proof of alternating linear clobber conj.

hayward @ ualberta . ca

XChen TFolkersen KHasham RH DLee

ORandall LSchultz EVandermeer

Bonn Forschungsinst-Diskrete-Math

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who wins chess?

who wins chess?

- maybe 1st player ?
- maybe draw ?
- can you prove either of these ?

who wins chess?

- Zermelo's theorem 1913:
- finite alt-turn win-lose-draw game, either

  1st-player can always win, or

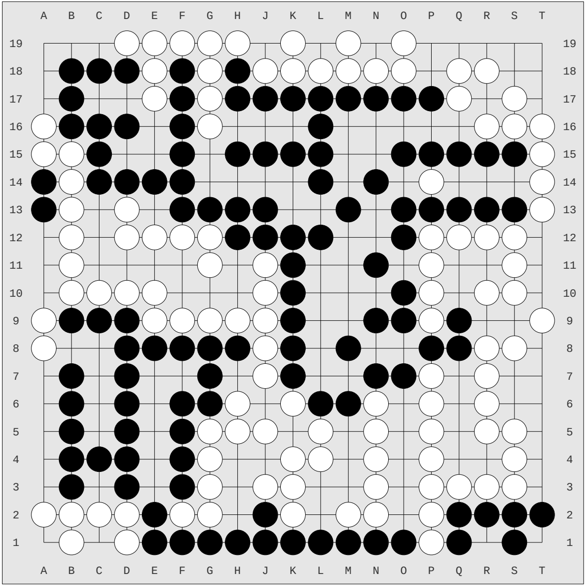
  2nd-player can always win, or

  each player can always at-least-draw.

combinatorial games history

- Nim
- Sprague-Grundy
- watching Go end games, 1970s
- Berlekamp-Conway-Guy: Winning Ways
- Conway: On Numbers and Games

Berlekamp/Wolfe 9-dan stumper



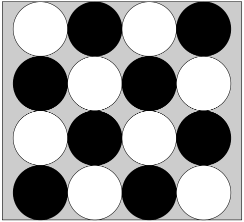
Mathematical Go:

Chilling Gets the Last Point

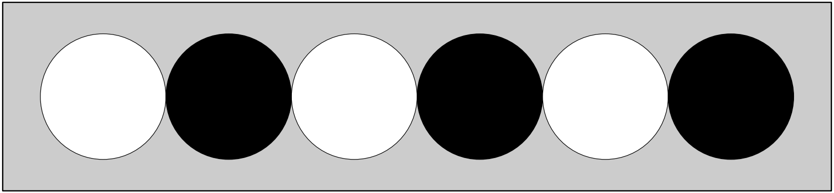
(Berlekamp/Wolfe)

2001 AGNW clobber

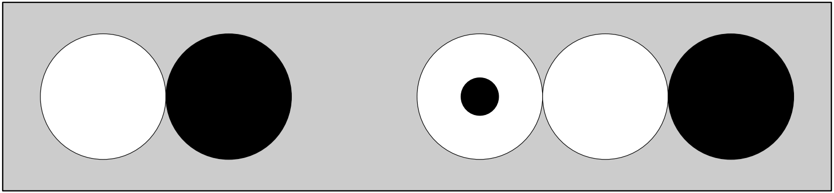
x plays first, who wins?



example Alternating Linear Clobber game



example Alternating Linear Clobber game



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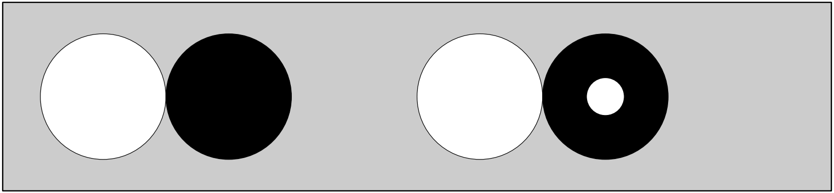

