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Resonant three-wave interactions in non-linear hyper-elastic fluid-filled tubes

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1. Introduction

Nonlinear interactions play a central role in the development of wave spectra in a dispersive medium. In the early stages of the spectral energy transfer, the resonant wave-wave interactions are important (see, for example, Craik (1985)). The principle objective of the present paper is to show that it is possible for a triad of resonantly interacting dispersive waves to exist in nonlinear hyper-elastic fluid-filled tubes.

The study of wave propagation in elastic tubes is of interest particularly with regard to its application to, among others, pulse propagation in blood vessels (see Pedley, 1980). There is an abundant literature, both theoretical and applied, on wave propagation in compliant tubes. While a thorough review is beyond the scope of this article, we point out that most studies that have focussed on the *dispersive* properties of the waves (e.g., Rubinow and Keller, 1971, 1978; Moodie *et al.*, 1984, 1986 and Moodie and Barclay, 1986), have tended to ignore nonlinearity, and on the other hand studies on the role of *nonlinearity* in pulse propagation have generally adopted a low wavenumber approximation and thereby ignored the effects of dispersion (e.g., Moodie and Haddow, 1976; Anliker *et al.*, 1971 and Seymour and Mortell, 1973).

Johnson (1970) and Cowley (1982, 1983) have shown that in the weakly nonlinear and weakly dispersive limit, pulse propagation in elastic tubes is governed by a (perturbed) K-dV equation. In *large* amplitude pulse propagation the analytic problem is very difficult and numerical solutions are generally required (e.g. Elad *et al.*, 1984). Finally, we point out that in the theory of *collapsible* tubes, the effects of nonlinearity, non-axisymmetry, and dispersion (associated with longitudinal tension and bending moments) appear to be very important (e.g., Flaherty *et al.*, 1972; Kececioglu *et al.*, 1981 and McClurken *et al.*, 1981). In contrast to – and indeed complementing – the above studies, the wave-wave interaction equations derived here will correspond to the propagation of *small-but-finite amplitude strongly dispersive* waves in compliant elastic tubes.

We will consider only axi-symmetric deformations of an incompressible homogeneous fluid-filled elastic tube. The tube wall will be assumed to be a nonlinear membraneous shell which is axially tethered so as to prevent axial motion during deformation. To provide a degree of generality to our analysis the azimuthal and longitudinal resultant stresses in the tube wall will not be specifically given but rather, following Cowley (1982, 1983), will be determined by a *strain-energy* functional denoted $W(\lambda_1, \lambda_2)$ where λ_1 and λ_2 are the principal stretches in the azimuthal and longitudinal directions, respectively (see (2.6), (2.7a) and (2.7b)). For a general account of the nonlinear theory of deformed cyclindrical shells see Green and Zerna (1954).

The plan of the paper is as follows. In Section 2.1 the non-dimensional problem is formulated. In Section 2.2, the appropriate asymptotic solution is constructed and the wave-wave interaction equations are derived using the method of multiple-scales. In Section 3 various properties and special solutions to the interaction equations are presented. Section 4 summarizes the paper and outlines possible future research.

2. Formulation of the governing equations and derivation of the interaction equations

2.1. Problem Formulation

We begin by assuming that the inviscid, homogeneous fluid within the tube is perturbed by an axi-symmetric disturbance and that the tube wall can be described in terms of a homogeneous, membraneous hyper-elastic material which is tethered so as to prevent along tube (i.e., longitudinal) wall motion. Inertial effects in the tube wall are neglected. Cowley (1982) has shown that this approximation is valid even if the fluid is unsteady provided

$$\frac{\varrho_m H}{\varrho a_0} \ll 1 , \tag{2.1}$$

where ϱ, ϱ_m, a_0, H are the fluid and wall densities, and undeformed wall radius and thickness, respectively.

Under the above approximation the nonlinear dimensional equation describing the fluid are given by

$$(r^* u^*)_{x^*} + (r^* v^*)_{r^*} = 0 , \tag{2.2}$$

$$u^*_{t^*} + u^* u^*_{x^*} + v^* u^*_{r^*} + \frac{1}{\varrho} p^*_{x^*} = 0 , \tag{2.3}$$

$$v^*_{t^*} + u^* v^*_{x^*} + v^* v^*_{r^*} + \frac{1}{\varrho} p^*_{r^*} = 0 , \tag{2.4}$$

with the wall boundary conditions

$$v^* = a^*_{t^*} + u^* a^*_{x^*} \quad \text{on} \quad r^* = a^*, \tag{2.5a}$$

$$p^* = \pi^*(x^*, t^*) \quad \text{on} \quad r^* = a^*, \tag{2.5b}$$

where x^*, r^*, u^*, v^* and p^* are the longitudinal and radial coordinates, longitudinal and radial velocities, and fluid pressure, respectively. The time-dependent radial position of the tube wall is denoted $a^*(x^*, t^*)$, and $\pi^*(x^*, t^*)$ is the pressure drop across the tube wall due to the wall elasticity.

It can be shown for axi-symmetric cyclindrical membraneous shells (Green and Zerna, 1954; in the present context see also Cowley, 1982, 1983), that

$$\pi^*(x^*, t^*) = \frac{H}{a^*(1+e)} \frac{\partial W^*}{\partial \lambda_1} - \frac{H}{a^*} \frac{\partial}{\partial x^*} \left[\frac{a_0 a^*_{x^*}}{(1+(a^*_{x^*})^2)^{1/2}} \frac{\partial W^*}{\partial \lambda_2} \right], \tag{2.6}$$

where subscripts indicate differentiation, and where λ_1 and λ_2 are the azimuth and longitudinal stretches, respectively, i.e.,

$$\lambda_1 \equiv a^*/a_0, \tag{2.7a}$$

$$\lambda_2 \equiv (1+e)(1+(a^*_{x^*})^2)^{1/2}, \tag{2.7b}$$

where e is the imposed axial pre-strain due to the tethering force. The dimensional strain-energy function is denoted $W^*(\lambda_1, \lambda_2)$.

