Perturbation Theory for the Solitary Wave Solutions to a Sasa-Satsuma Model Describing Nonlinear Internal Waves in a Continuously Stratified Fluid

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The adiabatic evolution of perturbed solitary wave solutions to an extended Sasa-Satsuma (or vector-valued modified Korteweg–de Vries) model governing nonlinear internal gravity propagation in a continuously stratified fluid is considered. The transport equations describing the evolution of the solitary wave parameters are determined by a direct multiple-scale asymptotic expansion and independently by phase-averaged conservation relations for an arbitrary perturbation. As an example, the adiabatic evolution associated with a dissipative perturbation is explicitly determined. Unlike the case with the dissipatively perturbed modified Korteweg–de Vries equation, the adiabatic asymptotic expansion for the Sasa-Satsuma model considered here is not exponentially nonuniform and no shelf region emerges in the lee-side of the propagating solitary wave.

1. Introduction

Notwithstanding the important role that nonlinearity plays in the evolution and propagation of internal waves of moderate amplitude in a stably stratified fluid, the development of a weakly nonlinear theory has been a difficult problem because of the well-known property that internal gravity plane waves are exact solutions to the full nonlinear equations of motion. This is a problem because,

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on the face of it, this property implies that it is not possible to determine the wave-induced mean flow in a straightforward application of perturbation theory (e.g., Craik [1]) and thereby systematically derive a nonlinear amplitude evolution equation. An important breakthrough in this problem was made by Sutherland [2] who, exploiting the underlying Hamiltonian structure of the governing equations (Scinocca and Shepherd [3]), was able to connect the wave-induced mean flow with the conserved pseudomomentum per unit mass (McIntyre [4]) and thereby give the first rational derivation of a nonlinear Schrödinger (NLS) equations describing the evolution of weakly nonlinear but strongly dispersive internal gravity waves in a Boussinesq fluid.

Dosser and Sutherland [5] extended this work to describe the weakly nonlinear evolution of internal wavepackets in a non-Boussinesq fluid. This model, which corresponds to an extended or modified Sasa-Satsuma equation (SSE), has the added feature of including higher order nonlinear and dispersive effects. The Sasa-Satsuma model may be considered as a vector-valued or complex generalization of the modified Korteweg–de Vries (mKdV) equation. Swaters et al. [6] have recently described the conservation laws, Hamiltonian structure, modulational stability properties and both the bright and dark solitary wave solutions to this new model. The principal purpose of this paper is to develop a singular perturbation theory for the bright (compactly supported) solitary wave solutions to this model.

The plan of this paper is as follows. In Section 2, the extended Sasa-Satsuma model that describes nonlinear internal gravity waves is briefly introduced and recast into a more amenable form for our discussion. The energy and momentum conservation laws are briefly described for the recast model and the one-parameter family of bright solitary wave solutions is given. In Section 3, the direct multiple-scales (fast phase and slow time) singular asymptotic theory is developed for an arbitrary perturbation. The appropriate transport equation describing the slow time evolution of the solitary wave amplitude is determined by the application of solvability conditions associated with the first-order perturbation equations and the corresponding homogeneous adjoint problem. It is shown that the transport equation for the slow time evolution of the solitary wave amplitude also follows from the leading-order fast-phase averaged energy conservation balance associated with the perturbed model.

In Section 4, an explicit example is considered for the perturbed model with a Rayleigh damping term with a time dependent dissipation rate. For this simple example, the solitary wave amplitude exponentially decays to zero. However, unlike the case for the related perturbed mKdV equation where the solitary wave translation velocity vanishes in the zero amplitude limit, the translation velocity for the dissipatively perturbed SSE model considered here either monotonically accelerates or de-accelerates toward a negative value depending on the initial amplitude of the solitary wave. In Section 5, the first order perturbation equations are considered. Necessary and sufficient conditions on the perturbation term are established in order that no shelf region emerges in the lee of the propagating solitary wave, i.e., the singular adiabatic perturbation theory is *not* exponentially nonuniform in the far field. It is shown that the dissipative example considered here satisfies these conditions, in contrast to the known adiabatic perturbation theory for the related perturbed mKdV problem. The paper is summarized in Section 6.