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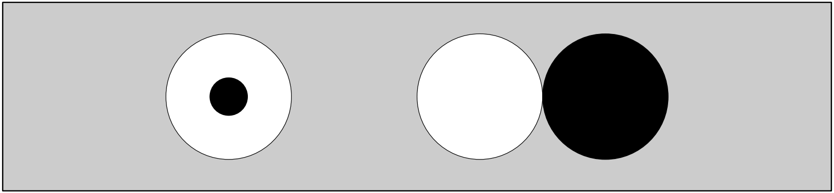
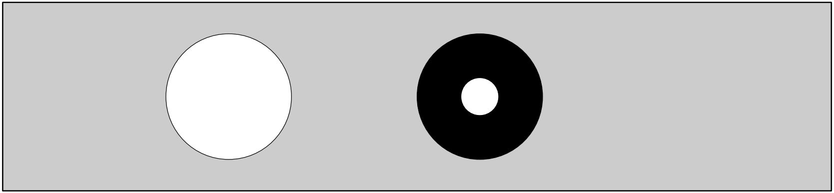
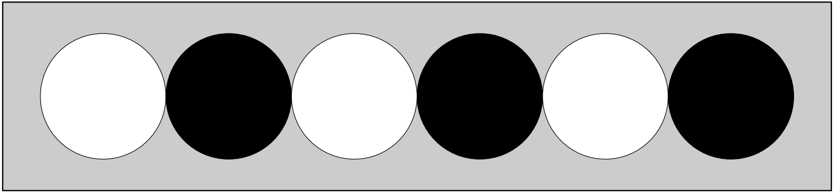


example Alternating Linear Clobber game



now White has no legal moves, White loses

play a game with your neighbour



clobber

- *intro to clobber* 2001 AGNW

  Albert Grossman Nowakowski Wolfe

  `https://webdocs.cs.ualberta.ca/~hayward/papers/AGNW.pdf`

  this talk:

  `https://webdocs.cs.ualberta.ca/~hayward/talks/cgtcv.pdf`

alternating linear clobber

- *linear clobber* clobber on a path
- *alternating linear clobber*

  start from alternating sequence

  `ox, oxox, oxoxox, oxoxoxox, ...`

conjecture

conjecture 3.2 AGNW 2004:

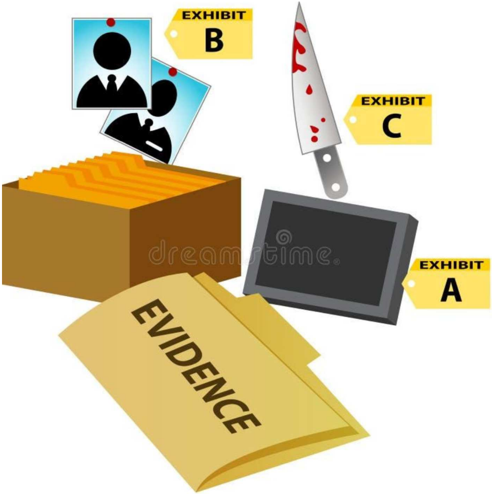
for each ALC start position except `oxoxox`,

first player can win

we prove this

how to prove this? a visual riddle

how to prove this? a visual riddle



proof by induction

combinatorial games: zero, neg, sum

- zero $\{ \mid \}$
- sum $g + h$
- negation
- equivalence $\mathrm{g} \equiv \mathrm{h} \quad \text{iff} \quad \mathrm{g} + -\mathrm{h}$ is $\mathcal{P}$-position

combinatorial games: outcome classes

- $\mathcal{P}$
- $\mathcal{L}$
- $\mathcal{R}$
- $\mathcal{N}$

combinatorial games: equality

- g = h iff game trees isomorphic
- example
- g $= \{a, b, c \,|d, e\}$
- h $= \{b, a, c \,|e, e\}$

proof sketch

- players L, R, assume L plays 1st
- $\mathcal{A}$ set of starting ALC positions
- K some games reachable from $\mathcal{A}$
- for each k in K, for each R-move $k \to h$,

  find L-move $h \to k'$ or $h \to 0$
- details: K? L-win strategy?

