A nonlinear stability theorem for baroclinic quasigeostrophic flow

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The baroclinic quasigeostrophic equations describe the essential dynamics of large-scale, low-frequency atmospheric and oceanic flow. A nonlinear stability theorem is given based on a convexity argument of Arnold [Am. Math. Soc. Transl. **19**, 267 (1969)], complementing a linear analysis by Blumen [J. Atmos. Sci. **25**, 929 (1968)]. An *a priori* estimate bounding the growth of perturbations is derived.

Recently, several studies have attempted to establish the nonlinear stability of various planetary flows.[1–5] All of these analyses have been based on deriving sufficient conditions for the positive definiteness of the second variation of a relevant constrained Hamiltonian describing the basic flow. However, it is known[6–8] that this method fails to establish nonlinear stability for infinite-dimensional dynamical systems. Rigorous nonlinear stability theorems require certain convexity hypotheses on the constrained Hamiltonian.[9–11] For flows of relevance to geophysical fluid dynamics, correct nonlinear stability proofs have been given for the stratified Euler equations,[12] compressible barotropic flows,[13] circular vortex patches,[14] and multilayer quasigeostrophic flows.[15] The principle purpose of this letter is to provide a rigorous nonlinear stability theorem for continuously stratified quasigeostrophic flow including the effects of topography in a bounded or unbounded horizontal domain. (See note added in proof.) These equations describe the essential dynamics for large-scale, low-frequency atmospheric and oceanic motions.[16,17]

The nondimensional quasigeostrophic equations for a vertically stratified fluid on the beta plane in the absence of heating or dissipation are

$$[\partial_t + J(p,\cdot)][\Delta p + \bar\rho^{-1}(\bar\rho S^{-1}p_z)_z + \beta y] = 0, \quad (1)$$

with

$$[\partial_t + J(p,\cdot)][p_z - S(z)h(x,y)] = 0 \quad \text{on } z=0, \quad (2)$$

$$[\partial_t + J(p,\cdot)]p_z = 0 \quad \text{on } z=1, \quad (3)$$

where $p = C_0$ on the smooth horizontal boundary D, or $\nabla p \to 0$ as $(x^2+y^2)\to\infty$ if the fluid is horizontally unbounded. The symbols are standard[16–18]; however, we point out here that $h(x,y)$ is the topography, p is the geostrophic pressure, the two-dimensional geostrophic velocity field $(v,v) = (-p_y, p_x)$, $J(*,\cdot) = \partial(*,\cdot)/\partial(x,y)$, and derivatives are denoted by subscripts. Note that $\nabla = (\partial_x, \partial_y)$ and $\Delta = \nabla^2$.

The nonlinear stability analysis presented here is based on a Hamiltonian formulation of (1)–(3) given in a previous linear analysis by Blumen,[19] modified to include the effects of topography and a smooth arbitrary horizontal boundary. The analysis begins by noting that the functions

$$E_p = \frac{1}{2}\iiint \bar\rho[\nabla p\cdot\nabla p + S^{-1}(p_z)^2]\, dV,$$

$$F_1 = \iiint \bar\rho\Phi_1(z;q)\, dV,$$

$$F_2 = -\iint [\bar\rho S^{-1}\Phi_2(p_z)]_{z=1}\, dA,$$

$$F_3 = \iint [\bar\rho S^{-1}\Phi_3(p_z - Sh)]_{z=0}\, dA,$$

$$F_4 = -\int \lambda\bar\rho \oint_D \mathbf{n}\cdot\nabla p\, ds\, dz,$$

are conserved by (1)–(3),[16] where Φ_1, Φ_2, and Φ_3 are smooth functions of their arguments; the quasigeostrophic potential vorticity is given by $q = \Delta p + \bar\rho^{-1} \times (\bar\rho S^{-1}p_z)_z + \beta y$, $\mathbf{n}$ is the outward unit normal on D, $dV = dx\,dy\,dz$, and $dA = dx\,dy$.

The Hamiltonian is given by

$$H = E_p + F_1 + F_2 + F_3 + F_4, \quad (4)$$

and an equilibrium solution of (1)–(3) is denoted as p^s. The Hamiltonian can be chosen as (4) since F_1, F_2, and F_3 are Casimir functions with respect to the mass-constrained Hamiltonian $E_p + F_4$ and Poisson bracket given in Ref. 20. The derivative of H evaluation at p^s is given by (after integration by parts)

$$DH(p^s)\delta p = \iiint \bar\rho\, \delta q[\Phi_1'(q^s) - p^s]\, dV + \iint \{\bar\rho S^{-1}\delta p_z[p^s - \Phi_2'(p_z^s)]\}_{z=1}\, dA + \iint \{\bar\rho S^{-1}\delta p_z \times [\Phi_3'(p_z^s - Sh) - p^s]\}_{z=0}\, dA + \int \bar\rho \oint_D (C_0 - \lambda)\mathbf{n}\cdot\nabla\delta p\, ds\, dz,$$

where $\delta p = p - p^s$, $\delta q = q - q^s$, and $(\ )'$ means differentiation with respect to the argument. The stationary solution p^s is a critical point of H when

$$\Phi_1'(q^s) = p^s, \quad (5)$$

$$\Phi_2'(p_z^s) = p^s \quad \text{on } z=1, \quad (6)$$

$$\Phi_3'(p_z^s - Sh) = p^s \quad \text{on } z=0, \quad (7)$$

and when $\lambda = C_0$. Equations (5)–(7) serve to define the functions Φ_1, Φ_2, and Φ_3 given p^s.