In the small-amplitude low-wavenumber limit the first term in (2.6) is the hoop stress which is responsible for the non-dispersive Korteweg-Moens wave velocity for pressure pulses. Similarly, the second term is principally responsible for the dispersive effects associated with longitudinal tension. Cowley (1982) analyzed the propagation characteristics of weakly nonlinear and weakly dispersive solutions to the above set of equations. Here, we focus on *strongly* dispersive but weakly nonlinear disturbances in order to study the resonant wave-wave interactions. There is abundant experimental evidence that dispersion is an essential feature of wave propagation in fluid-filled distensible elastic tubes.

Equations (2.2)–(2.6) are put into nondimensional form by defining the nondimensional (unasterisked) variables

$$\begin{aligned} (r^*, x^*) &= a_0(r, x), \quad & (u^*, v^*) &= \varepsilon c_*(u, v), \\ p^* &= \varrho c_*^2 p, & W^* &= W_* W(\lambda_1, \lambda_2), \\ \pi^*(x^*, t^*) &= \hat{\pi}(x, t), & t^* &= (a_0/c_*)\, t, \\ a^* &= a_0(1 + \varepsilon \varphi), \end{aligned} \tag{2.8}$$

where c_* is the Korteweg-Moens wave velocity given by

$$c_*^2 = \frac{H W_*}{a_0 \varrho},$$

and ε is a non-dimensional amplitude parameter. It will be assumed that $0 < \varepsilon \ll 1$.

Substitution of (2.8) into the governing equations will imply that the nonlinear wave-wave interactions will occur over a space-time scale of $0(\varepsilon^{-1})$. Thus the auxiliary slow space and time scales

$$T = \varepsilon t\,, \quad X = \varepsilon x \tag{2.9}$$

will be introduced. Consequently, nondimensional derivatives will be re-written

$$\partial_x \to \partial_x + \varepsilon \partial_X\,, \tag{2.10a}$$

$$\partial_t \to \partial_t + \varepsilon \partial_T\,. \tag{2.10b}$$

Substitution of (2.8), (2.9) and (2.10) into (2.2)–(2.6) yields the nondimensional problem

$$(ru)_x + (rv)_r = -\varepsilon(ru)_X\,, \tag{2.11a}$$

$$u_t + \frac{1}{\varepsilon} p_x = -\varepsilon u u_x - \varepsilon v u_r - p_X - \varepsilon u_T + 0(\varepsilon^2)\,, \tag{2.11b}$$

$$v_t + \frac{1}{\varepsilon} p_r = -\varepsilon u v_x - \varepsilon v v_r - \varepsilon v_T + 0(\varepsilon^2)\,, \tag{2.11c}$$

with the Taylor expanded (about $\varepsilon = 0$) boundary conditions,

$$p = \pi_0 + \varepsilon \pi_1 + \varepsilon^2 \pi_2 - \varepsilon \varphi p_r - \tfrac{1}{2}\varepsilon^2 \varphi^2 p_{rr} + 0(\varepsilon^3)\,, \tag{2.12a}$$

$$v - \varphi_t = \varepsilon \varphi_T + \varepsilon u \varphi_x - \varepsilon \varphi v_r + 0(\varepsilon^2)\,, \tag{2.12b}$$

evaluated at $r = 1$. The functions π_0, π_1 and π_2 are the first three terms in the small-amplitude Taylor expansion of $\hat{\pi}(x, t)$ given by, respectively,

$$\pi_0 \equiv (1+e)^{-1}\, W_1^0\,, \tag{2.13}$$

$$\pi_1 \equiv (1+e)^{-1}\,(W_{11}^0 - W_1^0)\,\varphi - W_2^0\, \varphi_{xx}\,, \tag{2.14}$$

$$\pi_2 \equiv (1+e)^{-1}\,(W_1^0 - W_{11}^0 + W_{111}^0/2)\,\varphi^2 \tag{2.15}$$

$$- W_{12}^0 (\varphi_x)^2/2 - 2 W_2^0\, \varphi_{xX} \tag{2.16}$$

$$- (W_{21}^0 - W_2^0)\, \varphi\, \varphi_{xx}\,,$$

where

$$W_{i_1 \ldots i_n}^0 \equiv \partial^{(n)} W(1, 1+e)/\partial\lambda_{i_1} \ldots \partial\lambda_{i_n}\,,$$

with

$$\lambda_1 \equiv 1 + \varepsilon \varphi \tag{2.17}$$

$$\lambda_2 \cong (1+e)\,(1 + \varepsilon^2 (\varphi_x)^2/2) + 0(\varepsilon^3)\,. \tag{2.18}$$

2.2. Asymptotic Solution

The nonlinear problem (2.11)–(2.18) can be solved with a straight-forward asymptotic expansion of the form

$$p = \pi_0 + \varepsilon p^{(0)}(r, x, t; X, T) + \varepsilon^2 p^{(1)}(r, x, t; X, T) + \ldots\,, \tag{2.19a}$$

$$u = u^{(0)}(r, x, t; X, T) + \varepsilon u^{(1)}(r, x, t; X, T) + \ldots\,, \tag{2.19b}$$

$$v = v^{(0)}(r, x, t; X, T) + \varepsilon v^{(1)}(r, x, t; X, T) + \dots, \tag{2.19c}$$

$$\varphi = \varphi^{(0)}(x, t; X, T) + \varepsilon\, \varphi^{(1)}(x, t; X, T) + \dots. \tag{2.19d}$$