2. Extended Sasa-Satsuma model for nonlinear internal waves in a continuously stratified fluid

After a suitable transformation (see [6]) the Dosser and Sutherland [5] model, in the Boussinesq limit, can be written in the nondimensional form

$$u_t - u(|u|^2)_x + \alpha u_{xxx} + 2i\delta\beta u|u|^2 = 0, \tag{1}$$

where t is time and, for our purposes, x is the spatial coordinate and u is the envelope amplitude of the underlying neutrally stable dispersive internal gravity wave. The parameters α, β and δ are all real-valued. In the Dosser and Sutherland model (see [6]) α is sign indefinite, $\delta = sgn\,(\alpha)$ and $\beta > 0$. The $\alpha < 0$ (i.e., $\delta = -1$) limit corresponds to the "bright" version of the model and $\alpha > 0$ (i.e., $\delta = 1$) corresponds to the "dark" version of the model. The solitary wave solutions associated with the bright limit decay exponentially rapidly to zero as $|x| \to \infty$ (see [6] and below). The envelope of the oscillatory solitary wave solutions associated with the dark limit approach a constant as $|x| \to \infty$ [6].

While (1) can be identified as a version of the SSE, as written it may be alternatively considered as a complex or vector-valued generalization of the mKdV equation (see, e.g., Foursov [7] or Sergyeyev and Desmkoi [8]). Closely related models have been examined, for example, by Yang [9], Slunyaev [10], and Grimshaw and Helfrich [11].

Note that in the limit that $\beta = 0$ and assuming that u is real-valued then (1) reduces to the classical mKdV equation with cubic nonlinearity. In the case where $\alpha = 0$, then (1) is a (nonintegrable) modified derivative nonlinear Schrödinger equations; see, for example, Refs. [12] and [13].

We will focus attention here on the "bright" limit of the model and thus assume that $\alpha < 0$ ($\Longrightarrow \delta = -1$) with both α and $\beta > 0$ considered to be $O(1)$ parameters. With this specific choice in mind it is convenient in what follows to introduce the new tilde variables

$$x = \sqrt[3]{-3\alpha}\widetilde{x},\ t = -3\widetilde{t},\ u = \sqrt[6]{-3\alpha}\widetilde{u} \text{ and } \widetilde{\beta} = \sqrt[3]{-3\alpha}\beta > 0,$$

into (1) yielding, after dropping the tildes,

$$u_t + 3u(|u|^2)_x + u_{xxx} + 6i\beta u|u|^2 = 0. \tag{2}$$

Swaters et al. [6] found two conservation laws for (2). The "energy" and "momentum" conservation laws are given by, respectively,

$$(|u|^2)_t + (\overline{u}u_{xx} + u\overline{u}_{xx} - |u_x|^2 + 3|u|^4)_x = 0. \tag{3}$$

$$\begin{aligned}&[|u_x|^2 - |u|^4 + i\beta(\overline{u}u_x - u\overline{u}_x)]_t + \Big[i\beta(\overline{u}_t u - u_t\overline{u}) + |u_x|^2|u|^2 \\ &- 4|u|^6 - |u_{xx}|^2 - \overline{u}_t u_x - u_t\overline{u}_x - \frac{1}{2}\left(\overline{u}^2 u_x^2 + u^2\overline{u}_x^2\right) - 2|u|^2(\overline{u}u_{xx} + u\overline{u}_{xx}) \\ &+ 2i\beta(u_x\overline{u}_{xx} - \overline{u}_x u_{xx}) + 6\beta^2|u|^4 + 6i\beta|u|^2(\overline{u}u_x - u\overline{u}_x)\Big]_x = 0,\end{aligned} \tag{4}$$

where $\overline{*}$ is the complex-conjugate of $*$. Equations (3) and (4) above are identical to Equations (23) and (24) in Swaters et al. [6] with the mapping $\alpha = -1/3$, $\delta = -1$ and $t \to -3t$ (there are two minor typographical errors in the flux in Equation (24) in [6]; the flux term $-4\,|u|^6\,/\,(3\alpha)$ should be $-4\,|u|^6\,/\,(9\alpha)$ (also to be corrected in Equation (A2) in [6]) and the last flux term $\delta\beta i\,|u|^2\,(\overline{u}u_x - u\overline{u}_x)$ in Equation (24) in [6] should be $2\delta\beta i\,|u|^2\,(\overline{u}u_x - u\overline{u}_x)$).

