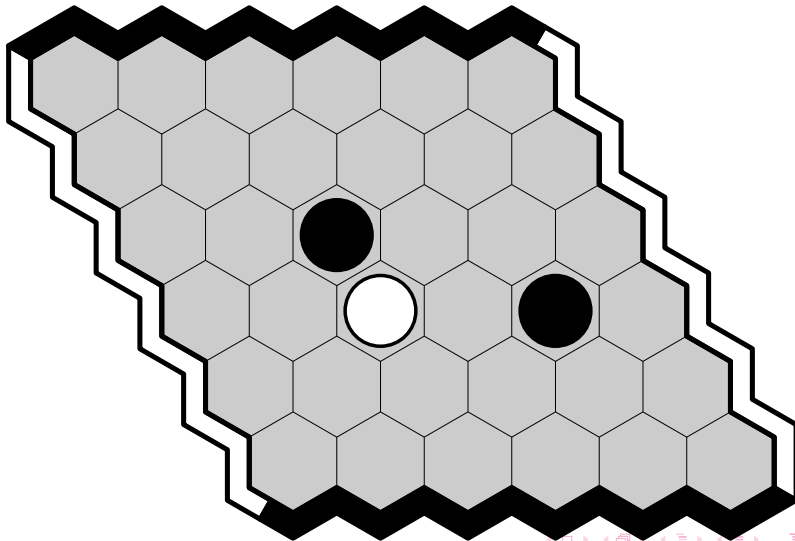
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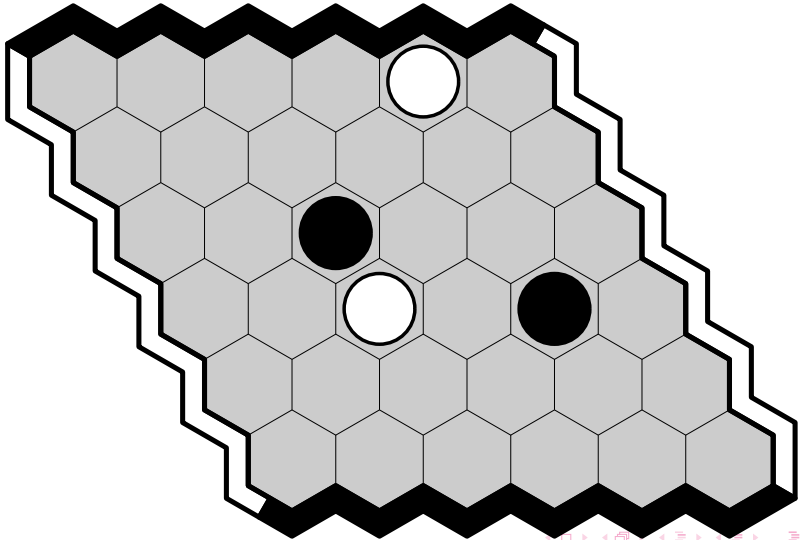
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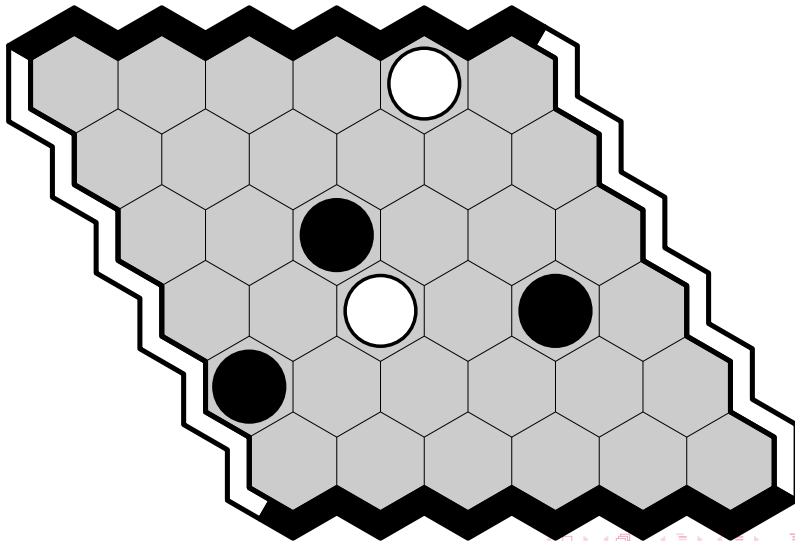
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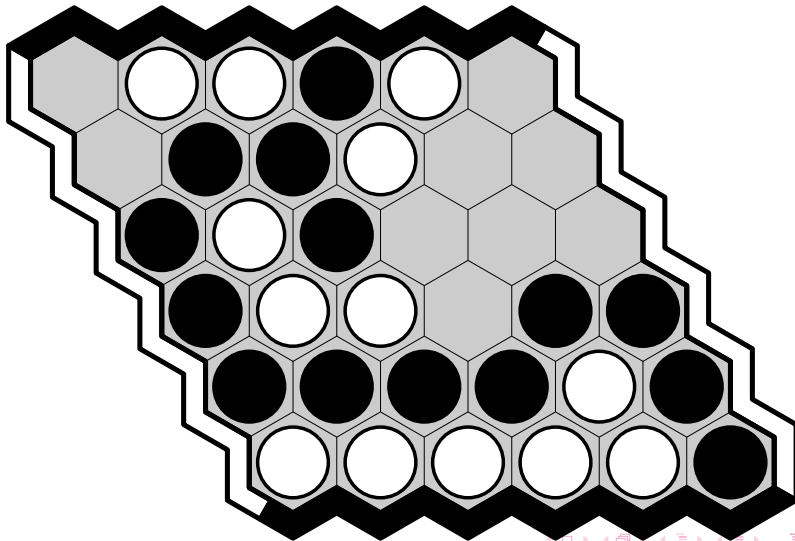
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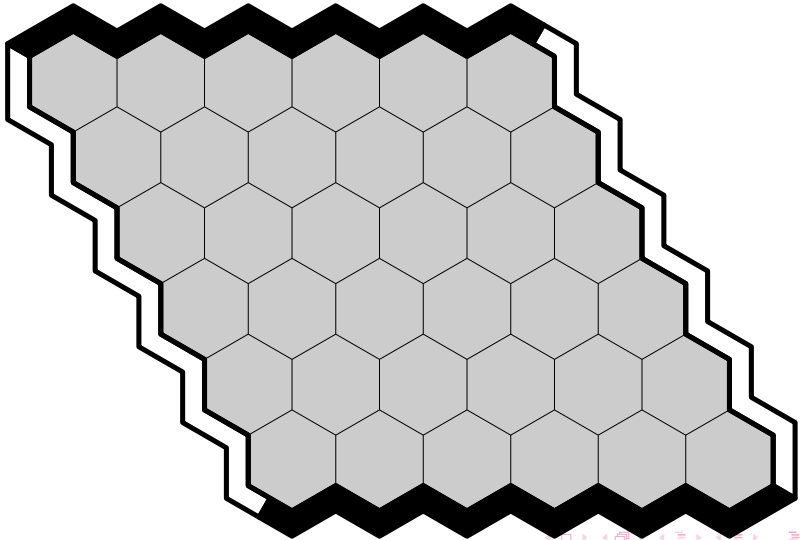
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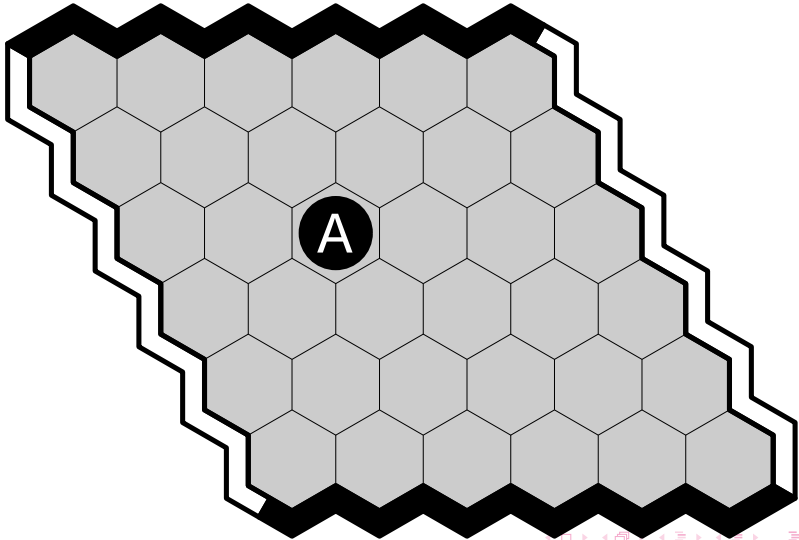
in this talk, we assume **classic Hex rules**, so  
1st player can win (easily on small boards)