Substitution of (2.19) into the governing equations gives the $0(1)$ problem

$$(r p_r^{(0)})_r + r p_{xx}^{(0)} = 0 \tag{2.20}$$

with

$$\mathscr{L} p_r^{(0)} + p_{tt}^{(0)} = 0 \quad \text{on} \quad r = 1, \tag{2.21}$$

where $\mathscr{L}$ is the operator given by

$$\mathscr{L} \equiv (1 + e)^{-1}(W_{11}^0 - W_1^0) - W_2^0\, \partial_{xx}^2. \tag{2.22}$$

The $0(1)$ velocity field and tube wall displacement will be determined by

$$u_t^{(0)} = -p_x^{(0)}, \tag{2.23}$$

$$v_t^{(0)} = -p_r^{(0)}, \tag{2.24}$$

$$\varphi_{tt}^{(0)} = p_r^{(0)}(r = 1). \tag{2.25}$$

The linearity of the $0(1)$ problem allows the superposition of three dispersive waves in the form

$$p^{(0)} = \sum_{n=1}^{3} A_n(X, T)\, I_0(|k_n| r) \exp(i k_n x - i\omega_n t) + \text{c.c.}, \tag{2.26}$$

where (c.c) denotes complex conjugate, and where *each* wavenumber/frequency doublet (k_n, ω_n) satisfies the dispersion relationship

$$\omega_n^2 = |k_n|\, I_1(|k_n|)\, [W_{11}^0 - W_1^0)/(1 + e) + W_2^0 k_n^2]/I_0(|k_n|), \tag{2.27}$$

where $I_i(*)$ denotes the modified Bessel function of the first kind of order i.

The amplitude of each wave in the triad is a function of the slow space/time variables. It is the evolution of $A_i(X, T)$ and its coupling to the other two wave packets in the triad which determines the wave-wave energy transfer. The dynamics of the $A_i(X, T)$ are determined by secularity conditions in the $O(\varepsilon)$ problem.

It follows from (2.23), (2.24), (2.25) and (2.26) that

$$u^{(0)} = \sum_{n=1}^{3} k_n A_n(X, T)\, I_0(|k_n| r) \exp(i\theta_n)/\omega_n + \text{c.c.}, \tag{2.28}$$

$$v^{(0)} = \sum_{n=1}^{3} |k_n| A_n(X, T)\, I_1(|k_n| r) \exp(i\theta_n - i\pi/2)/\omega_n + \text{c.c.}, \tag{2.29}$$

$$\varphi^{(0)} = \sum_{n=1}^{3} |k_n| A_n(X, T)\, I_1(|k_n| r) \exp(i\theta_n)/\omega_n^2 + \text{c.c.}, \tag{2.30}$$

where $\theta_n \equiv k_n x - \omega_n t$ is the rapidly varying phase of the n-th wave packet.

The $0(\varepsilon)$ problem for $p^{(1)}(r, x, t; X, T)$ can be put into the form

$$(r p_r^{(1)})_r + r p_{xx}^{(1)} = H_0(p^{(0)}) \tag{2.31}$$

with the boundary condition

$$\mathscr{L} p_r^{(1)} + p_t^{(1)} = \mathscr{L} F_1(p^{(0)}) + \partial^2 F_0(p^{(0)})/\partial t^2 \quad \text{on} \quad r = 1\,. \tag{2.32}$$

The nonlinear operators $H_0(p^{(0)})$, $F_0(p^{(0)})$ and $F_1(p^{(0)})$ are given by, respectively,

$$\begin{aligned} H_0(p^{(0)}) \equiv {} & - r[u^{(0)} u_x^{(0)} + v^{(0)} v_r^{(0)} + p_X^{(0)} + u_T^{(0)}]_x \\ & - [r(u^{(0)} v_x^{(0)} + v^{(0)} v_r^{(0)} + v_T^{(0)})]_r - r p_{xX}^{(0)}\,, \end{aligned} \tag{2.33}$$

$$\begin{aligned} F_0(p^{(0)}) \equiv {} & (1+e)^{-1} (W_1^0 - W_{11}^0 + W_{111}^0/2)\, \varphi^{(0)2} - \varphi^{(0)} p_r^{(0)} \\ & - W_{12}^0 (\varphi_x^{(0)})^2/2 - 2 W_2^0 \varphi_{xX}^{(0)} - (W_{21}^0 - W_2^0)\, \varphi^{(0)} \varphi_{xx}^{(0)}\,, \end{aligned} \tag{2.34}$$

$$F_1(p^{(0)}) \equiv - \varphi_{tT}^{(0)} - (u^{(0)} \varphi_x^{(0)})_t + (\varphi^{(0)} v_r^{(0)})_t - u^{(0)} v_x^{(0)} - v^{(0)} v_r^{(0)} - v_T^{(0)}\,, \tag{2.35}$$

with $u^{(0)}$, $v^{(0)}$, and $\varphi^{(0)}$ understood to be implicit functions of $p^{(0)}$ via (2.23), (2.24) and (2.25), respectively. Substitution of $p^{(0)}$ into $H_0(p^{(0)})$, $F_0(p^{(0)})$ and $F_1(p^{(0)})$ allows them to be expressed in the respective forms,

$$\begin{aligned} H_0(p^{(0)}) \equiv {} & - 2ir \sum_{n=1}^{3} k_n I_0(|k_n| r) A_{nX} \exp(i\theta_n) \\ & + \sum_{n=1}^{3} \sum_{m=1}^{3} \gamma_{nm}(r; X, T) A_n^* A_m^* \exp(-i\theta_n - i\theta_m) \\ & + \sum_{n=1}^{3} \sum_{m=1}^{3} \bar{\gamma}_{mn}(r; X, T) A_n A_m^* \exp(i\theta_n - i\theta_m) + \text{c.c.}\,, \end{aligned} \tag{2.36}$$