The one-parameter (bright) solitary wave solution to (2) is given by (see Refs. [11] and [6]; and Yang [9] for a solitary wave solution to a related model with an asymptotically restricted set of parameter values)

$$u_{\text{solitary}} = a\, sech\{a[x - x_0 + (3\beta^2 - a^2)t]\}\ \exp\{i\beta[x - x_1 + (\beta^2 - 3a^2)t]\}, \tag{5}$$

where the amplitude a is the free solitary wave parameter and (x_0, x_1) are arbitrary phase shift parameters. In the limit $\beta = 0$, (5) reduces to the well-known real-valued soliton solution to the mKdV equation

$$u_t + 6u^2u_x + u_{xxx} = 0.$$

The solitary wave solution (5) differs from its mKdV cousin in three principal respects. First, of course, we note that (5) is complex-valued when $\beta \neq 0$. The presence of the "β term" in (2) also introduces an oscillatory core within the *sech* envelope function, which does not occur in the mKdV soliton. In addition, unlike the mKdV soliton, the solitary wave solution (5) does not possess a positive definite translation velocity associated with the *sech* envelope function. If $a^2 < 3\beta^2$, the solitary wave solution (5) is leftward-travelling whereas if $a^2 > 3\beta^2$, the solitary wave solution (5) is rightward-travelling.

3. Perturbation theory

Our goal here is to develop a leading order perturbation theory when (2) is replaced by

$$u_t + 3u(|u|^2)_x + u_{xxx} + 6i\beta u|u|^2 = \varepsilon F(u, \varepsilon t), \tag{6}$$

where $0 < \varepsilon << 1$, F is a smooth (possibly complex-valued) function of u (and/or its derivatives) and the slow time $T \equiv \varepsilon t$, and where the solution to (6) is subject to the initial condition $u(x, 0) = u_{\text{solitary}}(x, 0)$.

3.1. Direct singular perturbation expansion

To determine the adiabatic evolution of the perturbed solitary wave solution (5) subject to (6), it is convenient to introduce the fast phase and slow time variables given by, respectively,

$$\theta = x + \frac{1}{\varepsilon}\int^{\varepsilon t} 3\beta^2 - a^2(\xi)\, d\xi,\; T = \varepsilon t, \tag{7}$$

which implies that

$$\partial_t \to [3\beta^2 - a^2(T)]\partial_\theta + \varepsilon\partial_T,\; \partial_x \to \partial_\theta, \tag{8}$$

together with the decomposition

$$u = q(\theta, T)\exp[i\beta(\theta - \sigma)], \tag{9}$$

where $q(\theta, T)$ is complex-valued, and where

$$\sigma(T) \equiv \frac{2}{\varepsilon}\int^{T} \beta^2 + a^2(\xi)\, d\xi \Longrightarrow \sigma_T = \frac{2}{\varepsilon}[\beta^2 + a^2(T)]. \tag{10}$$

In terms of these new variables the solitary wave solution (5) takes the form

$$u_{\text{solitary}} = a\, sech\,(a\,\theta)\exp[i\beta(\theta - \sigma)] \Longrightarrow q_{\text{solitary}} = a\, sech\,(a\,\theta). \tag{11}$$

In (11) it has been assumed that $x_0 = x_1 = 0$. As in the perturbed NLS or KdV problems (Refs. [14–17]), the leading order solvability conditions, or equivalently, phase-averaged conservation relations do not determine the evolution of the phase shift parameters and thus, without loss of generality for the analysis presented here, they may be set to zero. Their evolution is determined by higher order solvability conditions, or equivalently, higher order fast phase-averaged conservation relations, which are not examined here.