when you play Hex, to balance the game,  
use the **swap rule**

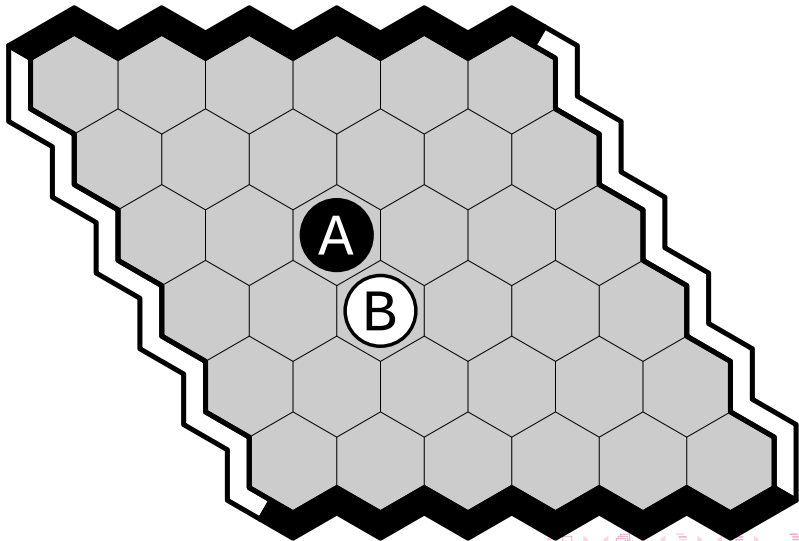
## SWAP RULE: NO-SWAP EXAMPLE



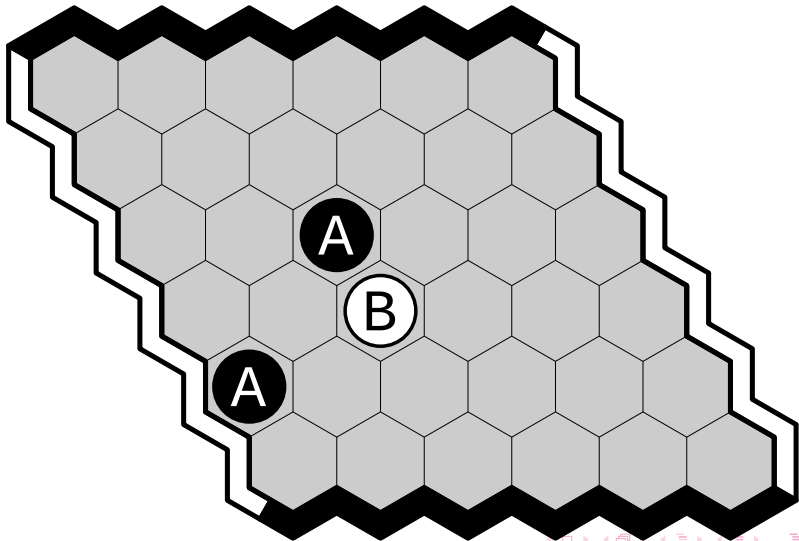
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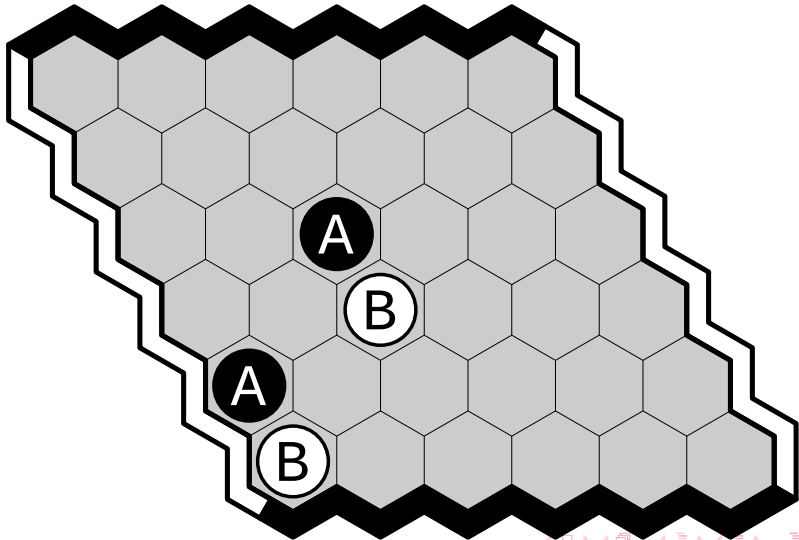
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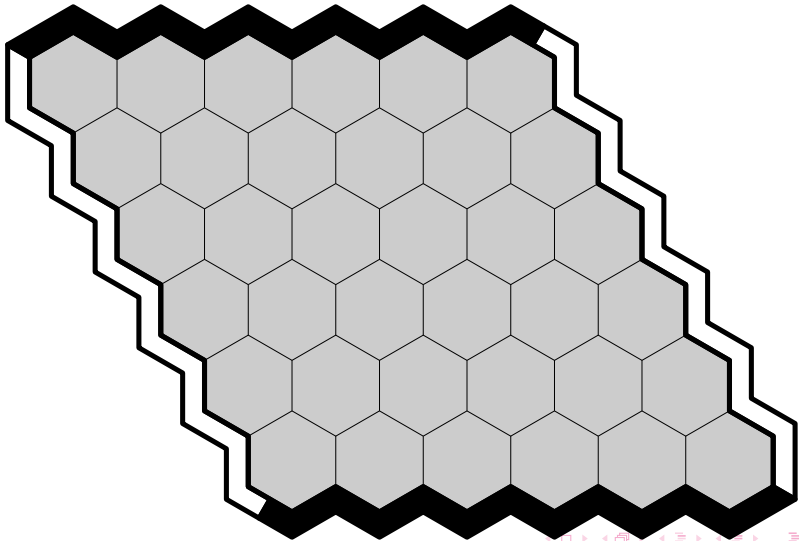
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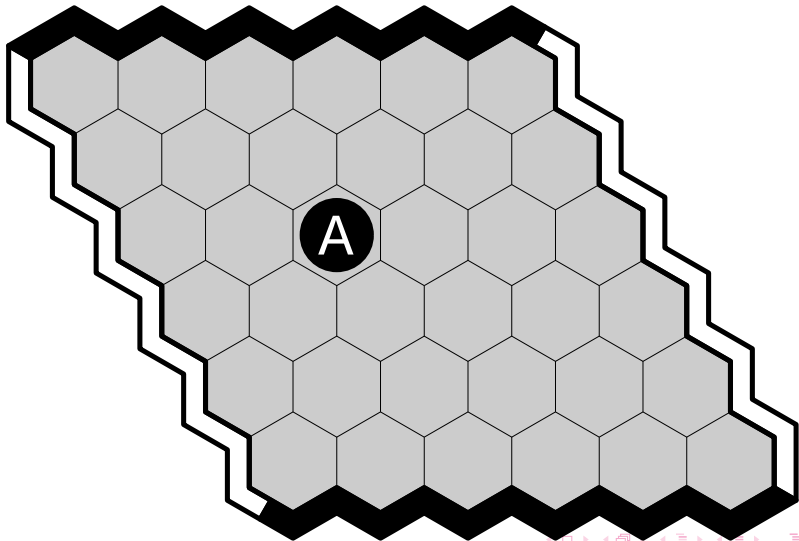