$$\begin{aligned} \{\mathscr{L} F_1(p^{(0)}) + \partial^2 F_0(p^{(0)})/\partial t^2\}_{r=1} \equiv {} & \sum_{n=1}^{3} [2iI_0(|k_n|)\, \omega_n A_{nT} \\ & + 2W_2^0 k_n |k_n| i I_1(|k_n|) A_{nX}] \exp(i\theta_n) \\ & + \sum_{n=1}^{3} \sum_{m=1}^{3} \nu_{nm} A_n^* A_m^* \exp(-i\theta_n - i\theta_m) \\ & + \sum_{n=1}^{3} \sum_{m=1}^{3} \bar{\nu}_{nm} A_n A_m^* \exp(i\theta_n - i\theta_m) + \text{c.c.}\,, \end{aligned} \tag{2.37}$$

where $(\cdot)^*$ is the complex conjugate of $(\cdot)$ and where the interaction coefficients in (2.36) and (2.37) are given by, respectively,

$$\begin{aligned} \gamma_{nm} \equiv {} & k_n k_m^2 (k_n + k_m)\, r I_0(|k_n| r)\, I_0(|k_m| r)/(\omega_n \omega_m) \\ & - |k_n k_m| k_m (k_n + k_m)\, r I_1(|k_n| r)\, I_1(|k_m| r)/(\omega_n \omega_m) \\ & - k_n k_m |k_m| [r I_0(|k_n| r)\, I_1(|k_m| r)]_r/(\omega_n \omega_m) \\ & + |k_m k_n k_m| [r \dot{I}_1(k_m r)\, I_1(k_n r)]_r\,, \end{aligned} \tag{2.38}$$

$$
\begin{aligned}
v_{nm} \equiv & -(W_1^0 - W_{11}^0 + W_{111}^0/2)(\omega_n + \omega_m)^2 \,|k_n k_m|\, I_1(|k_n|)\, I_1(|k_m|)/ \\
& \cdot [(1+e)\,\omega_n^2\,\omega_m^2] - W_{12}^0(\omega_n + \omega_m)^2 \,|k_n k_m|\, k_n k_m\, I_1(|k_n|)\, I_1(|k_m|)/ \\
& \cdot [2\omega_n^2\,\omega_n^2] - (W_{21}^0 - W_2^0)(\omega_n + \omega_m)^2 \,|k_n k_m|\, k_m^2\, I_1(|k_n|)\, I_1(|k_m|)/ \\
& \cdot [\omega_n^2\,\omega_m^2] + (\omega_n + \omega_m)^2 \,|k_n k_m|\, I_1(|k_n|)\, I_1(|k_m|)/\omega_n^2 \\
& \cdot [(1+e)^{-1}(W_{11}^0 - W_1^0) + W_2^0(k_n + k_m)^2] \\
& \cdot \{|k_m|\, k_n k_m\, I_0(|k_m|)\, I_1(|k_n|)/[\omega_n\,\omega_m] + |k_n|\, k_m^2\, I_1(|k_n|)\, \dot{I}_1(|k_m|) \\
& - k_n k_m\, |k_m|\, (\omega_n + \omega_m)\, I_0(|k_n|)\, I_1(|k_m|)/(\omega_n\,\omega_m^2) \\
& - |k_n|\, k_m^2(\omega_n + \omega_m)\, I_1(|k_n|)\, \dot{I}_1(|k_m|)/(\omega_n^2\,\omega_m)\}\,,
\end{aligned} \tag{2.39}
$$

and where $\bar{\gamma}_{nm}$ and $\bar{v}_{nm}$ are obtained by simply replacing (k_n, ω_n) by $-(k_n, \omega_n)$ in γ_{nm} and v_{nm}, respectively, and where $\dot{I}_i(*)$ indicates differentation with respect to the argument.

The solution for $p^{(1)}(r, x, t; X, T)$ can be written in the general form

$$
\begin{aligned}
p^{(1)} = & \sum_\mu A_\mu^{(1)}(t; X, T) \exp(i k_\mu x)\, I_0(|k_\mu| r) \\
& + \sum_\alpha \phi_\alpha(r; X, T) \exp(i\theta_\alpha) + \text{c.c.}
\end{aligned} \tag{2.40}
$$

The μ-summation term (2.40) will correspond to those (nonresonant) frequencies/wavenumber in the Kernel of (2.31) which are excited by the dynamics associated with the tube wall. The α-summation term in (2.40) corresponds to a particular solution of (2.31). It follows from (2.36) that the "α" sum will be over the wavenumber/frequency doublets defined by

$$
\begin{aligned}
(k_\alpha, \omega_\alpha) = & \{(k_i, \omega_i) : i = 1, 2, 3\} \cup \{-(k_l, \omega_l) \\
& \pm (k_m, \omega_m) : l = 1, 2, 3, \quad m = 1, 2, 3\}\,.
\end{aligned} \tag{2.41}
$$

The amplitude function $\phi_\alpha(r; X, T)$ in (2.40) is easily expressed in terms of the Green's function for (2.31) which is regular at the origin, i.e.,

$$
\begin{aligned}
\phi_\alpha(r; X, T) \equiv & -K_0(|k_\alpha| r) \int_0^r I_0(|k_\alpha| \xi)\, f_\alpha(\xi; X, T)\, \mathrm{d}\xi \\
& - I_0(|k_\alpha| r) \int_r^1 K_0(|k_\alpha| \xi)\, f_\alpha(\xi; X, T)\, \mathrm{d}\xi
\end{aligned}
$$

where $f_\alpha(\xi; X, T)$ are the coefficients of those terms in $H_0(p^{(0)})$ which have phases with frequency/wavenumber $(\omega_\alpha, k_\alpha)$.