Substitution of (7) through to (10) into (6) yields, after a little algebra,

$$q_{\theta\theta\theta} - a^2 q_\theta + 3q(|q|^2)_\theta + 3i\beta(q_{\theta\theta} - a^2 q + 2q|q|^2)$$
$$= \varepsilon[e^{-i\beta(\theta-\sigma)}F(qe^{i\beta(\theta-\sigma)}, T) - q_T]. \tag{12}$$

Introducing the straightforward expansion

$$q(\theta, T) \simeq q^{(0)}(\theta, T) + \varepsilon q^{(1)}(\theta, T) + \cdots, \tag{13}$$

into (12) leads to the $O(1)$ and $O(\varepsilon)$ problems given by, respectively,

$$q^{(0)}_{\theta\theta\theta} - a^2 q^{(0)}_\theta + 3q^{(0)}(|q^{(0)}|^2)_\theta + 3i\beta\big(q^{(0)}_{\theta\theta} - a^2 q^{(0)} + 2q^{(0)}|q^{(0)}|^2\big) = 0, \tag{14}$$

$$q^{(1)}_{\theta\theta\theta} - a^2 q^{(1)}_{\theta} + 6q^{(0)}q^{(0)}_{\theta}q^{(1)} + 3q^{(0)}[q^{(0)}(q^{(1)} + \overline{q^{(1)}})]_{\theta}$$
$$+3i\beta[q^{(1)}_{\theta\theta} - a^2 q^{(1)} + (q^{(0)})^2(4q^{(1)} + 2\overline{q^{(1)}})]$$
$$= e^{-i\beta(\theta-\sigma)}F(q^{(0)}e^{i\beta(\theta-\sigma)}, T) - q^{(0)}_T. \quad (15)$$

The solution to the $O(1)$ problem is taken to be

$$q^{(0)}(\theta, T) = a(T)\,sech\,[a(T)\,\theta], \quad (16)$$

because it follows from (16) that

$$q^{(0)}_{\theta\theta} - a^2 q^{(0)} + 2q^{(0)}|q^{(0)}|^2 = q^{(0)}_{\theta\theta} - a^2 q^{(0)} + 2(q^{(0)})^3 = 0, \quad (17)$$

$$q^{(0)}_{\theta\theta\theta} - a^2 q^{(0)}_{\theta} + 3q^{(0)}(|q^{(0)}|^2)_{\theta} = q^{(0)}_{\theta\theta\theta} - a^2 q^{(0)}_{\theta} + 6(q^{(0)})^2 q^{(0)}_{\theta} = 0, \quad (18)$$

so that both the real and imaginary parts of (14) are satisfied.

For the $O(\varepsilon)$ problem (15) the solution is written in the form

$$q^{(1)}(\theta, T) = \phi(\theta, T) + i\psi(\theta, T), \quad (19)$$

where $\phi(\theta, T)$ and $\psi(\theta, T)$ are real-valued. Substitution of (19) into (15) yields, after extracting the real and imaginary parts, respectively,

$$[\phi_{\theta\theta} - a^2\phi + 6(q^{(0)})^2\phi]_{\theta} - 3\beta[\psi_{\theta\theta} - a^2\psi + 2(q^{(0)})^2\psi]$$
$$= Re[e^{-i\beta(\theta-\sigma)}F(q^{(0)}e^{i\beta(\theta-\sigma)}, T)] - q^{(0)}_T, \quad (20)$$

$$\psi_{\theta\theta\theta} - a^2\psi_{\theta} + 6q^{(0)}q^{(0)}_{\theta}\psi + 3\beta[\phi_{\theta\theta} - a^2\phi + 6(q^{(0)})^2\phi]$$
$$= Im[e^{-i\beta(\theta-\sigma)}F(q^{(0)}e^{i\beta(\theta-\sigma)}, T)]. \quad (21)$$

The appropriate transport equation that will determine the evolution of $a(T)$ is obtained from the following solvability condition on the non–self-adjoint system (20) and (21). If (20) and (21) are multiplied, respectively, through by the arbitrary functions $f(\theta, T)$ and $g(\theta, T)$, added together and the result integrated with respect to $\theta \in (-\infty, \infty)$ assuming f and g together with all their derivatives vanish as $|\theta| \to \infty$, one obtains the balance

