

# Stability conditions and *a priori* estimates for equivalent-barotropic modons

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(Received 13 November 1985; accepted 7 February 1986)

Linear Liapunov stability conditions are obtained for all classes of equivalent-barotropic modons. The stability conditions are stated in terms of a parameter  $\eta$  called the "generalized disturbance wavenumber" related to the ratio of the initial values of disturbance enstrophy to energy. It is shown that  $c > 0$  modons ( $c$  is the drift speed, nondimensionalized by the long wave speed) are stable when  $\eta < \kappa$  ( $\kappa$  is the modon wavenumber), and that  $c < -1$  modons are stable when  $\eta > \kappa$ . This dependency of the stability on the initial spectral structure of the disturbance is observed in numerical calculations. *A priori*  $L^2$ -type estimates bounding the growth of perturbations are derived. The instability mechanism is interpreted in terms of Fjørtoft's energy cascade theorem [Tellus 5, 225 (1953)].

## I. INTRODUCTION

Modons or drift vortices have been obtained as solutions for a variety of fluid and plasma models.<sup>1-6</sup> However, relatively little is known about their stability. Numerical experiments<sup>7,8</sup> indicate that  $c > 0$  ( $c$  is the drift speed, nondimensionalized by the long wave speed, see Ref. 2) modons are stable for small amplitude perturbations, while for larger amplitude disturbances instability ensues. The numerical experiments of McWilliams *et al.*<sup>8</sup> indicate that the onset of instability is determined by the initial rms vorticity amplitude and "scale content" (i.e., the geometric mean of the initial wavenumber interval, see Ref. 8) of the perturbations. Similar observations have been made about the stability of other two-dimensional solitary planetary waves.<sup>9</sup> Laedke and Spatschek<sup>10</sup> have argued that  $c < -1$  ground-state density gradient drift waves are spectrally stable. Pierini<sup>11</sup> has recently shown (in work done independently of that reported here) that very large length scale  $c < -1$  modons are linearly Liapunov stable when the mean scale of the perturbations is small compared to the modon length scale.

Relatively recent advances in Hamiltonian descriptions of two-dimensional (e.g., geostrophic) hydrodynamics have enabled linear and nonlinear Liapunov stability theorems to be established for several classes of fluid and plasma flows.<sup>12</sup> These results are based on a development of the Liapunov method by Arnold<sup>13</sup> for infinite-dimensional dynamical systems. A comprehensive review of and applications of these methods can be found in Ref. 12. The stability of one-dimensional solitary waves has been obtained by a variant of these procedures.<sup>14,15</sup> The principal purposes of this article are to establish sufficient conditions for the linear Liapunov stability for all classes of equivalent-barotropic modons, to describe the instability mechanism and to derive *a priori* bounds on the growth of perturbations. The advantage of the Liapunov method over establishing spectral stability is that the former also eliminates possible algebraic instabilities and establishes an explicit norm that measures the growth of perturbations.

The results presented here differ from Laedke-Spatschek and Pierini in several important respects. In contrast to Laedke-Spatschek, it is argued here that  $c < -1$  modons are in fact linearly stable for a certain class of disturbances (those for which the "generalized disturbance wavenumber" denoted  $\eta$ , defined in Sec. III, is greater than the modon wavenumber  $\kappa$ ).

In contrast to Pierini, our results will contain no assumption about the length scale of the modon. The result of Pierini is recovered as a special case of the more general theory developed here. Moreover, our analysis will apply to both  $c < -1$  and  $c > 0$  drift waves and include all modon states (ground and higher). It is shown that  $c > 0$  modons are stable with  $\eta < \kappa$ . The analysis presented here qualitatively reveals the spectral evolution of exponentially growing modes and delineates the quite different instability mechanism between  $c < -1$  and  $c > 0$  drift waves. And finally, we provide  $L^2$ -type bounds on the growth of disturbances.