Nonlinear stability is proved by assuming the convexity hypotheses

$$0 < \alpha_1 \leqslant \Phi_1'' \leqslant \alpha_2 < \infty, \quad (8)$$

$$0 < \mu_1 \leqslant -\Phi_2'' \leqslant \mu_2 < \infty, \quad (9)$$

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$$0<\gamma_1\leqslant\Phi_3''\leqslant\gamma_2<\infty\,, \tag{10}$$

and examining the conserved functional

$$\hat{H}(\delta p)\equiv H(p^s+\delta p)-H(p^s)-DH(p^s)\,\delta p\,. \tag{11}$$

Here $\hat{H}$ is conserved since H is conserved, and $DH(p^s)\delta p=0$ for p^s satisfying (5)–(7). The conditions (8)–(10) are assumed to hold for all arguments and at all times for which smooth solutions exist to (1)–(3). When attention is restricted to a basic state corresponding to a zonal flow [i.e., $p^s=p^s(y,z)$], the convexity assumptions (8)–(10) can be shown to be equivalent to Pedlosky's sufficient linear stability conditions for a zonal flow.[21]

It follows from (11), exploiting the convexity hypotheses (8)–(10), that

$$2E_{\delta_p}+\alpha_1\int\!\!\int\!\!\int\bar{\rho}(\delta q)^2dV$$
$$+\mu_1\int\!\!\int[\bar{\rho}S^{-1}(\delta p_z)^2]_{z=1}dA$$
$$+\gamma_1\int\!\!\int[\bar{\rho}S^{-1}(\delta p_z)^2]_{z=0}dA\leqslant 2\hat{H}(\delta p)\leqslant 2E_{\delta_p}$$
$$+\alpha_2\int\!\!\int\!\!\int\bar{\rho}(\delta q)^2dV+\mu_2\int\!\!\int[\bar{\rho}S^{-1}(\delta p_z)^2]_{z=1}\,dA$$
$$+\gamma_2\int\!\!\int[\bar{\rho}S^{-1}(\delta p_z)^2]_{z=0}\,dA\,.$$

Since $\hat{H}(\delta p)=\hat{H}(\delta p_0)$, where $\delta p_0\equiv\delta p(t=0)$, it follows that

$$2E_{\delta p}+\alpha_1\int\!\!\int\!\!\int[\bar{\rho}(\delta q)^2]dV$$
$$+\mu_1\int\!\!\int[\bar{\rho}S^{-1}(\delta p_z)^2]_{z=1}dA$$
$$+\gamma_1\int\!\!\int[\bar{\rho}S^{-1}(\delta p_z)^2]_{z=0}\,dA$$
$$\leqslant 2E_{\delta p_0}+\alpha_2\int\!\!\int\!\!\int\bar{\rho}(\delta q_0)^2\,dV$$
$$+\mu_2\int\!\!\int[\bar{\rho}S^{-1}(\delta p_{0z})^2]_{z=1}\,dA$$
$$+\gamma_2\int\!\!\int[\bar{\rho}S^{-1}(\delta p_{0z})^2]_{z=0}\,dA\,, \tag{12}$$

which establishes nonlinear stability. The *a priori* estimate (12) implies Lyapunov stability of smooth solutions to (1)–(3) and is explicitly independent of the topography $h(x,y)$.[22]

The existence of classical solutions of (1)–(3) in a horizontally periodic domain has been proved only up to a finite time, which is inversely proportional to the norms of p_{0_z} on $z=0$ and $z=1$, and q_0 (Ref. 23). Stability can also be proved when $\Phi_1''<0$, and $\Phi_2''>0$ and $\Phi_3''<0$ by considering $-\hat{H}(\delta p)$ and requiring sufficiently large $\min(-\Phi_1'')$, $\min(\Phi_2'')$, and $\min(\Phi_3'')$. The stability theorem can be generalized to include "islands" (i.e., non-simply-connected domains) by introducing other circulation functions similar to F_4. Since the quasigeostrophic equations are zonally (i.e., in the x direction) Galilean invariant, these results also apply to (zonally) steadily translating fluid motions. Therefore the stability theorem presented here is of importance for the nonlinear stability of solitary planetary waves[24] (provided Φ_1, Φ_2, and Φ_3 are sufficiently smooth). On this latter application, a more detailed analysis will be published elsewhere.

Note added in proof: The author has become aware of a similar stability analysis by M. E. McIntyre and T. G. Shepherd.[25]

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