$$\int_{-\infty}^{\infty} \phi\{f_{\theta\theta\theta} - a^2 f_{\theta} + 6(q^{(0)})^2 f_{\theta} - 3\beta[g_{\theta\theta} - a^2 g + 6(q^{(0)})^2 g]\}$$
$$+\psi\left\{g_{\theta\theta\theta} - a^2 g_{\theta} - 6q^{(0)}q^{(0)}_{\theta}g + 3\beta[f_{\theta\theta} - a^2 f + 2(q^{(0)})^2 f]\right\} d\theta$$
$$= -\int_{-\infty}^{\infty} f\left\{Re[e^{-i\beta(\theta-\sigma)}F(q^{(0)}e^{i\beta(\theta-\sigma)}, T)] - q^{(0)}_T\right\}$$
$$+gIm[e^{-i\beta(\theta-\sigma)}F(q^{(0)}e^{i\beta(\theta-\sigma)}, T)]\,d\theta. \quad (22)$$

It follows from (22) that the homogeneous adjoint problem associated with (20) and (21) will be given by

$$f_{\theta\theta\theta} - a^2 f_\theta + 6(q^{(0)})^2 f_\theta - 3\beta[g_{\theta\theta} - a^2 g + 6(q^{(0)})^2 g] = 0, \quad (23)$$

$$g_{\theta\theta\theta} - a^2 g_\theta - 6q^{(0)} q_\theta^{(0)} g + 3\beta[f_{\theta\theta} - a^2 f + 2(q^{(0)})^2 f] = 0. \quad (24)$$

By inspection we see that

$$f = q^{(0)} \text{ and } g = 0, \quad (25)$$

is a homogeneous solution of the adjoint system (23) and (24) and thus it follows from (22) that, necessarily,

$$\frac{d}{dT}\int_{-\infty}^{\infty} \left(q^{(0)}\right)^2 d\theta = 2\int_{-\infty}^{\infty} q^{(0)} Re[e^{-i\beta(\theta-\sigma)} F(q^{(0)} e^{i\beta(\theta-\sigma)}, T)]\, d\theta, \quad (26)$$

which if (16) is substituted in, yields,

$$a_T = a\int_{-\infty}^{\infty} sech\,(a\,\theta)\, Re[e^{-i\beta(\theta-\sigma)} F(a\, sech(a\,\theta) e^{i\beta(\theta-\sigma)}, T)]\, d\theta, \quad (27)$$

which determines $a = a\,(T)$ subject to the initial condition $a\,(0) = a_0$. Finally, it is noted that we have not been able to find any other homogeneous solutions to (23) and (24).

3.2. Phase-averaged energy balance approach

The solvability condition (26) is simply the leading-order globally averaged energy balance relation when the adiabatic variables θ and T are introduced. If the sum $\overline{u} \times (6) + u \times \overline{(6)}$ is formed, the result can be written as

$$(|u|^2)_t + (\overline{u}u_{xx} + u\overline{u}_{xx} - |u_x|^2 + 3|u|^4)_x = 2\varepsilon Re\,[\overline{u}F\,(u, \varepsilon t)]\,. \quad (28)$$

When $\varepsilon = 0$, (28) reduces to (3). If (7) and (8) is introduced into (28), it follows that

$$[(3\beta^2 - a^2)\partial_\theta + \varepsilon\partial_T]|u|^2 + (\overline{u}u_{\theta\theta} + u\overline{u}_{\theta\theta} - |u_\theta|^2 + 3\,|u|^4)_\theta$$
$$= 2\varepsilon Re\,[\overline{u}F\,(u, T)]\,. \quad (29)$$

Finally, if

$$u\,(\theta, T) \simeq u^{(0)}\,(\theta, T) + \varepsilon u^{(1)}\,(\theta, T) + \cdots, \quad (30)$$

is introduced into (29) with the result integrated with respect to $\theta \in (-\infty, \infty)$ assuming $u^{(0)}$ and $u^{(1)}$ together with all their derivatives vanish as $|\theta| \to \infty$, one obtains the leading-order balance

$$\frac{d}{dT}\int_{-\infty}^{\infty} |u^{(0)}|^2\, d\theta = 2\int_{-\infty}^{\infty} Re[\overline{u^{(0)}} F(u^{(0)}, T)]\, d\theta, \quad (31)$$

and because

$$u^{(0)}(\theta, T) = q^{(0)}(\theta, T) \exp[i\beta(\theta - \sigma)],$$

it follows that (31) is identical to (26) and hence (27).