Substitution of (2.40) into the $0(\varepsilon)$ boundary condition (2.32) implies

$$
\begin{aligned}
& \sum_\mu \{[(1+e)^{-1}(W_{11}^0 - W_1^0) + W_2^0 k_\mu^2]\, |k_\mu|\, I_1(|k_\mu|)\, A_\mu^{(1)} \\
& \quad + I_0(|k_\mu|)\, \partial_{tt}^2 A_\mu^{(1)}\} = \{\mathscr{L} F_1(p^{(0)}) + \partial_{tt}^2 F_0(p^{(0)})\}\ (r = 1) \\
& \quad - \sum_\alpha [\mathscr{L}\phi_{\alpha_r}(1; X, T) + \partial_{tt}^2 \phi_\alpha(1; X, T)] \exp(i\theta_\alpha)\,.
\end{aligned} \tag{2.42}
$$

It follows from (2.42) if *any* of the resonance conditions

$$\pm\,\theta_1 \pm \theta_2 \pm \theta_3 = 0$$

holds, the requirement that the secular terms be removed from the right-hand-side of (2.42) will lead to a wave-wave coupling in the slow space/time evolution of $A_1(X, T)$, $A_2(X, T)$ and $A_3(X, T)$. With no loss of generality we assume

$$\theta_1 + \theta_2 + \theta_3 = 0\,, \tag{2.43a}$$

i.e.,

$$k_1 + k_2 + k_3 = 0\,. \tag{2.43b}$$

$$\omega_1(k_1) + \omega_2(k_2) + \omega_3(k_3) \tag{2.43c}$$

Thus, setting the coefficients of the secular terms to zero implies

$$(\partial_T + c_1\,\partial_X)\,A_1 = i\beta_1\,A_2^*\,A_3^*\,, \tag{2.44}$$

$$(\partial_T + c_2\,\partial_X)\,A_2 = i\beta_2\,A_1^*\,A_3^*\,, \tag{2.45}$$

$$(\partial_T + c_3\,\partial_X)\,A_3 = i\beta_3\,A_1^*\,A_2^*\,, \tag{2.46}$$

where the interaction coefficients can be written in the form

$$\beta_1 = (\Gamma_{23} + \Gamma_{32})/[2I_0(|k_1|)\,\omega_1] \tag{2.47}$$

$$\beta_2 = (\Gamma_{31} + \Gamma_{13})/[2I_0(|k_2|)\,\omega_2] \tag{2.48}$$

$$\beta_3 = (\Gamma_{12} + \Gamma_{21})/[2I_0(|k_3|)\,\omega_3] \tag{2.49}$$

where Γ_{nm} is given by

$$\begin{aligned}\Gamma_{nm} \equiv \nu_{nm} - [(1+e)^{-1}\,(W_{11}^0 - W_1^0) + W_2^0(k_n + k_m)^2]\,;\\ K_1(|k_n + k_m|)\cdot\langle r I_0(|k_n + k_m|\,r)\,\gamma_{nm}\rangle\\ -(\omega_n + \omega_m)^2\,K_0(|k_n + k_m|)\,\langle r I_0(|k_n + k_m|\,r)\,\gamma_{nm}\rangle\end{aligned} \tag{2.50}$$

with $\langle f(r)\rangle$ defined as

$$\langle f\rangle \equiv \int_0^1 f(\xi)\,\mathrm{d}\xi\,.$$

The coefficients c_1, c_2, c_3 are the individual wave packet group velocities, i.e.,

$$\begin{aligned}c_n \equiv \frac{\partial\omega_n}{\partial k_n} \equiv \operatorname{sgn}(k_n)\,\omega_n[I_0^2(|k_n|) - I_1^2(|k_n|)]/\\ \cdot[2I_0(|k_n|)\,I_1(|k_n|)] + W_2^0\,k_n|k_n|\,I_1(|k_n|)/[I_0(|k_n|)\,\omega_n]\,,\end{aligned} \tag{2.59}$$

where $\operatorname{sgn}(x) = +1$ for $x \geqq 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$. Note that if the resonance conditions do not hold then the secularity conditions will imply that the amplitude of each wave packet is conserved following the group velocity and there is *no* interaction. With the secular terms removed $A_\mu^{(1)} \sim \exp(i\omega_\mu)$ where the μ-summation in (2.42) will be over the remaining non-resonant wavenumber/frequency doublets in the right-hand-side of (2.42).

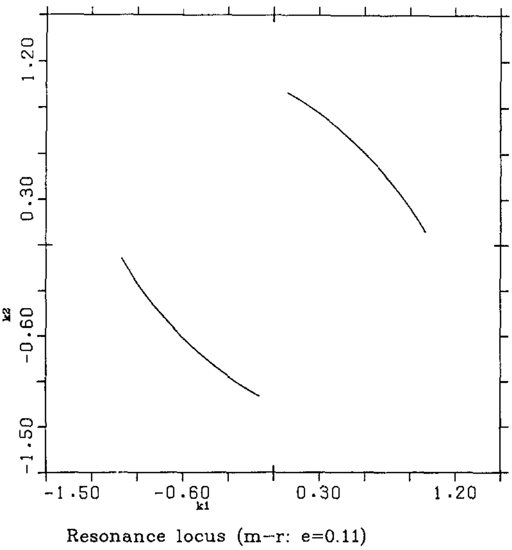

Figure 1
Resonance locus in k_1, k_2 space for the Mooney-Rivlin strain-energy function (3.1) with $a_1 \equiv a_2 \equiv 1.0$.