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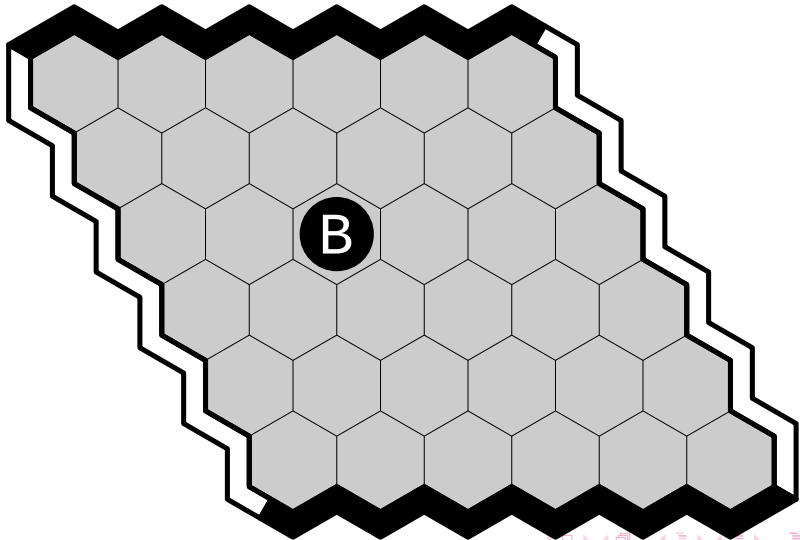
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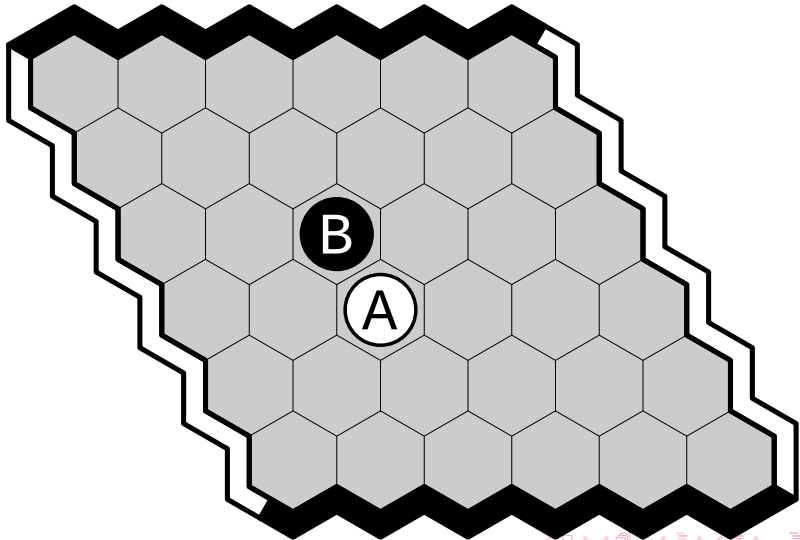
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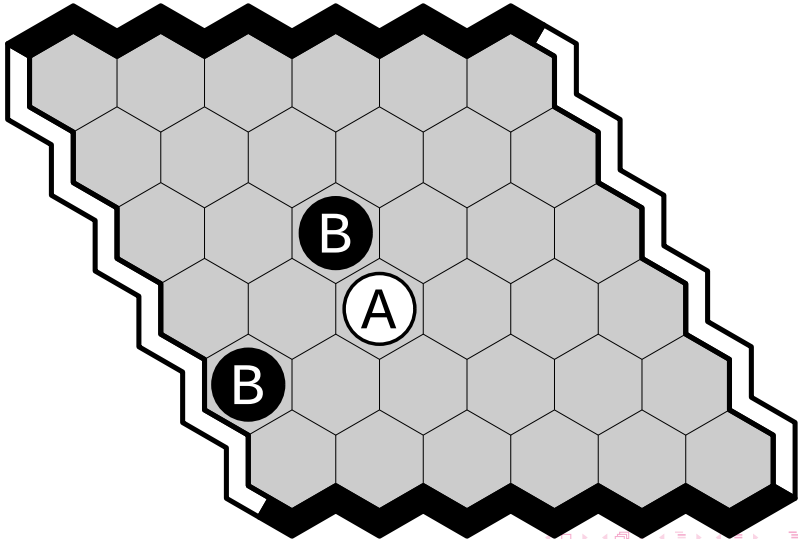
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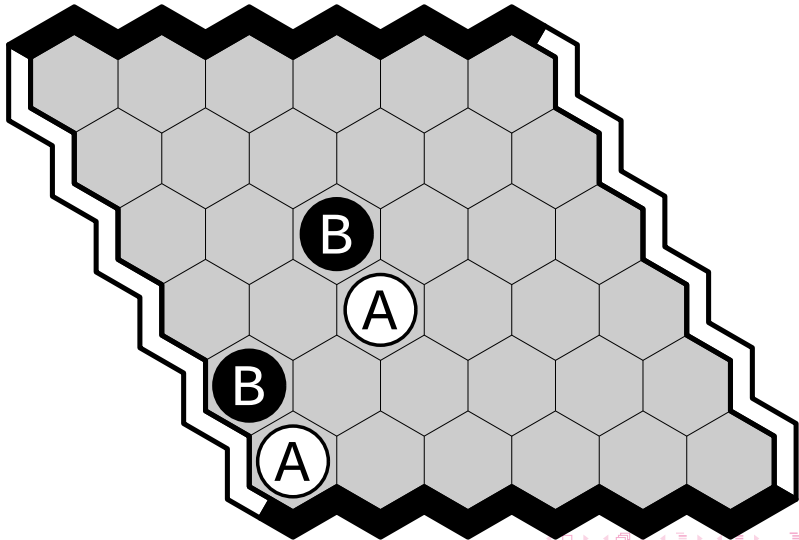
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## LEARN ABOUT HEX