4. Example with a dissipative perturbation

To provide an explicit example calculation, we consider a Rayleigh-like dissipative perturbation in the form

$$F(u, \varepsilon t) = -\gamma(\varepsilon t)\, u, \tag{32}$$

where $\gamma(T) > 0$. (For an oceanographic example of the derivation of solitary wave equation containing a dissipative perturbation term for internal gravity waves in a continuously stratified fluid see Timko and Swaters [18].) It follows from (27) that

$$a_T = -\gamma(T)\, a^2 \int_{-\infty}^{\infty} sech^2(a\,\theta)\, d\theta = -2\gamma(T)\, a$$

$$\Longrightarrow a(T) = a_0 \exp\left[-2\int_0^T \gamma(\xi)\, d\xi\right]. \tag{33}$$

For Rayleigh dissipation of the form (32), β has no effect on the evolution of $a(T)$. Consequently, the exponential decay in (33) is identical for the amplitude decay associated with the Rayleigh dissipation of the mKdV soliton [15].

The solitary wave translation velocity, denoted by $c(T)$, and given by (see (7)),

$$c(T) = a^2(T) - 3\beta, \tag{34}$$

will satisfy the limit $c(T) \to -3\beta$ as $T \to \infty$. If $a_0^2 > 3\beta$, the solitary wave is initially rightward-propagating, i.e., $c(0) > 0$, and as time proceeds the solitary wave de-accelerates eventually acquiring a negative propagation velocity, which ultimately approaches the finite value -3β. On the other hand if $a_0^2 < 3\beta$, the solitary wave is always leftward-travelling, i.e., $c(0) < -3\beta$, continuously accelerating but nevertheless approaching the finite propagation velocity -3β.

5. First-order perturbation equations

Unlike the situation associated with the mKdV equation (i.e., the real-valued $\beta = 0$ limit of the model considered here), when $\beta \neq 0$ there is no "shelf" region (see, e.g., Refs. [15] and [19]) produced in the lee side of the propagating solitary wave. That is, the direct adiabatic perturbation theory constructed here does not result in an exponentially nonuniform asymptotic expansion. To see this let us suppose that a shelf region does emerge, that is, suppose that

$$\lim_{\theta \longrightarrow -sgn(c)\infty} \phi = \phi_\infty \text{ and } \lim_{\theta \longrightarrow -sgn(c)\infty} \psi = \psi_\infty,$$

where ϕ_∞ and ψ_∞ are constants with respect to θ, i.e., are at most functions of the slow time T. It therefore follows after taking the limits of (20) and (21) that

$$3\beta a^2 \psi_\infty = \lim_{\theta \longrightarrow -sgn(c)\infty} Re[e^{-i\beta(\theta-\sigma)} F(q^{(0)} e^{i\beta(\theta-\sigma)}, T)],$$

$$-3\beta a^2 \phi_\infty = \lim_{\theta \longrightarrow -sgn(c)\infty} Im[e^{-i\beta(\theta-\sigma)} F(q^{(0)} e^{i\beta(\theta-\sigma)}, T)],$$

where the fact that

$$\lim_{\theta \longrightarrow -sgn(c)\infty} q^{(0)} = 0,$$

exponentially rapidly has been used. Hence, provided

$$\lim_{\theta \longrightarrow -sgn(c)\infty} Re[e^{-i\beta(\theta-\sigma)} F(q^{(0)} e^{i\beta(\theta-\sigma)}, T)]$$

$$= \lim_{\theta \longrightarrow -sgn(c)\infty} Im[e^{-i\beta(\theta-\sigma)} F(q^{(0)} e^{i\beta(\theta-\sigma)}, T)] = 0, \tag{35}$$

it will follow that

$$\psi_\infty = \phi_\infty = 0,$$

and hence no shelf develops.