3. Discussion and special solutions

The existence of wavenumber triplets which can satisfy the triad conditions (2.43) is easily demonstrated by direct calculation, or by a geometrical argument. For example, Fig. 1 illustrates the nontrivial resonance locus in k_1, k_2 wavenumber space ($k_3 = -k_1 - k_2$ cf., (2.43)) assuming the tube wall is described by a Mooney-Rivlin strain-energy function in the form

$$\begin{aligned} W(\lambda_1, \lambda_2) = {} & (a_1/2)(\lambda_2^1 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3) \\ & + (a_2/2)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2 - 3) \end{aligned} \tag{3.1}$$

(Ogden, 1984), where a_1 and a_2 are nondimensional parameters, and λ_1, λ_2 are the principal stretches given by (2.17) and (2.18). Note that for a Mooney-Rivlin tube wall, the allowed wavenumber triplets have a high-wavenumber cutoff. Other strain-energy functions may, of course, imply other configurations for the resonance locus, including perhaps the non-existence of resonance triplets. We have been unable to rigorously established any necessary (or sufficient) conditions on the strain-energy function which will gurarantee the existence or non-existence of resonant triads.

Because there are no energy loss terms in our governing equations, the *total* energy must be conserved even though energy is being continuously transfered from wave to wave. The leading order nondimensional energy conservation equation for (2.11)–(2.18) can be put into the form

$$(\partial_t + \varepsilon\partial_T)E + (\partial_x + \varepsilon\partial_X)F = 0\,, \tag{3.2}$$

where E and F are leading order energy density and flux given by, respectively,

$$E \cong (1+e)^{-1}(W_{11}^{0} - W_{1}^{0})\,\varphi^2 + W_2^0 (a_x)^2 + \int_0^1 r(v^2 + v^2)\,\mathrm{d}r + 0(\varepsilon)\,, \tag{3.3}$$

$$F \cong -2W_2^0\,\varphi_t\,\varphi_z + 2\int_0^1 rup\,\mathrm{d}r + 0(\varepsilon)\,. \tag{3.4}$$

If the leading order solutions are substituted into (3.2) and the *fast-phase* oscillations are averaged out, it follows that the slowly-varying amplitudes satisfy

$$\sum_{n=1}^{3} \{\partial_T + c_n\,\partial_X\}\,[|A_n|^2\,|k_n|\,I_1(|k_n|)\,I_0(|k_n|)/\omega_n^2] = 0\,. \tag{3.5}$$

Consequently, the averaged energy of the i-th wave packet is simply

$$|A_i|^2\,|k_i|\,I_1(|k_i|)\,I_0(|k_i|)/\omega_i^2\,. \tag{3.6}$$

Equation (3.5) states simply that the sum of the energies following the wave packet envelopes is conserved. It follows from (3.5) and the interaction Eqs. (2.44), (2.45) and (2.46) that the interaction coefficients *must* satisfy

$$\sum_{n=1}^{3} |k_n|\,I_1(|k_n|)\,I_0(|k_n|)\,\beta_n/\omega_n^2 = 0\,. \tag{3.7}$$

It is tedious but straight-forward to show that the interaction coefficients satisfy (3.7). Note that it follows from (3.7) that

$$\operatorname{sgn}(\beta_1\,\beta_2\,\beta_3) = -1\,. \tag{3.8}$$

The condition (3.8) guarantees that the wave packet amplitudes remain continuously bounded for all time for the initial-value problem for sufficiently smooth initial data (Craik, 1985).

The general initial-value problem for (2.44)–(2.46) for compact initial data can be solved by the method of inverse scattering (Kaup, 1976) using the Zakharov and Manakov (1973, 1975) scattering problem. If, however, the initial wave envelopes are well separated, Kaup (1976) has shown how to treat the subsequent problem using a sequence of the much simpler Zakharov and Shabat (1972) scattering problem. We will not reproduce these details here. Excellent reviews and further details can be found in, for example, Kaup, Reiman and Bers (1979), Ablowitz and Segur (1981) or Craik (1985).

In some practical situations the interaction equations can be well approximated by retaining only the time dependence (or equivalently the space dependence, e.g. Hsieh and Mysak (1980)). It is easy to show, when the above approximation can be made, that maximum energy exchange occurs when the wave packet amplitudes are of the form

$$A_k(T) = i\alpha_k(T)\,, \tag{3.9}$$

with $\alpha_K(T)$ real-valued and where $i^2 = -1$. Substitution of (3.9) into (2.44), (2.45) and (2.46) implies

$$\alpha_{1_T} = -\beta_1 \alpha_2 \alpha_3 , \tag{3.10a}$$

$$\alpha_{2_T} = -\beta_2 \alpha_1 \alpha_3 , \tag{3.10b}$$

$$\alpha_{3_T} = -\beta_3 \alpha_2 \alpha_1 . \tag{3.10c}$$

Solutions to (3.10) may be written in terms of Jacobi elliptic functions (Bretherton, 1964). Without loss of generality we may assume that β_1, β_2 are of one sign and β_3 differs, and that at $T = 0$, $\alpha_1(0) \equiv \alpha_{10} > 0$, $\alpha_2(0) \equiv \alpha_{20} > 0$ and $\alpha_3(0) = 0$. Consequently, the solutions can be written

$$\alpha_1(T) = a_{10}\, \mathrm{d}n(\sigma T \mid m) \tag{3.11a}$$

$$\alpha_2(T) = a_{20}\, cn(\sigma T \mid m) \tag{3.11b}$$

$$\alpha_3(T) = a_{20}(-\beta_3/\beta_2)^{1/2}\, sn(\sigma T \mid m) \tag{3.11c}$$

where

$$\sigma \equiv a_{10}(-\beta_2 \beta_3)^{1/2} \tag{3.11d}$$

$$m \equiv (\beta_1/\beta_2)(a_{20}/a_{10})^2 . \tag{3.11e}$$

It is customary to assume that $0 \leqq m \leqq 1$. If in (3.11e) it turns out that $m > 1$ the following transformations (Abramowitz and Stegun, 1965) can be used:

$$\mathrm{d}n(* \mid m) = cn(m^{1/2} * \mid m^{-1})$$

$$cn(* \mid m) = \mathrm{d}n(m^{1/2} * \mid m^{-1})$$

$$sn(* \mid m) = sn(m^{1/2} * \mid m^{-1})/m^{1/2} .$$

Assuming $0 \leqq m \leqq 1$ the period of the energy transfer is easily computed to be

$$T_p \equiv 2(-\beta_2 \beta_3)^{1/2} K(m)/a_{10} \tag{3.12}$$

where $K(m)$ is a complete elliptic integral of the first kind with parameter m.