6 sets of parts: $\mathcal{A}$, $\mathcal{O}$, $o\mathcal{A}$

$$
\begin{array}{rcllll}
\mathcal{A} & = & \{\text{ox}, & \text{oxox}, & \text{oxoxox}, & \ldots\ \} \\
 & & \{\text{a2}, & \text{a4}, & \text{a6}, & \ldots\ \} \\
\mathcal{O} & = & \{\text{o}, & \text{oxo}, & \text{oxoxo}, & \ldots\ \} \\
 & & \{\text{o}, & \text{o3}, & \text{o5}, & \ldots\ \} \\
o\mathcal{A} & = & \{\text{o}, & \text{oox}, & \text{ooxox}, & \ldots\ \} \\
 & & \{\text{o}, & \text{oo3}, & \text{oo5}, & \ldots\ \}
\end{array}
$$

6 sets of parts: $o\mathcal{O}$, $o\mathcal{O}o$, $o\mathcal{A}x$

$$
\begin{array}{rcllll}
o\mathcal{O} & = & \{\mathsf{oo}, & \mathsf{ooxo}, & \mathsf{ooxoxo}, & \ldots\} \\
 & & \{\mathsf{oo}, & \mathsf{oo4}, & \mathsf{oo6}, & \ldots\} \\
o\mathcal{O}o & = & \{\mathsf{ooo}, & \mathsf{ooxoo}, & \mathsf{ooxoxoo}, & \ldots\} \\
 & & \{\mathsf{ooo}, & \mathsf{oo5oo}, & \mathsf{oo7oo}, & \ldots\} \\
o\mathcal{A}x & = & \{\mathsf{ooxx}, & \mathsf{ooxoxx}, & \mathsf{ooxoxoxx}, & \ldots\} \\
 & & \{\mathsf{ooxx}, & \mathsf{oo6xx}, & \mathsf{oo8xx}, & \ldots\}
\end{array}
$$

CGT names

$$\text{ox, oxo, ooxox, ooxoxoo} \quad = \quad *$$

$$\text{oxx} = \uparrow$$

$$\text{oxox} = \pm\{\uparrow, 0\}$$

$$\text{ooxo, oxoxoxoxo} \quad = \quad \uparrow *$$

$$\text{ooxoxx, oxoxoxoxoxox} \quad = \quad \text{oxox} *$$

$$\text{oxoxo} = \{\uparrow \mid *\}$$

standard form: apply after each move

o, oo, ooo, ooxx, oxoxox $\Rightarrow$ – (game 0)

oxo, ooxox, ooxoxoo $\Rightarrow$ ox

oxoxoxoxo $\Rightarrow$ ooxo

ooxoxx, oxoxoxoxoxox $\Rightarrow$ oxox ox

ooxoo $\Rightarrow$ oox

ooxo $\Rightarrow$ xxo ox

standard form

- also apply the negative rule forms

  e.g. xxox $\Rightarrow$ oox   xo

- also apply $p + -p \Rightarrow$ -   (game 0)

  e.g. xxo $+$ oox $\Rightarrow$ -

- repeatedly apply rules until no rule applies theorem 1:

all parts that arise in ALC game

in one of $\mathcal{A}$, $\mathcal{O}$, $o\mathcal{A}$, $o\mathcal{O}$, $o\mathcal{O}o$, $o\mathcal{A}x$

or their negatives.

$\mathcal{A}'$ to $o\mathcal{O}o'$

$$
\begin{array}{rcccc}
\mathcal{A}' & = & \mathcal{A} & \setminus & \{\text{a2, a4, a12}\} \\
\mathcal{O}' & = & \mathcal{O} & \setminus & \{\text{o, o3, o9}\} \\
o\mathcal{O}' & = & o\mathcal{O} & \setminus & \{\text{oo, oo4, oo6, oo8}\} \\
o\mathcal{A}' & = & o\mathcal{A} & \setminus & \{\text{o, oo3, oo5}\} \\
o\mathcal{O}o' & = & o\mathcal{O}o & \setminus & \{\text{ooo, oo7oo }\}
\end{array}
$$