As an example, for $F(u, \varepsilon t)$ given by (32), it follows that (20) and (21) take the form

$$[\phi_{\theta\theta} - a^2\phi + 6(q^{(0)})^2\phi]_\theta - 3\beta[\psi_{\theta\theta} - a^2\psi + 2(q^{(0)})^2\psi]$$

$$= \gamma\left(q^{(0)} + 2\theta q_\theta^{(0)}\right), \tag{36}$$

$$\psi_{\theta\theta\theta} - a^2\psi_\theta + 6q^{(0)}q_\theta^{(0)}\psi + 3\beta[\phi_{\theta\theta} - a^2\phi + 6(q^{(0)})^2\phi] = 0, \tag{37}$$

where (16) and (33) have been used. We see immediately that the right-hand sides of (36) and (37) satisfy (35) so that for the dissipation perturbation (32) no shelf region will emerge in the first order fields.

Moreover, when $\beta \neq 0$, (36) and (37) can be combined into the single equation

$$\left[\psi_{\theta\theta\theta} - a^2\psi_\theta + 6q^{(0)}q_\theta^{(0)}\psi\right]_\theta + 9\beta^2[\psi_{\theta\theta} - a^2\psi + 2(q^{(0)})^2\psi]$$

$$= -3\beta\gamma\left(q^{(0)} + 2\theta q_\theta^{(0)}\right). \tag{38}$$

In principle, one solves (38) for ψ and obtains ϕ from, say, (37).

In the case when $\beta = 0$, it follows from (37) that $\psi \equiv 0$ and (36) reduces to the well-known first-order perturbation equation associated with the perturbed mKdV problem with Rayleigh dissipation. In the $\beta = 0$ limit, (36) can be integrated to explicitly yield ϕ and it will follow that $\phi_\infty = -\pi\gamma/a^2$ (see [15]).

Notwithstanding the fact that clearly $\phi = q_\theta^{(0)}$ and $\psi = q^{(0)}$ correspond to a homogeneous solution to (36) and (37) or, equivalently, (38), we have been unable to solve the first-order perturbation equations and explicitly obtain ϕ and ψ. Again, we remark that in the case where $\beta = 0$, it follows that $\psi \equiv 0$ and the resulting (36) can be integrated to explicitly obtain ϕ (see [15]).

The emergence of a "shelf" region in the lee of the propagating soliton in the perturbed mKdV problem with Rayleigh dissipation, for example, is a consequence of the fact that the adiabatically deforming solitary wave is unable to simultaneously satisfy the appropriate leading-order fast-phase averaged mass *and* energy balance relations [19]. In perturbed soliton problems where no shelf region emerges (e.g., the dissipatively perturbed NLS or sine-Gordon equations, see [15] and [20]), the leading-order fast-phase averaged mass, energy and momentum balance relations are all satisfied.

In the situation considered here we have shown that no "shelf" region will emerge in the perturbed solitary wave problem with Rayleigh dissipation. This suggests that the adiabatic solution we have obtained for the decay in the solitary wave amplitude given by (33), which has been obtained by demanding that the leading-order fast-phase averaged energy balance relation (31) must hold, will imply that the appropriate leading-order fast-phase averaged momentum balance relation will also be satisfied. We now show this to be the case.

The appropriate momentum balance equation will be the analogue of (4) when the perturbation term $\varepsilon F(u, \varepsilon t)$ is retained in its derivation. Following the derivation described in [6], the momentum balance relation is obtained from the "sum"

$$-\left[\left(\overline{u}_{xx} + 2|u|^2\overline{u}\right) \times (6) + c.c.\right] - 2i\beta\left[\overline{u}_x \times (6) - c.c.\right], \tag{39}$$

which yields, after some algebra,

$$\begin{aligned}
&\left[|u_x|^2 - |u|^4 + i\beta\left(\overline{u}u_x - u\overline{u}_x\right)\right]_t + \Bigg[i\beta\left(\overline{u}_t u - u_t\overline{u}\right) + |u_x|^2|u|^2 \\
&-4|u|^6 - |u_{xx}|^2 - \overline{u}_t u_x - u_t\overline{u}_x - \frac{1}{2}\left(\overline{u}^2u_x^2 + u^2\overline{u}_x^2\right) - 2|u|^2\left(\overline{u}u_{xx} + u\overline{u}_{xx}\right) \\
&+2i\beta\left(u_x\overline{u}_{xx} - \overline{u}_x u_{xx}\right) + 6\beta^2|u|^4 + 6i\beta|u|^2\left(\overline{u}u_x - u\overline{u}_x\right)\Bigg]_x \\
&= 2\varepsilon\left\{2\beta Im\left[\overline{u}_x F(u, \varepsilon t)\right] - Re\left[\left(\overline{u}_{xx} + 2|u|^2\overline{u}\right)F(u, \varepsilon t)\right]\right\}.
\end{aligned} \tag{40}$$