Since we have assumed that $\alpha_3(0) = 0$, the solutions (3.11) describe the following process: during $0 < T < T_p/2$ the third wave extracts energy from the first two and during the last half of the cycle re-deposits it back again. Note that there is a singular limit $m \to 1$ for which $T_p \to +\infty$ and the amplitudes are described by hyperbolic functions.

There also exist steadily-propagating periodic and "explosively unstable" solutions to the interactions Eqs. (2.44), (2.45) and (2.46) (Case and Chiu, 1977). These are obtained by constructing solutions of the form

$$A_i \equiv A_i(X - \mu T)$$

for $i = 1, 2, 3$ with μ a common constant propagation velocity. If $c_i - \mu$ is one sign for $i = 1, 2, 3$, then it is known that the solutions develop a singularity in finite time (Craik and Adam, 1978). On the other hand if $c_i - \mu$ are of different sign bounded periodic solutions in terms of elliptic functions can be constructed similar to those obtained previously.

Other approximations are also of interest. In particular the "pump-wave approximation" which describes the situation in which one of the three wave packets in the triad has an initially large amplitude relative to the other two. For this configuration it is possible to assume that the large amplitude wave packet, say A_1, remains relatively constant (at least initially) in comparison to the rapid development of $A_2(X, T)$ and $A_3(X, T)$ and, consequently, the dynamics for $A_2(X, T)$ and $A_3(X, T)$ is linear. The wave packet with amplitude A_1 is called the pump wave.

The analysis is straight-forward and thus we only briefly present the main results for (2.44), (2.45) and (2.46). Under the pump-wave approximation, (2.44), (2.45) and (2.46) reduce to

$$(\partial_T + c_2\,\partial_X)\,A_2 = i\beta_2\,A_3^*\,A_{10}^* \tag{3.13}$$

$$(\partial_T + c_3\,\partial_X)\,A_3 = i\beta_3\,A_2^*\,A_{10}^* \tag{3.14}$$

where A_{10} is the constant pump wave amplitude.

Equations (3.13) and (3.14) can be combined to an equation of the form

$$[\partial_\tau^2 - \partial_\chi^2 - \operatorname{sgn}(\beta_2\,\beta_3)]\,A_i = 0 \tag{3.15}$$

where $i = 2, 3$, and where the *real* variables τ and χ are given by, respectively,

$$\tau \equiv |A_{10}|\,|\beta_2\,\beta_3|^{1/2}\,T \tag{3.16a}$$

$$\chi \equiv 2|A_{10}|\,|\beta_2\,\beta_3|^{1/2}\,[X - (c_2 + c_3)\,T/2]/(c_2 - c_3)\,. \tag{3.16b}$$

When $\operatorname{sgn}(\beta_2\,\beta_3) = -1$, (3.15) is a Klein-Gordon equation and the initial-value problem may be solved by Fourier integrals (e.g., Whitham, 1974). If $\operatorname{sgn}(\beta_2\,\beta_3) = +1$, (3.15) is the *telegraph* equation and the initial-value problem may be solved by Riemann's method (see, e.g., Craik and Adam, 1978).

4. Summary

It has been shown that small-but-finite amplitude dispersive waves in non-linear hyper-elastic fluid-filled tubes can form a resonant triad. The three-wave interaction equations (*TWI*) were derived using a multiple-scales method. The total energy associated with the propagating wave packets is conserved, although there is a continuous process of energy transfer between the three waves. For a Mooney-Rivlin strain-energy functional there is a $0(1)$ nondimensional high wavenumber cut-off for the allowed triads. This implies that the wave-wave interactions described here may be of importance in pressure pulse with characteristic wavelengths on the order of the tube radius and longer. We expect, therefore, that resonant energy exchange will be of some importance in the evolution of pressure pulse spectra in fluid-filled elastic tubes.

Some important approximations to the *TWI* equations and their corresponding solutions were also discussed. In some practical situations only the temporal or spatial evolution of the wave packet amplitudes need be considered. For this configuration bounded, periodic solutions written in terms of Jacobi elliptic functions are well known (Bretherton, 1964). Under a pump wave approximation, it is possible to derive a linear Klein-Gordon or telegraph equation to describe the initial evolution of the (non-pump) wave packet amplitudes (Craik and Adam, 1978). There also exist steadily-traveling (with a common propagation speed) bounded periodic and explosively unstable solutions to the *TWI* equations (Case and Chiu, 1977).

In many practical configurations, however, the tube wall may have variable mechanical properties in the axial direction or a small resonance mis-match may occur. Thus the question of how the interaction process is modified by the presence of perturbations is of interest. Kaup (1976 b) has developed a singular perturbation theory for the Zakharov-Shabat scattering transform. This method will be useful for investigating the interaction of perturbed localized well-separated wave packets. Similarly, other effects have been neglected here: visco-elasticity in the tube wall and fluid viscosity are two and, of course, we have not retained any azimuthal dependence in our solutions. These questions need to be answered before the role of nonlinear wave-wave interactions in pulse propagation in elastic fluid-filled tubes is completely understood.

References

[1] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform,* S.I.A.M. Philadelphia, 1981.

[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions,* Dover, New York 1965.

[3] M. Anliker, R. L. Rockwell and E. Odgen, *Nonlinear analysis of flow pulses and shock waves in arteries. Part I: Derivation and properties of mathematical model.* J. Appl. Math. Phys. (ZAMP) *22,* 217–246 (Part I), (1971).