$\mathcal{A}'$ to $o\mathcal{O}o'$

$$
\begin{array}{lllllll}
\mathcal{A}' & \{ & & \texttt{a8} & \texttt{a10} & & \texttt{a14} & \ldots\} \\
\mathcal{O}' & \{\texttt{o5} & & \texttt{o7} & & \texttt{o11} & \texttt{o13} & \ldots\} \\
o\mathcal{O}' & \{ & & & \texttt{oo10} & \texttt{oo12} & \texttt{oo14} & \ldots\} \\
o\mathcal{A}' & \{\texttt{oo7} & & \texttt{oo9} & \texttt{oo11} & \texttt{oo13} & \texttt{oo15} & \ldots\} \\
o\mathcal{O}o' & \{\texttt{oo5oo} & & & \texttt{oo9oo} & \texttt{oo11oo} & \texttt{oo13oo} & \ldots\}
\end{array}
$$

definition of irregular games $\mathcal{I}$

$$\mathcal{I} = \{\texttt{a2},\ \texttt{a4},\ \texttt{oo6}\}$$

$$= \{\texttt{ox},\ \texttt{oxox},\ \texttt{ooxoxo}\}$$

definition of set K

K = games g, each part in any of

$$\mathcal{O}' \quad o\mathcal{O}' \quad o\mathcal{O}o' \quad \mathcal{I} \quad \{\text{xxo}\} \quad \{\text{oo8}\} \quad \mathcal{A}' \quad o\mathcal{A}'$$

(a, b, c, d, e, f, y, z)

count vector $\mu(g)$

definition of games to be avoided Q

- $Q = \{$o5 + a2, o7 + a4, o15 + a4 + a2$\}$
- thm: $\mathcal{O}' \subseteq \mathcal{L}$
- sum $\mathcal{O}'+$ ox $\setminus Q \subseteq \mathcal{L}$
- sum $\mathcal{O}'+$ oxox $\setminus Q \subseteq \mathcal{L}$
- sum $\mathcal{O}'+$ oxox + ox $\setminus Q \subseteq \mathcal{L}$

definition of sets $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$

$\mathcal{S}$ games of $K \setminus Q$

$\mathcal{S}_0$: $a \geq c, \quad y \leq 1, \quad z = 0$ or

$a \geq c + 1, \quad y = 0, \quad z = 1$

$\mathcal{S}_1$: $y, z = 0,\ a \geq c + 1$

$\mathcal{S}_2$: $a, b, c, y, z = 0,\ e \geq 1$ and

$d = 0$ or $d + e \geq 3$

definition of set LL

- LL is set of three games

  $\{$ oo8,

  o5 $+$ oox $+$ a4 $+$ a2,

  a14 $+$ xxo $+$ a4 $+$ a2 $\}$

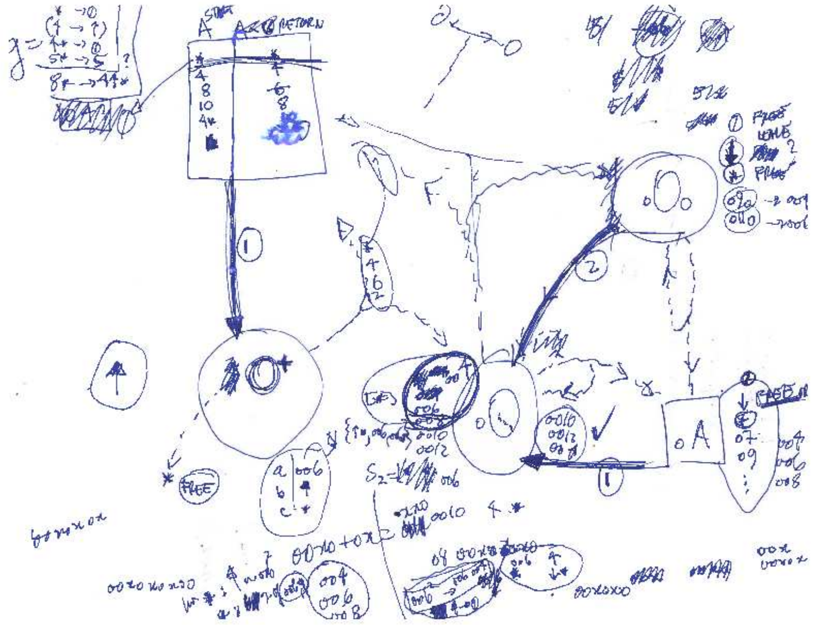
main results

- $g \in \mathcal{S}_1 \cup \mathcal{S}_2$: R-move $\to \quad \mathcal{S}_0$
- $g \in \mathcal{S}_0$: L-move $\to \quad \mathcal{S}_1 \cup \mathcal{S}_2 \cup LL \cup \{0\}$
- $\mathcal{S}_0 \subset \mathcal{L} \cup \mathcal{N}$
- $(\mathcal{S}_1 \cup \mathcal{S}_2) \subset \mathcal{L}$
- $\mathcal{A} \setminus \{\texttt{oxoxox}\} \subset \mathcal{N}$