The plan of this article is as follows. In Sec. II the linear stability problem is formulated and a brief description of the modon is given. The stability results are established in Sec. III. In Sec. IV the results are interpreted and a qualitative assessment of the stability of planetary modons to observed oceanic and atmospheric fluctuation spectra is given. The results are summarized in Sec. V.

## II. GOVERNING EQUATIONS

In a coordinate system moving with the modon, the nondimensional equivalent-barotropic potential vorticity equation can be written

$$(\Delta - 1)\Psi_t + J[\Psi + cy, (\Delta - 1)\Psi + y] = 0. \quad (1)$$

The scalings have been chosen as in Flierl *et al.*<sup>2</sup> The symbols are standard<sup>16</sup>; however, we note that  $J[* , \cdot ]$  is the Jacobian determinant  $\partial(* , \cdot )/\partial(x, y)$ , and  $x, y$ , and  $t$  are the eastward, northward, and time coordinates. The two-dimensional velocity field  $(u, v) = (-\Psi_y, \Psi_x)$  with  $\Psi$  as the streamfunction. In geophysical fluid dynamics,  $\Psi$  is the geostrophic pressure<sup>16</sup> whereas in plasma dynamics,  $\Psi$  is proportional to the electrostatic potential.<sup>4</sup> The translation speed is denoted  $c$ . Derivatives are denoted by subscripts.

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The modon is a stationary solution of (1) with the boundary conditions:  $\Psi + cy = 0$  on  $r^2 = x^2 + y^2 = a^2$ ,  $\Psi \rightarrow 0$  sufficiently rapidly as  $r \rightarrow \infty$  so that the energy and enstrophy of the modon is bounded, and  $\nabla\Psi$  is continuous on  $r = a$ . The modon is given by (see Refs. 2 or 7 for details)

$$\Delta\Psi - (c^{-1} + 1)\Psi = 0, \quad \text{in } R_0, \quad (2)$$

$$\Delta\Psi + \kappa^2\Psi = -[c(\kappa^2 + 1) + 1]y, \quad \text{in } R_1, \quad (3)$$

where  $R_0 = \{(x,y): r > a\}$  and  $R_1 = \{(x,y): r < a\}$ , yielding the expressions

$$\Psi = \frac{-acK_1[(c^{-1} + 1)^{1/2}r]\sin\theta}{K_1[(c^{-1} + 1)^{1/2}a]}, \quad r > a, \quad (4a)$$

$$\Psi = a(c + 1)\kappa^{-2}J_1(\kappa r)\sin(\theta)/J_1(\kappa a) - (c\kappa^2 + c + 1)\kappa^{-2}r\sin\theta, \quad r < a, \quad (4b)$$

where  $\tan\theta = y/x$ , and  $\kappa$  is called the modon wavenumber and is a nonzero solution of (obtained from demanding continuity of  $\nabla\Psi$  on  $r = a$ )

$$-(c^{-1} + 1)^{1/2}J_2(\kappa a)K_1[(c^{-1} + 1)^{1/2}a] = \kappa J_1(\kappa a)K_2[(c^{-1} + 1)^{1/2}a]. \quad (4c)$$

Note that  $c > 0$  or  $c < -1$  for solutions of the form (4). The  $n$ th-state modon wavenumber will satisfy  $J_{1,n} \leq \kappa a \leq J_{2,n}$ , where  $J_{1,n}$  and  $J_{2,n}$  are the  $n$ th zeros of  $J_1$  and  $J_2$ , respectively. Hence the  $n$ th-state modon will contain  $n - 1$  radially symmetric nodal lines in the interior region (i.e.,  $r < a$ ). Figure 1 is a three-dimensional plot of the streamfunction  $\phi_m$  vs  $(x,y)$  for the ground-state (i.e.,  $n = 1$ ) modon.