If (7), (8), (30), and (32) are introduced into (40), and the result integrated with respect to $\theta \in (-\infty, \infty)$ assuming $u^{(0)}$ and $u^{(1)}$ together with all their derivatives vanish as $|\theta| \to \infty$, one obtains the leading-order balance

$$\frac{d}{dT}\int_{-\infty}^{\infty} |u_\theta^{(0)}|^2 - |u^{(0)}|^4 + 2\beta Im[\overline{u_\theta^{(0)}}u^{(0)}]\, d\theta$$

$$= -2\gamma \int_{-\infty}^{\infty} |u_\theta^{(0)}|^2 - 2|u^{(0)}|^4 + 2\beta Im[\overline{u_\theta^{(0)}}u^{(0)}]\, d\theta. \quad (41)$$

Our goal is to show that when

$$u^{(0)} = q^{(0)}(\theta, T)\exp[i\beta(\theta - \sigma)] = a\, sech\,(a\,\theta)\exp[i\beta(\theta - \sigma)], \quad (42)$$

is substituted into (41), the transport Equation (33) follows. Substitution of (42) into (41) yields

$$\frac{d}{dT}\int_{-\infty}^{\infty} (q_\theta^{(0)})^2 - (q^{(0)})^4 - \beta^2 (q^{(0)})^2\, d\theta$$

$$= -2\gamma \int_{-\infty}^{\infty} (q_\theta^{(0)})^2 - 2(q^{(0)})^4 - \beta^2 (q^{(0)})^2\, d\theta,$$

which can be evaluated to give

$$\frac{d}{dT}\left(\frac{a^3}{3} + a\beta^2\right) = -2\gamma a(a^2 + \beta^2), \quad (43)$$

which implies that

$$a_T = -2\gamma a, \quad (44)$$

which is exactly (33). Consequently, the exponential decay in the solitary wave amplitude $a(T)$ given by (33) ensures that *both* the leading-order fast-phase averaged energy *and* momentum balance relations associated with (6) are satisfied for the dissipative perturbation (32).

6. Conclusions

Recent theoretical work has shown that the propagation of weakly nonlinear but strongly dispersive internal gravity waves in a Boussinesq fluid is described by an extended or modified SSE. The SSE may be considered as a vector-valued or complex generalization of the mKdV equation. It is of physical interest to determine the propagation characteristics of perturbed solitary wave solutions to this new model.

A direct multiple-scales (fast phase and slow time) adiabatic singular asymptotic theory has been developed for the model equation assuming an arbitrary perturbation. The appropriate transport equation describing the slow time evolution of the solitary wave amplitude were determined by the application of solvability conditions associated with the first-order perturbation equations and the corresponding homogeneous adjoint problem. It was shown that the transport equation for the slow time evolution of the solitary wave amplitude also followed from the leading-order fast-phase averaged energy conservation balance associated with the perturbed model.

An explicit example was considered for the perturbed model with a Rayleigh damping term with a time dependent dissipation rate. For this simple example, the solitary wave amplitude exponentially decays to zero. However, unlike the case for the related perturbed mKdV equation where the solitary wave translation velocity vanishes in the zero amplitude limit, the translation velocity for the dissipatively perturbed SSE model considered here either monotonically accelerates or de-accelerates toward a finite negative value depending on the initial amplitude of the solitary wave.

The first order perturbation equations were also considered. Necessary and sufficient conditions on the perturbation term were established in order that no shelf region emerges in the lee of the propagating solitary wave, i.e., the singular adiabatic perturbation theory is *not* exponentially nonuniform in the far field. It was shown that the dissipative example considered here satisfies these conditions, in contrast to the known adiabatic perturbation theory for the related perturbed mKdV problem.

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