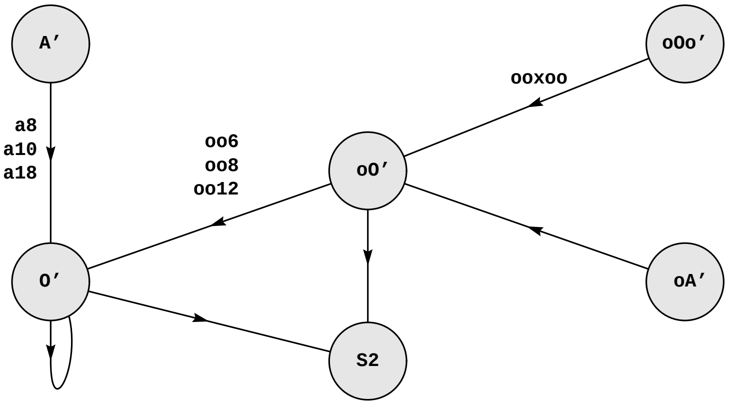
how L wins on $\mathcal{A} \setminus \{$oxoxox$\}$: outline

- play any part in $\mathcal{A}' \cup o\mathcal{O}'$ to $\mathcal{O}'$
- play any part in $o\mathcal{O}o' \cup o\mathcal{A}'$ to $o\mathcal{O}'$
- avoid Q
- ignore $\mathcal{I}$ until the end

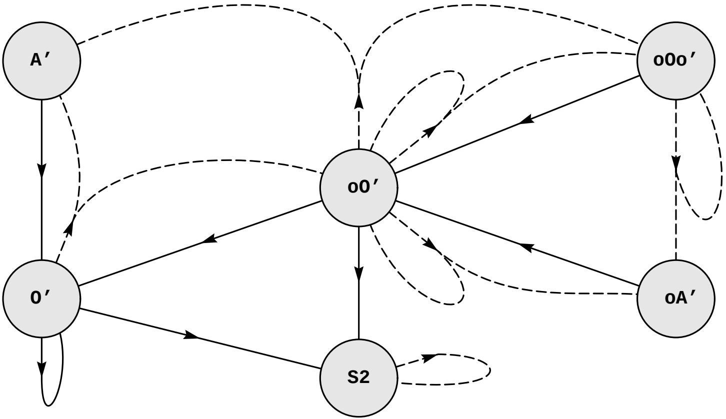
how L wins



how L wins



how L and White play



where is the devil?

where is the devil?

in the details

https://arxiv.org/abs/2509.08985

how L wins

rule	{game} or subgame	result
1	$\mathcal{A}'$ non-empty	
a	{ a8 + a2}	xxo + a4 + a2 (avoid o5 + a2)
b	{a10 + a4}	o5 + xxox + a4 (avoid o7 + a4)
c	{a18 + a4 + a2}	a14 + xxo + a4 + a2 (avoid o15 + a4 + a2)
d	any of a8, a10, a14, ...	o5, o7, o11, ...

how L wins

rule	$\{$game$\}$ or subgame	result
2	$\mathcal{A}'$ empty, $o\mathcal{A}'$ non-empty	
	oo7, oo9, $\ldots$	oo4, oo6, $\ldots$

how L wins

rule	$\{$game$\}$ or subgame	result
3	$\mathcal{A}', o\mathcal{A}'$ empty, $o\mathcal{O}o'$ non-empty	
a	$\{$o5 $+$oox$\}$	xxo $+$oox
b	o5 $+$a4 $+$a2 $+$oox	o5 $+$a4 $+$a2 $+$a2
c	a4 $+$oox	xxo $+$oox
d	oox	a2
e	oo9oo, oo11oo, ...	oo4, oo6, ...

how L wins

rule	{game} or subgame	result
4	$\mathcal{A}', o\mathcal{A}', o\mathcal{O}o'$ empty, $o\mathcal{O}'$ non-empty	
a	{oo10 +a2}	o7 +a2 +a2 (avoid o5 +a2)
b	{oo12 +a4}	oo8 +xxo +a4 (avoid o7 +a4)
c	oo10	o5
d	oo12	o7
e	oo14	o11 +a2
f	oo16	o11
g	oo18	o13 (avoid o15 +a4 +a2)
h	oo20	o17 a2 (avoid o15 +a4 +a2)
i	oo22, oo24, ...	o17, o19, ...