[https://en.wikipedia.org/wiki/Hex\\_\(board\\_game\)](https://en.wikipedia.org/wiki/Hex_(board_game))

[http://www.mseymour.ca/hex\\_book/hexstrat.html](http://www.mseymour.ca/hex_book/hexstrat.html)

[http://www.mseymour.ca/hex\\_puzzle/hexpuzzle.html](http://www.mseymour.ca/hex_puzzle/hexpuzzle.html)

Hex, the Full Story !   H + Toft 2019

<https://webdocs.cs.ualberta.ca/~hayward/hexbook>

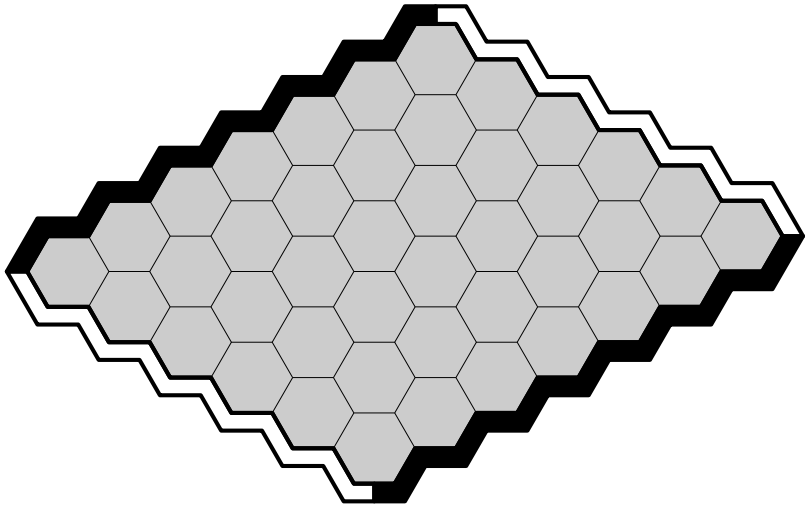
Hex, a Playful Intro !                      coming 2021

a high school / uni. level intro to math/strategy

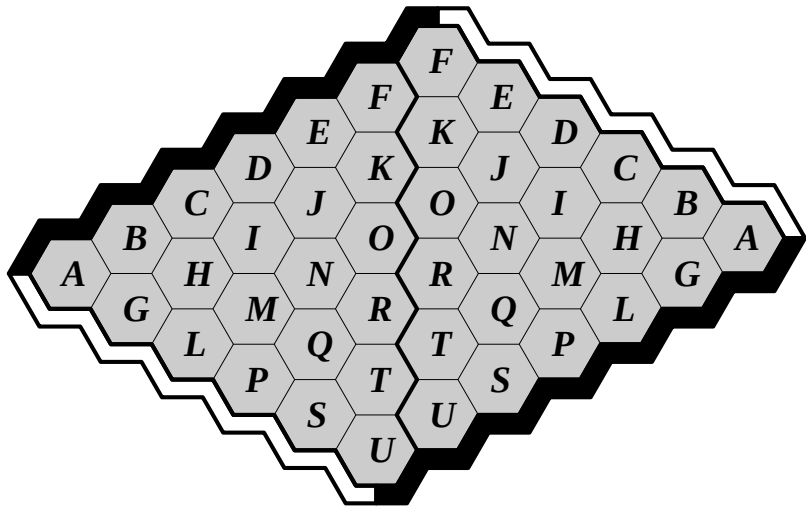
## CLASSIC Hex THEOREMS

- no draws
- $n \times n$  Hex,  $n \geq 1$ , exists 1st-player-wins strategy
- $n \times m$  Hex,  $1 \leq n < m$ ,  
exists shorter-distance-wins strategy  
(as 1st or 2nd player)
- solving (who wins?) arbitrary positions is  
P-space complete