[4] F. P. Bretherton, *Resonant interactions between waves. The case of discrete oscillations,* J. Fluid Mech. *20,* 457–479 (1964).

[5] K. M. Case and S. C. Chiu, *Three-wave resonant interactions of gravity-capillary waves,* Phys. Fluids *20,* 742–745 (1977).

[6] S. J. Cowley, *Elastic jumps on fluid-filled elastic tubes,* Fluid Mech. *116,* 459–473 (1982).

[7] S. J. Cowley, *On the wavetrains associated with elastic jumps on fluid-filled elastic tubes,* Q. Jl. Mech. Appl. Math. *36,* 284–312 (1983).

[8] A. D. D. Craik, *Wave Interactions and Fluid Flows,* Cambridge Universiy Press, Cambridge 1985.

[9] A. D. D. Craik and J. A. Adam, *Evolution in space and time of resonant wave triads. I. The "pump-wave approximation",* Proc. Roy. Soc. Lond. *A 363,* 243–255 (1978).

[10] D. Elad, A. Foux, Y. Kivity and Y. Lanir, *Inflation waves in nonlinear distensible tubes,* Int. J. Engng. Sci. *22,* 103–118 (1984).

[11] J. E. Flaherty, J. B. Keller and S. I. Rubinow, *Post-buckling behaviour of elastic tubes and rings with opposite sides in contact,* SIAM J. Appl. Math. *23,* 446–455 (1972).

[12] A. E. Green and W. Zerna, *Theoretical Elasticity,* Oxford University Press, London 1954.

[13] W. W. Hsieh and L. A. Mysak, *Resonant interactions between shelf waves with applications to the Oregon Shelf,* J. Phys. Oceanogr. *10,* 1729–1741 (1980).

[14] R. S. Johnson, *A non-linear equation incorporating damping and dispersion*, J. Fluid Mech. *42*, 49–60 (1970).
[15] D. J. Kaup, *The three-wave interaction – a nondispersive phenomenon*, Stud. Appl. Math. *55*, 9–44 (1976a).
[16] D. J. Kaup, *A perturbation expansion for the Zakharov-Shabat inverse scattering transform*, S.I.A.M. J. Appl. Math. *31*, 121–133 (1976b).
[17] D. J. Kaup, A. Reiman and A. Bers, *Space-time evolution of nonlinear three-wave interactions. I. Interactions in a homogeneous medium*, Rev. Mod. Phys. *51*, 275–310 (1979).
[18] I. Kececioglu, M. E. McClurken, R. D. Kamm and A. H. Shapiro, *Steady, super-critical flow in collapsible tubes. Part. I. Experimental observations*, J. Fluid Mech. *109*, 367–389 (1981).
[19] A. H. Shapiro, *Steady, Super-critical flow in collapsible tubes. Part II. Theoretical studies*, J. Fluid Mech. *109*, 391–415 (1981).
[20] T. B. Moodie, D. W. Barclay, S. E. Greenwald and D. L. Newman, *Waves in fluid filled tubes: Theory and experiment*, Acta Mech. *54*, 107–119 (1984).
[21] T. B. Moodie, D. W. Barclay and S. E. Greenwald, *Impulse propagation in liquid filled distensible tubes: Theory and experiment for intermediate to long wavelengths*, Acta Mech. *59*, 47–58 (1986).
[22] T. B. Moodie and D. W. Barclay, *Wave propagation and reflection in liquid filled distensible tube systems exhibiting dissipation and dispersion*, Acta Mech. *59*, 139–155 (1986).
[23] T. B. Moodie and J. B. Haddow, *Waves in thin-walled elastic tubes containing an incompressible inviscid fluid*, Int. J. Non-Lin. Mech. *12*, 223–231 (1977).
[24] R. W. Ogden, *Non-linear Elastic Deformations*, Ellis Horwood, Chichester 1984.
[25] T. J. Pedley, *The Fluid Mechanics of Large Blood Vessels*, Cambridge University Press, Cambridge 1980.
[26] S. I. Rubinow and J. B. Keller, *Wave propagation in a fluid-filled tube*. J. Acoust. Soc. Am. *50*, 198–223 (1971).
[27] S.I. Rubinow and J. B. Keller, *Wave Propagation in a visco-elastic tube containing a viscous fluid*. J. Fluid Mech. *88*, 181–203 (1978).
[28] B. R. Seymour and M. P. Mortell, *Nonlinear resonant oscillations in open tubes*, J. Fluid Mech. *58*, 733–749 (1973).
[29] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York 1974.
[30] V. E. Zakharov and S. V. Manadov, *Resonant interaction of wave packets in nonlinear media*, Soviet Phys. – J.E.T.P. Lett. *18*, 243–247 (1973).
[31] V. E. Zakharov and S. V. Manakov, *The theory of resonance interaction of wave packets in nonlinear media*, Soviet Phys. – J.E.T.P. *42*, 842–850 (1976).
[32] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-modulating waves in nonlinear media*, Soviet Phys. – J.E.T.P. *34*, 62–69 (1972).

Abstract

A multiple-scales method is used to derive the Three-Wave Interaction (*TWI*) equations describing resonantly interacting triads in nonlinear hyper-elastic fluid-filled tubes. The tube wall is assumed to be an axially-tethered nonlinear membraneous cylindrical shell for which the resultant stresses can be determined by a strain-energy functional. The fluid within the tube is assumed to be two-dimensional, axi-symmetric and inviscid. We show that small-but-finite amplitude strongly dispersive pressure wave packets can continuously exchange energy in a resonant triad while conserving total energy. For a Mooney-Rivlin shell wall the theory presented predicts a short wavelength cutoff on the order of the tube radius. Thus pressure pulses containing wavelengths on the order of the tube radius and longer may contain resonantly interacting modes. Special solutions are presented: temporally developing modes, pump-wave approximations and explosively unstable steadily-traveling wave packets.

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