Assuming a solution to (1)  $\Psi = \phi_m + \phi$  and linearizing about  $\phi_m$  [the modon solution (4)] gives the linear barotropic instability problems<sup>10,17</sup>

$$(\Delta - 1)\phi_t + \mathbf{U} \cdot \nabla[(\Delta + \lambda)\phi] = 0, \quad (5a)$$

where  $\lambda = -(c^{-1} + 1)$  in  $R_0$ ,  $\lambda = \kappa^2$  in  $R_1$ , and  $\mathbf{U} = \mathbf{e}_3 \times \nabla(\phi_m + cy)$ . The boundary conditions on the perturbations follow those used in the instability theory of inviscid shear flow<sup>18,19</sup>; namely that the pressure (i.e.,  $\phi$ ) and the mass flux (i.e.,  $\nabla\phi$ ) be continuous at the perturbed modon boundary  $r = a + a_1(\theta, t)$ . To leading order in the assumed small perturbation  $a_1(\theta, t)$ , these reduce to simply

$$\phi(r = a^-) = \phi(r = a^+), \quad (5b)$$

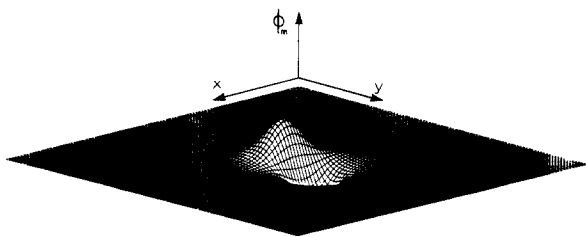


FIG. 1. Three-dimensional contour plot of the stream function  $\phi_m(x,y)$  versus the horizontal coordinates  $(x,y)$  for a modon with  $c > 0$  [see (4)]. Thus the modon is moving parallel to the  $x$  axis in the direction of increasing  $x$ . In atmospheric or oceanic dynamics,  $\phi_m$  can be interpreted as the deflection from equilibrium of a geopotential surface. Typical scales in the ocean (atmosphere) would be:  $c \sim 0.1(10) \text{ m sec}^{-1}$ , the maximum absolute deflection about 0.1 (100) m, and the horizontal distance between the maximum and minimum deflection about 80 (800) km. The Eulerian velocity field is given by  $\mathbf{u} = \mathbf{e}_3 \times \nabla\phi_m \sim 0.1(10) \text{ m sec}^{-1}$ .

$$\nabla\phi(r = a^-) = \nabla\phi(r = a^+). \quad (5c)$$

In addition, we assume that the energy and enstrophy of the perturbation is bounded. Laedke-Spatschek and Pierini have used these boundary conditions as well.

The linear dynamics described by (5) admits the conserved quantity<sup>10,17</sup>

$$L = \int_R \int \nabla\phi \cdot \nabla\phi + \phi^2 dx dy + c \int_{R_0} \int [(\Delta - 1)\phi]^2 dx dy - (\kappa^2 + 1)^{-1} \int_{R_1} \int [(\Delta - 1)\phi]^2 dx dy, \quad (6)$$

where  $R = R_0 \cup R_1$ . Physically,  $L$  describes the conservation of disturbance energy constrained by the enstrophy. The constants  $c$  and  $-(\kappa^2 + 1)^{-1}$  are the derivatives of the "streaklines"  $\phi_m + cy$  with respect to the (potential) vorticity  $q_m = \Delta\phi_m - \phi_m + y$  in the domains  $R_0$  and  $R_1$ , respectively.

### III. STABILITY THEOREMS

The stability results presented in this section are obtained by establishing *a priori* bounds on the conserved functional  $L$ . The analyses of Laedke-Spatschek and Pierini attempted to show the negative definiteness of  $L$  for  $c < -1$ . Our analysis is based on examining the spectral structure of  $L$  at  $t = 0$  as suggested by the numerical stability calculations. Although not rigorously correct, it is heuristically useful to think of  $L$  as the second variation of a constrained Hamiltonian describing the nonlinear dynamics (see Ref. 12, Sec. 4 for details of such a construction). From this point of view, our analysis is seen as providing conditions for the definiteness of the second variation of such a Hamiltonian. In this sense, we are proving "formal" stability<sup>12</sup> from which linear stability follows. Thus, the stability results presented here will eliminate both exponential and algebraic instabilities.