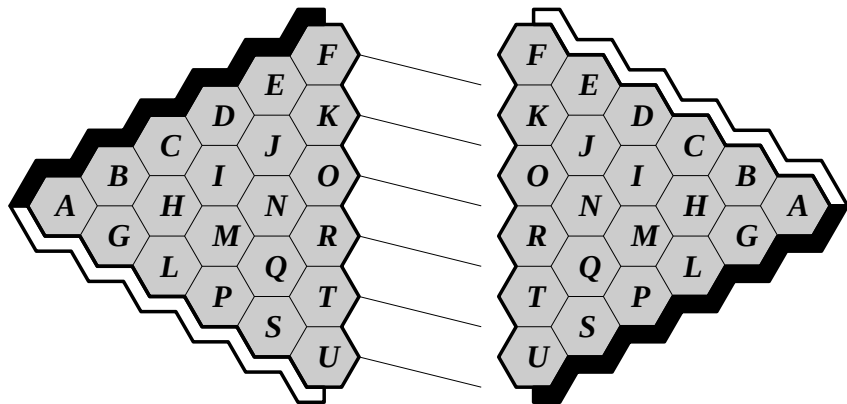
## SHORTER-DISTANCE-WINS PROOF



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## CLASSIC HEX PROBLEMS: STRATEGIES

# classic hex problems: strategies

## CLASSIC HEX PROBLEMS: STRATEGIES

for  $n \times n$  boards with  $n \geq 11$ :

find a 1st-player winning strategy ?

find a winning 1st move ?

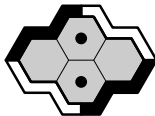
1st-move: short-diagonal-centermost always wins ?

1st-move: short-diagonal-row-2 always wins ?

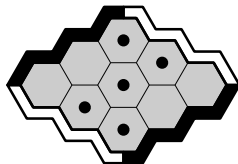
# WINNING 1ST-MOVES



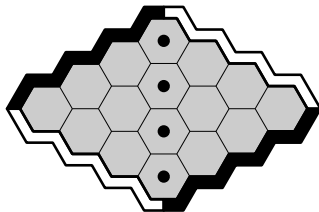
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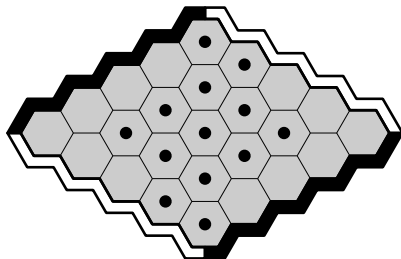
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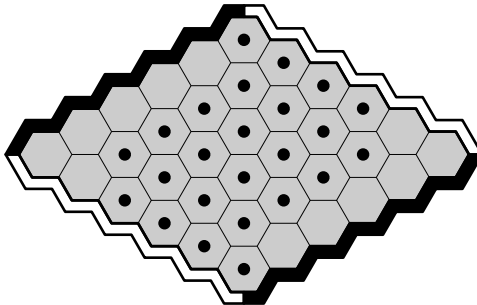
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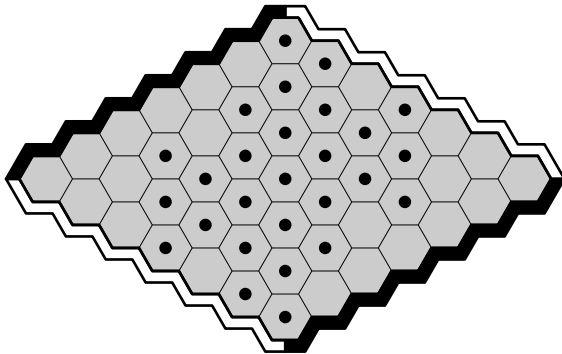
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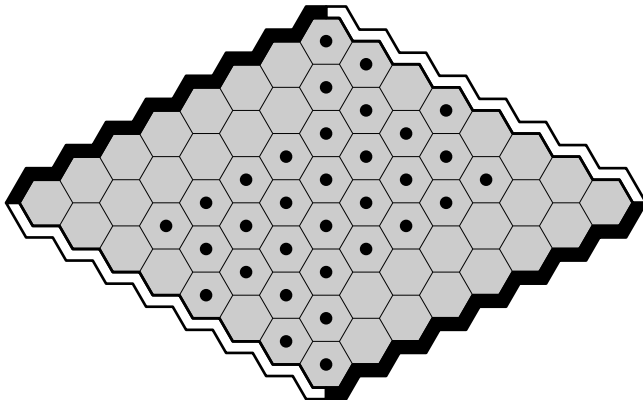
# WINNING 1ST-MOVES 1995 ENDERTON