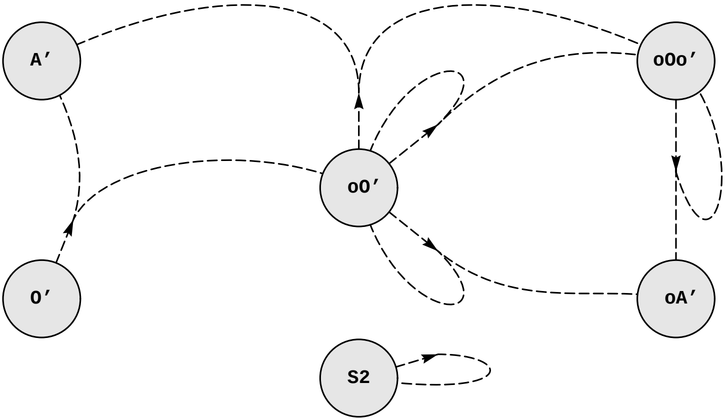
how L wins

rule	{game} or subgame	result
5	$\mathcal{A}'$, $o\mathcal{A}'$, $o\mathcal{O}o'$, $o\mathcal{O}'$ empty, $\mathcal{I}$ non-empty	
a	oo6 $+$oo6 $+$a4	xxo $+$oo6 $+$a4
b	oo6 $+$oo6 $+$a2	xxo $+$oo6 $+$a2
c	oo6 $+$oo6	ooxo $+$oo6 $=$ xxo $+$oo6 $+$a2
d	oo6 $+$a4 $+$a2	xxo $+$a4 $+$a2
e	oo6 $+$a4	ooxo $+$a4 $=$ xxo $+$a4 $+$a2
f	oo6 $+$a2	ooxo $+$a2 $=$ xxo
g	a4 $+$a2	a2 $+$a2
h	oo6	xxo
i	a4	xxo
j	a2	0

how L wins

rule	{game} or subgame	result
6	$\mathcal{A}'$, $o\mathcal{A}'$, $o\mathcal{O}o'$, $o\mathcal{O}'$, $\mathcal{I}$ empty	
a	o13, o15, ...	o11, o13 ...
b	o11	o7 xxo
c	o7	o5
d	o5	xxo
7	$\mathcal{A}'$, $o\mathcal{A}'$, $o\mathcal{O}o'$, $o\mathcal{O}'$, $\mathcal{I}$, $\mathcal{O}'$ empty	
a	oo8	ooxo xxo
b	xxo	0

how R can move



e.g. how R can move on oxoxoxoxo

can assume R clobbers to right

(oxoxoxoxo has left-right symmetry)

oxoxoxoxo $\rightarrow$ _ooxoxoxo

oxoxoxoxo $\rightarrow$ ox_ooxoxo

oxoxoxoxo $\rightarrow$ oxox_ooxo

oxoxoxoxo $\rightarrow$ oxoxox_oo

example game

a26	ox. . . oxoxoxoxoxoxox
L $\rightarrow$ o23	ox. . . oxoxoxoxoxo xx
R $\rightarrow$ a12 oo10	oxoxoxoxoxox ooxoxoxoxo
st.form a4 a2 oo10	oxox ox ooxoxoxoxo
L $\rightarrow$ a4 a2 o7	oxox ox oxoxoxo
R $\rightarrow$ o3 a2 o7	oox ox oxoxoxo
L $\rightarrow$ a2 a2 o7	ox ox oxoxoxo

example game (cont.)

L $\rightarrow$ a2 a2 o7	ox ox oxoxoxo
st.form o7	oxoxoxo
R $\rightarrow$ a4	oxox oo
L $\rightarrow$ xxo	o xxo
R $\rightarrow$ a2	xo
L $\rightarrow$ 0	x

thank you

email hayward@ualberta.ca