Define  $\eta$  as the *generalized wavenumber of the initial disturbance*, given by

$$\eta^2 = \frac{\int_R \int (\Delta\phi_0)^2 + \nabla\phi_0 \cdot \nabla\phi_0 dx dy}{\int_R \int \nabla\phi_0 \cdot \nabla\phi_0 + (\phi_0)^2 dx dy},$$

where  $\phi_0 = \phi(t = 0)$ . Physically,  $\eta^2 + 1$  is the ratio of the initial values of the disturbance enstrophy to energy. The enstrophy at  $t = 0$  is

$$\int_R \int [(\Delta - 1)\phi_0]^2 dx dy = \int_R \int (\Delta\phi_0)^2 + 2\nabla\phi_0 \cdot \nabla\phi_0 + (\phi_0)^2 dx dy,$$

thus

$$\frac{\int_R \int [(\Delta - 1)\phi_0]^2 dx dy}{\int_R \int \nabla\phi_0 \cdot \nabla\phi_0 + (\phi_0)^2 dx dy} = \eta^2 + 1.$$

The role played by  $\eta$  in our stability conditions is analogous to the "scale content" dependency observed in the numerical stability experiments.<sup>7-9</sup> The following stability results hold.

**Theorem I:** If  $c > 0$  and  $\eta \leq \kappa$ , then  $(L)^{1/2}$  defines a norm  $\|\phi\|$  and the modon is stable.

**Theorem II:** If  $c < -1$  and  $\eta \geq \kappa$ , then  $(-L)^{1/2}$  defines a norm  $\|\phi\|$  and the modon is stable.

In order to prove Theorem I, the estimates

$$L(\phi) = L(\phi_0) \geq \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy - (\kappa^2 + 1)^{-1} \int_R [(\Delta - 1)\phi_0]^2 dx dy,$$

$$L(\phi) = L(\phi_0) \leq \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy + c \int_R [(\Delta - 1)\phi_0]^2 dx dy$$

are needed. Note the implicit assumption that  $c > 0$ . Also, it is important to note that  $L(\phi) = L(\phi_0)$  does *not* imply that each of the three integrals in (6) is time invariant. Swaters<sup>17</sup> has shown that, in general, the advection of perturbation vorticity by the modon results in a sink (or source) for the area-integrated perturbation energy and a source (or sink) for the area-integrated perturbation enstrophy.

Rewriting the above estimates in terms of  $\eta$  gives

$$(\kappa^2 - \eta^2)(\kappa^2 + 1)^{-1} \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy \leq L(\phi) < [1 + c(\eta^2 + 1)] \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy. \quad (7)$$

Assuming the disturbance has bounded initial energy,  $\eta \leq \kappa$  implies that  $[L(\phi)]^{1/2}$  defines a norm  $\|\phi\|$  and linear Liapunov stability immediately follows [i.e., for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\|\phi_0\| < \delta \Rightarrow \|\phi\| < \epsilon$  for  $t > 0$ , where  $\phi$  is governed by the linear dynamics (5)].

To establish Theorem II first note that

$$\int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy + c \int_R [(\Delta - 1)\phi_0]^2 dx dy = [1 + c(\eta^2 + 1)] \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy < 0,$$

since  $c < -1$  [recall  $-1 \leq c \leq 0$  is not allowed for the modon, see (4)]. Here  $L(\phi)$  can be estimated by

$$L(\phi) \geq \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy + c \int_R [(\Delta - 1)\phi_0]^2 dx dy, \\ L(\phi) < \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy - (\kappa^2 + 1)^{-1} \int_R [(\Delta - 1)\phi_0]^2 dx dy.$$

Both inequalities have used the fact that  $c + (\kappa^2 + 1)^{-1} < 0$  since  $c < -1$ . Hence  $L$  is estimated *a priori* by

$$(\eta^2 - \kappa^2)(\kappa^2 + 1)^{-1} \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy < -L(\phi) < -[1 + c(\eta^2 + 1)] \int_R \nabla \phi_0 \cdot \nabla \phi_0 + (\phi_0)^2 dx dy. \quad (8)$$

Therefore, for bounded initial energy,  $\eta \geq \kappa$  (recall that  $c < -1$ ) implies that  $(-L)^{1/2}$  defines a norm  $\|\phi\|$  and stability is again immediately established.