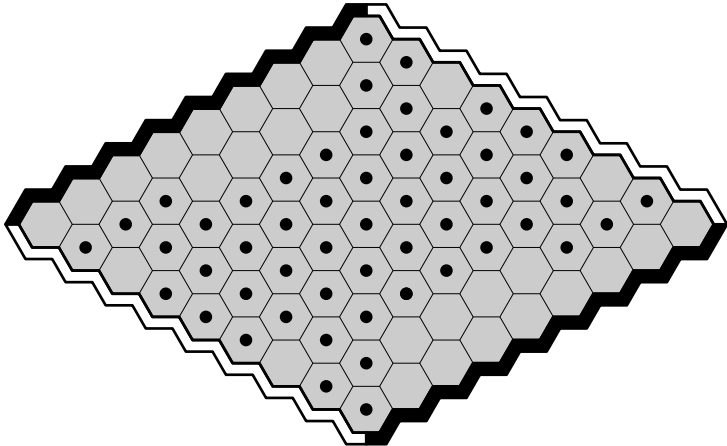
# WINNING 1ST-MOVES 2004 HBJPvR



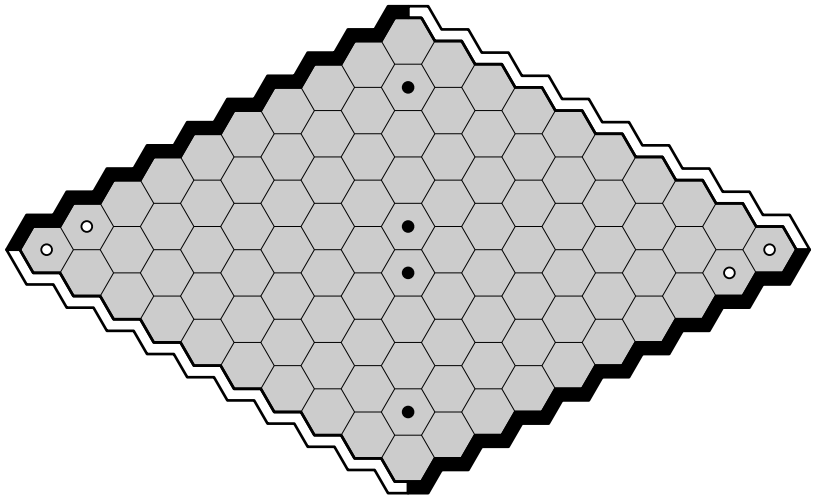
# WINNING 1ST-MOVES 2009 HAH



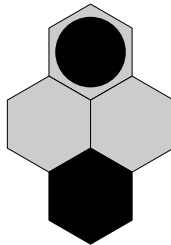
# WINNING 1ST-MOVES 2013 AHHP



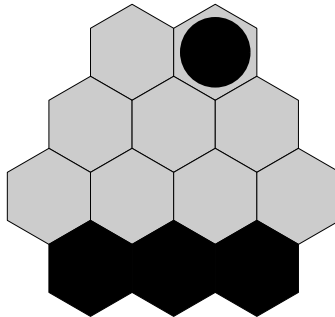
# WINNING 1ST-MOVES 2014 PH



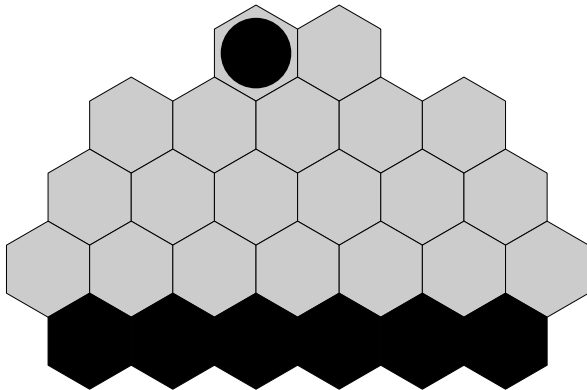
# SAFE EDGE CONNECTIONS (TEMPLATES)



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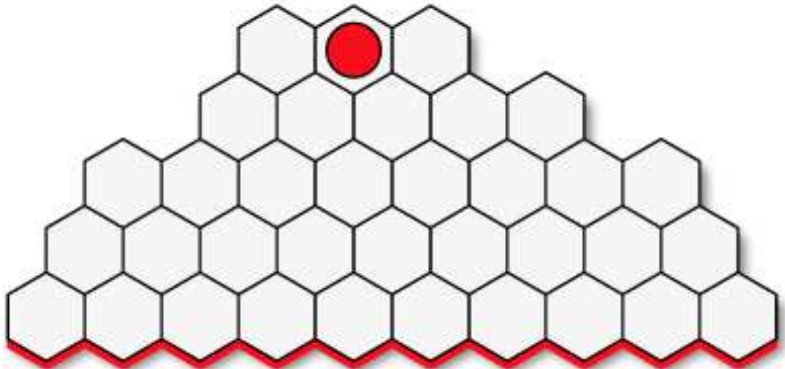


image from

[https://www.hexwiki.net/index.php/Edge\\_template\\_V1a](https://www.hexwiki.net/index.php/Edge_template_V1a)

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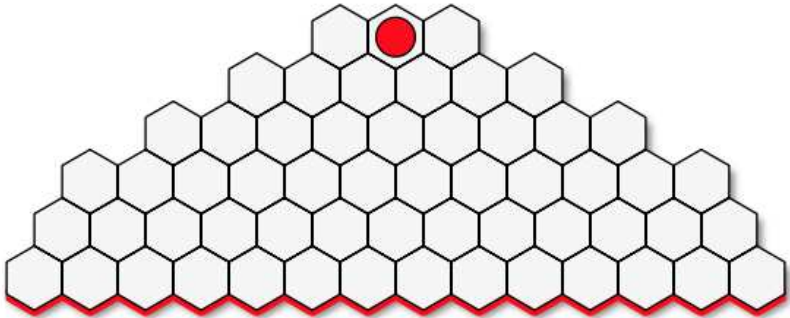


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## CLASSIC HEX PROBLEM: EDGE TEMPLATES

# classic hex problems: templates

## CLASSIC HEX PROBLEM: EDGE TEMPLATES

- for some  $n \geq 7$ ,

is there a row- $n$  Hex edge template ?

# RANDOM HEX RULES

**random hex**

# RANDOM HEX RULES

- one or both players plays uniform randomly (each legal move equi-probable)
- Random vs Random
- Random vs Deterministic

# RANDOM HEX THEOREMS

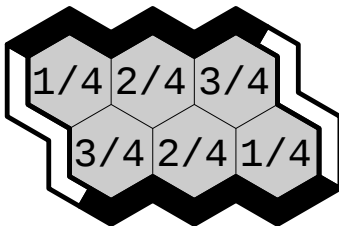
- random Hex on even  $(2t \times 2t)$  boards, 1st-player wins with probability .5
- random Hex on odd  $((2t + 1) \times (2t + 1))$  boards, 1st-player win probability:

| $n$     | 3    | 5    | 7    | 9    | 11   | 13   | 15   | 17   |
|---------|------|------|------|------|------|------|------|------|
| winrate | .667 | .573 | .544 | .531 | .523 | .518 | .515 | .513 |

- in the limit, does this probability equal  $1/2$ ?

# RANDOM-VS-DETERMINISTIC HEX

- Assume Black (random) plays 1st on 2-row 3-column board, White (deterministic, perfect) plays 2nd. For each 1st move, find expected Black winrate.



# RANDOM-VS-DETERMINISTIC HEX

for  $t \times n$  Hex assume

**Black (random, 1st player)** owns closer edges and

**White (det.)** owns further edges.

B's expected winrate ?

| $n$       | 2                 | 3 | 4 | 5 or more*      |
|-----------|-------------------|---|---|-----------------|
| B winrate | $1 - (n + 1)/2^n$ | ? | ? | $\rightarrow 0$ |

\* Barrett, Brustle, Clusiau, Narayan, Ndiaye, Reed, 2020 (unpublished)

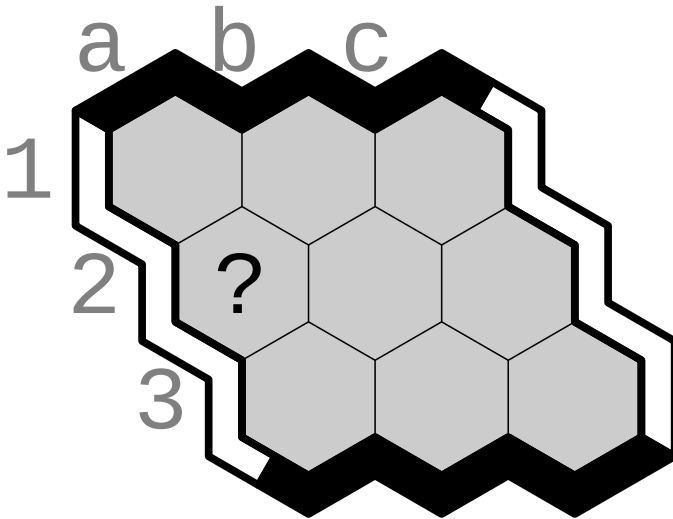
# KRIEGSPIEL HEX (AKA DARK HEX)

**dark hex**

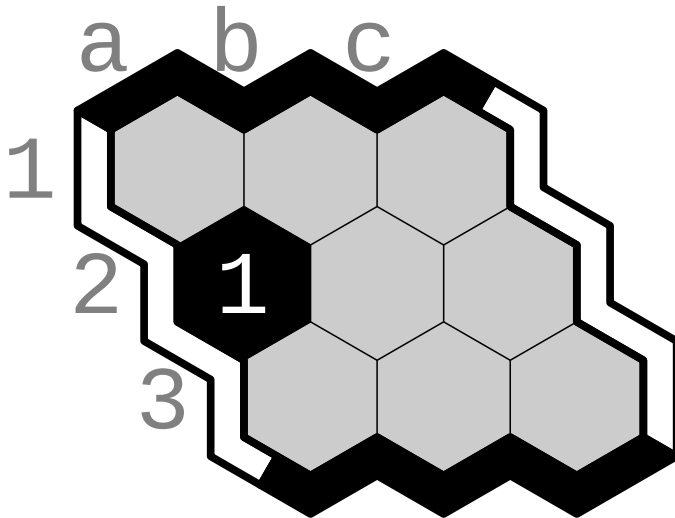
# KRIEGSPIEL HEX (AKA DARK HEX)

- usual board, usual alternating turns
- usual 2 players **but also a referee**
- each player sees only their stones
- referee sees all, reports when someone wins
- on your turn, tell referee your move choice
- referee replies privately **ok** or **no**  
(keep trying until referee says ok)