#### IV. INTERPRETATION OF STABILITY CONDITIONS AND PHYSICAL SIGNIFICANCE

The assumptions of Theorems I and II may not be the sharpest stability conditions possible. However, the numerical experiments<sup>7-9</sup> clearly indicate that modon stability is determined by the initial spectral structure of the perturbations. The above results propose stability criteria consistent with that observation.

The *a priori* estimates provided by (7) and (8) are, of course, only valid in the context of smooth solutions. Classical solutions to (1) have only been proved up to a finite time.<sup>20</sup>

Another interpretation of the stability conditions in Theorems I and II can be obtained by rewriting  $\eta < \kappa$  and  $\eta \geq \kappa$  using Parseval's identity into the respective forms

$$\iint |\phi_0^T|^2 (\mu^2 + 1)(\mu^2 - \kappa^2) d\mu < 0 \quad (c > 0), \quad (9)$$

$$\iint |\phi_0^T|^2 (\mu^2 + 1)(\mu^2 - \kappa^2) d\mu \geq 0 \quad (c < -1), \quad (10)$$

where  $\phi_0^T$  is the spatial Fourier transform of  $\phi_0$  and  $\mu = (\mu_1, \mu_2)$  is the two-dimensional wavenumber vector.<sup>21</sup> Since  $|\phi_0^T|^2 (\mu^2 + 1) d\mu$  is approximately the initial disturbance energy between  $\mu$  and  $\mu + d\mu$ , the inequality (9) implies that  $c > 0$  modons are stable when the initial perturbation energy is principally contained in wavenumbers satisfying  $|\mu| \leq \kappa$ . Analogously, (10) implies that  $c < -1$  modons will be stable when the initial perturbation energy is principally contained in wavenumbers satisfying  $|\mu| \geq \kappa$ . Pierini established the linear stability of very large length scale  $c < -1$  modons against high wavenumber perturbations. In our notation, Pierini's stability result can be recovered as the special asymptotic case  $0 < \kappa \ll 1$  and  $\eta \gg \kappa$ .

Since  $L$  is conserved, it follows  $L = 0$  if the modon is exponentially unstable. This necessary instability condition can be put into the form

$$\iint |\phi^T|^2 (\mu^2 + 1)(\mu^2 - \kappa^2) d\mu = cL,$$

where

$$I = 4\pi^2 (c^{-1} + 1 + \kappa^2) \int_{R_0} [(\Delta - 1)\phi]^2 dx dy \geq 0,$$

for all time. Thus when  $c > 0$ , instability can only occur if the perturbation energy cascade is principally (but not necessarily exclusively) into length scales *smaller* than  $\kappa^{-1}$ . For  $c < -1$  modons, instability can only occur if the perturbation energy cascade is principally into length scales *greater* than  $\kappa^{-1}$ . The instability mechanism for  $c > 0$  and  $c < -1$  modons is, therefore, quite different, with the principal energy transfer occurring toward distinct regions of the disturbance spectrum relative to the modon wavenumber. This preferential direction in the disturbance energy cascade during instability is reminiscent of Fjørtoft's<sup>22</sup> energy cascade theorem.

It is of interest to estimate  $\eta$  based on observed atmospheric transient eddies<sup>23-25</sup> (i.e., the synoptic scale fluctuations). Here we only briefly report these calculations. A more detailed discussion is presented in Refs. 17 and 26.

Typical synoptic scales for the atmosphere give length scales of 1000 km and velocity scales of  $10 \text{ m sec}^{-1}$ . Based on observed two-dimensional spectra of the 300, 500, and 700 mb seasonally and annually averaged atmospheric transient eddies,  $a\eta \simeq 2$  (corresponding to a zonal wavenumber<sup>27</sup> of about 8) with a standard deviation of about 0.75. Thus we conclude (qualitatively) that  $a\eta < J_{1,1}$  (to about two standard deviations) and consequently that  $c > 0$  atmospheric modons are stable to the observed averaged transient eddies. Because of the strong suggestion that  $a\eta < J_{1,1}$  the stability of  $c < -1$  atmospheric modons cannot be determined (i.e., the sufficient stability condition is likely not satisfied).

We hasten to add a cautionary note. While the transient eddies do characterize the synoptic scale atmospheric fluctuations, other time averages may emphasize other synoptic scales and thus  $\eta$  is likely to be a function of the averaging process. Intermittency in the spectrum for the transient eddies will therefore tend to vary the value of  $\eta$  observed at any given time.

In the ocean typical mesoscale length and velocity scales are approximately 100 km and  $10 \text{ cm sec}^{-1}$ , respectively. Estimating  $\eta$  based on a wavenumber spectrum of oceanic mesoscale variability computed by Fu<sup>28</sup> we calculate  $a\eta \simeq 8$  (corresponding to an average fluctuation length scale of about 250 km) with standard deviation about 2. This calculation seems to (qualitatively) imply that ground state  $c < -1$  modons are stable as are the higher state  $c > 0$  modons. Again a cautionary note must be made. The extreme paucity of available oceanic data implies the above calculation may be completely inaccurate. However, we feel the above calculations at least qualitatively support the notion that modons (as an example of a coherent vortex) may be stable in a typically observed geophysical fluid regime.

## V. CONCLUSIONS

It is known that linear Liapunov stability implies spectral stability, but the converse is not in general true.<sup>12,19</sup> Thus it should not be surprising that the stability analysis presented here differs from Laedke–Spatschek and Pierini in several respects.

Our study has been based on the introduction of a parameter  $\eta$  called the “generalized disturbance wavenumber,” related to the ratio of the initial values of disturbance enstrophy to energy. We have shown that  $c < -1$  drift waves are Liapunov stable when  $\eta \gg \kappa$  (where  $\kappa$  is the modon wavenumber), in contrast to Laedke–Spatschek’s proof of spectral stability against all disturbances. However, our analysis applies to a much larger class of drift waves. The results presented here apply to both  $c < -1$  and  $c > 0$  modons and to all states (ground and higher). The stability of  $c > 0$  drift waves has been shown when  $\eta \leq \kappa$ . The theory developed here recovers Pierini’s result as a special case. We have also been able to provide  $L^2$ -type *a priori* bounds [i.e., (7) and (8)] on the growth of perturbations. The instability mechanism between  $c < -1$  and  $c > 0$  modons has been shown to be quite different, and satisfies a Fjørtoft-type<sup>22</sup> energy cascade theorem.

Qualitative estimates of the “generalized disturbance wavenumber”, based on seasonally and annually-averaged

fluctuation energy spectra, suggest that drift waves may be stable for typically observed geophysical fluid regimes.

Finally, we note that the necessary instability conditions suggested by Theorems I and II may be exploited in order to explicitly calculate asymptotic representations of exponential growth rates of an instability. For example, in the limit of very large disturbance wavenumbers (needed for the instability of  $c > 0$  modons), growth rates may be calculated using a WKB-type theory. On the other hand, in the limit of very small wavenumber disturbances (needed for the instability of  $c < -1$  modons), it may be possible to model the instability process as a dipole in a slowly oscillating current.

## ACKNOWLEDGMENTS

The author thanks Dr. Lawrence A. Mysak and Dr. Paul LeBlond for many helpful discussions on this research.

This research was supported by Natural Sciences and Engineering Research Council of Canada and U. S. Office of Naval Research grants awarded to Dr. Mysak, and by a U. B. C. Department of Mathematics Teaching Assistantship. Final preparation of the manuscript was supported by National Science Foundation grants awarded to Dr. Glenn Flierl of the Massachusetts Institute of Technology.

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