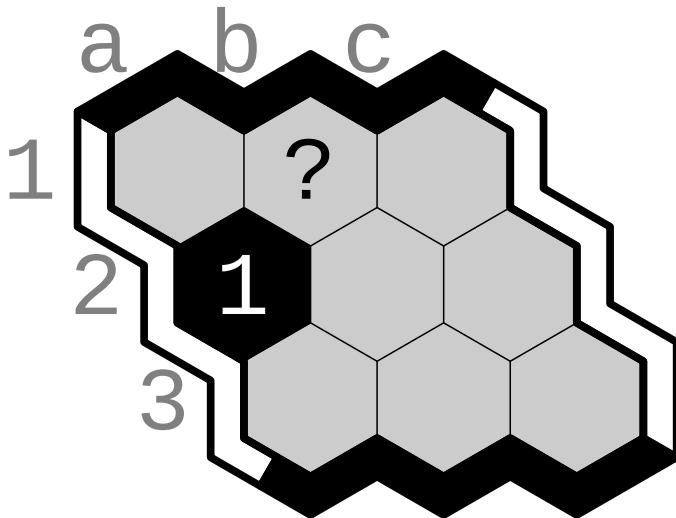
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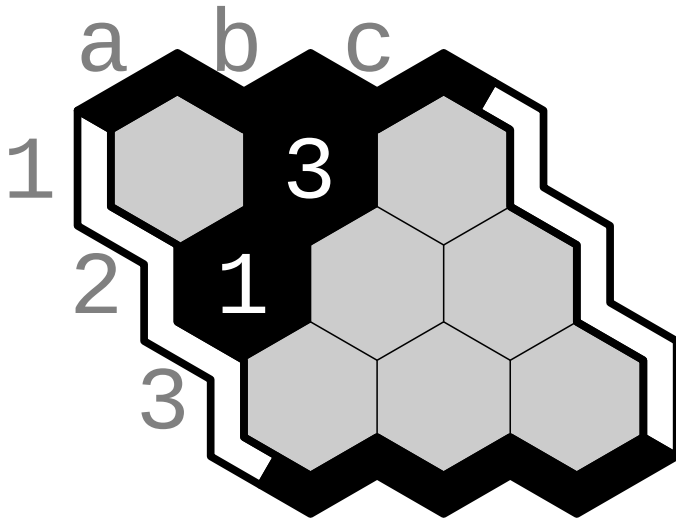
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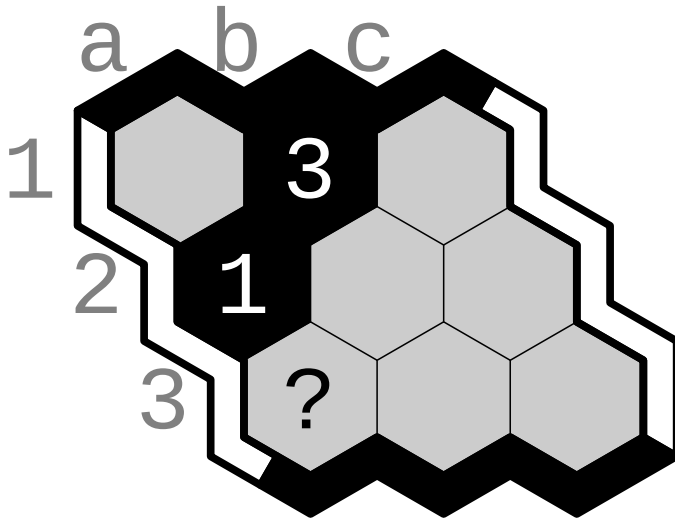
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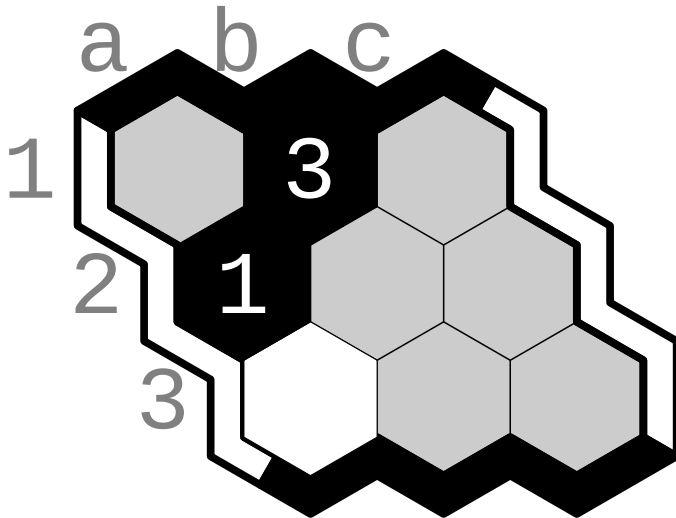
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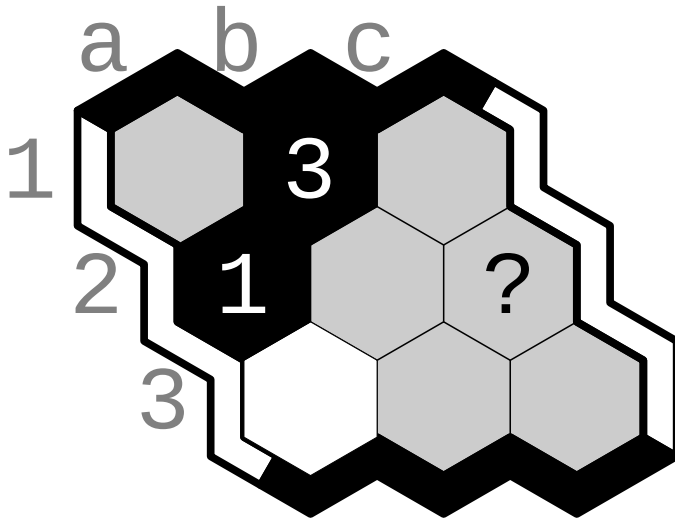
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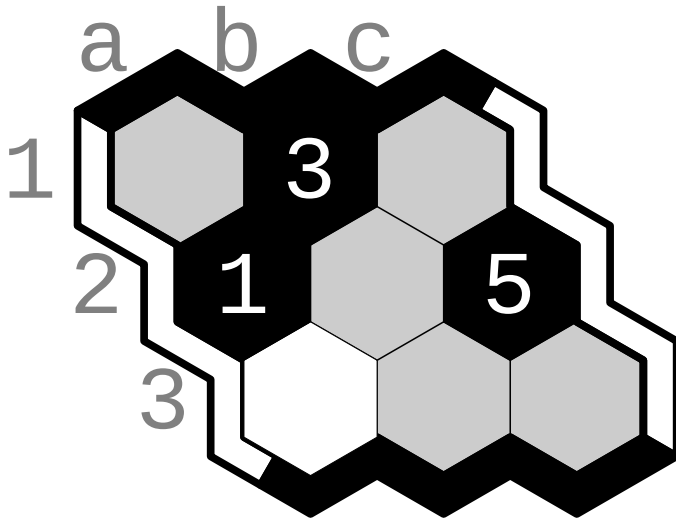
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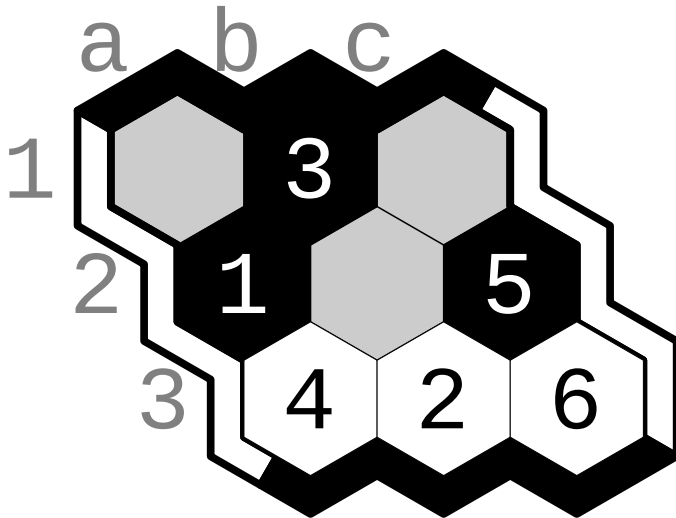
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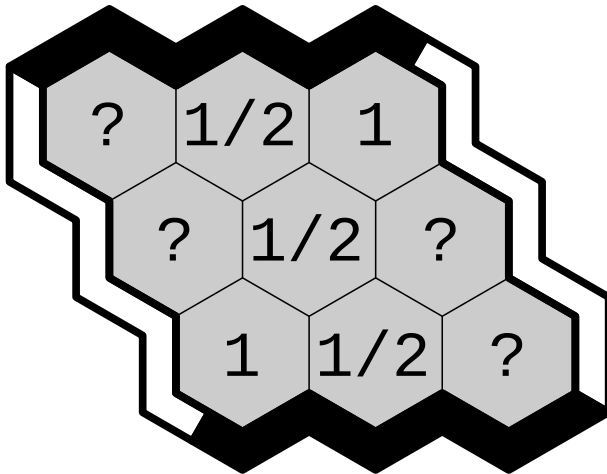


# KRIEGSPIEL HEX (AKA DARK HEX)

Black's minimax score: maximum, over all your strategies, of  
B's minimum score, over all possible opponent strategies

on the  $3 \times 3$  board, for each 1st Black move,  
what is Black's minimax score? and a best mixed strategy?

# KRIEGSPIEL HEX (AKA DARK HEX)



# KRIEGSPIEL HEX (AKA DARK HEX)

- on the  $4 \times 4$  board, for each 1st move,  
  
minimax score?  
  
a best mixed strategy?

## NOTIFICATION

to be notified when **Hex, A Playful Intro** available  
email: [hayward@ualberta.ca](mailto:hayward@ualberta.ca)  
subject: Hex book

to play a strong Hex player  
contact: [demer@mailbox.org](mailto:demer@mailbox.org)

# PAWLEWICZ H HUANG THANK YOU

