Nonlinear Stability of Baroclinic Fronts in a Channel with Variable Topography

By Richard H. Karsten and Gordon E. Swaters

The governing equations describing baroclinic bottom-trapped fronts in a channel with variable bottom topography are shown to be a noncanonical Hamiltonian system. The Hamiltonian formalism is exploited to derive a variational principle for arbitrary steady solutions based on an appropriately constrained energy functional. The variational principle is exploited to obtain formal and nonlinear stability conditions. In the infinitesimal amplitude limit, these stability conditions reduce to previously obtained normal mode results for the transverse gradient of the mean frontal potential vorticity.

1. Introduction

One of the problems in applying quasi-geostrophic (QG) theory to the dynamics of oceanographic fronts is that QG theory explicitly requires that the dynamic deflections of the frontal interface are small in comparison to the scale thickness of the front. This restriction is rarely satisfied in practice because these flows usually possess the property that the scale amplitude of the interface deflections is the same as the scale height of the front itself (see Figure 1 for the geometry of the shallow water system underlying our theory). There are many oceanographic examples of these kinds of flows

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(e.g., [1, 2]. A typical feature observed in the data associated with these flows is low frequency fluctuations which are interpreted as resulting from the underlying inertial instability of these currents. For example, we have recently applied the theory developed here to studying the instability of the gravity current associated with deep water replacement in the Strait of Georgia as an attempt to understand aspects of the low-frequency current fluctuations observed there [3].

Griffiths et al. [4] analyzed the stability of these flows using a reduced gravity model which focused on the transition to instability associated with the release of mean frontal kinetic energy. This model was not able to reproduce important aspects of the instability spectra observed in experiments and these differences were attributed to the presence of an unstable mode outside the applicability of the Griffiths et al. [4] theory.

Subsequently, Swaters [5] derived an alternative model for the instability of fronts on a sloping bottom that focused on the intrinsically baroclinic destablization associated with the release of mean frontal potential energy. This model was able to qualitatively reproduce many of the observed instability characteristics and in particular was able to reproduce a curious dipole-like mode observed in the Griffiths et al. [4] experiments but not previously explained. However, the Swaters model and stability analysis were restricted to a very simple topographic and current configuration. Recently, Karsten et al. [3] has generalized this model to study the instability of more realistic current and topographic configurations. The principal purpose of this paper is to develop a general stability theory for the frontal dynamics model of Karsten et al. [3].

2. Problem formulation and Hamiltonian structure

2.1. Derivation of the governing equations

Since a detailed derivation of the equations is given by Swaters [5] and Karsten et al. [3], our presentation is very brief here. The basic model is an f-plane, two-layer, shallow water system with varying cross-channel bottom topography (see Figure 1). Assuming that the bottom variations are not too large and that the upper layer is much thicker than the scale height of the lower layer, it can be shown that the leading order dynamics are described by

$$(\Delta\eta + h)_t + J(\eta, \Delta\eta + h + h_B) = 0, \tag{2.1a}$$

$$h_t + J(\eta + h_B, h) = 0, \tag{2.1b}$$

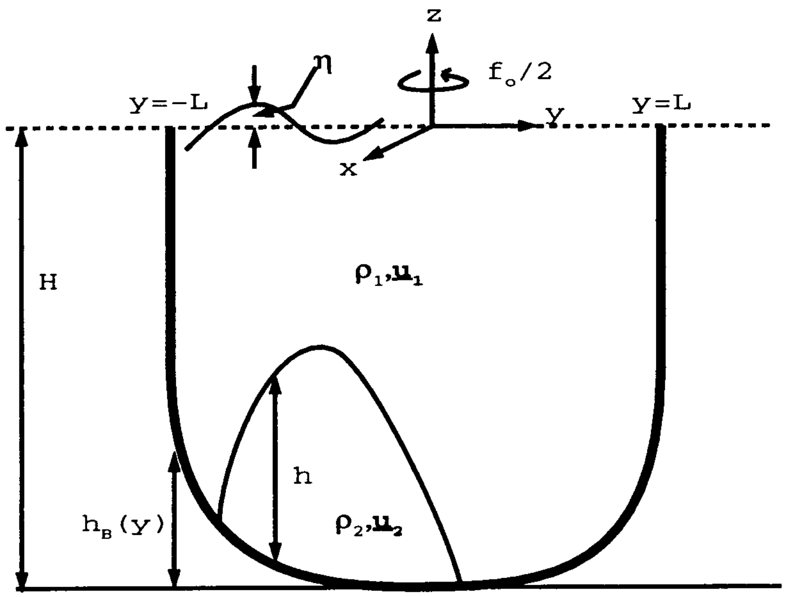

Figure 1. The geometry of the two-layer system, in a channel with variable bottom topography given by $h_B(y)$ and walls at $y = \pm L$.

where $J(A,B) \equiv A_x B_y - A_y B_x$, with the auxiliary geostrophic relations

$$\mathbf{u}_1 = (u_1, v_1) = \hat{e}_3 \times \nabla\eta, \tag{2.2a}$$

$$\mathbf{u}_2 = (u_2, v_2) = \hat{e}_3 \times \nabla(\eta + h + h_B), \tag{2.2b}$$

where $\mathbf{u}_1(x,y,t)$, $\mathbf{u}_2(x,y,t)$, $\eta(x,y,t)$, $h(x,y,t)$, and $h_B(y)$ are the nondimensional velocity in the upper layer, velocity in the lower layer, reduced pressure in the upper layer, lower layer height, and bottom topography, respectively. It should be noted that we have exaggerated the vertical scales of the bottom variations and frontal height in Figure 1 to effectively illustrate our geometry.

The spatial domain we work with is a periodic channel given by

$$\Omega = \{x_{\mathrm{L}} < x < x_{\mathrm{R}}, |y| < L\}. \tag{2.3}$$

The no-normal-flow boundary conditions on the channel walls, i.e., $v_1 = v_2 = 0$ on $|y| = L$, can be reduced using (2.2a,b) to

$$\eta(x, -L, t) = 0, \qquad \eta(x, L, t) = \eta_L, \tag{2.4a}$$

$$h_x(x, \pm L, t) = 0, \tag{2.4b}$$

where η_L is a constant. We assume that η and h are also smoothly periodic at $x = x_L$ and x_R.

2.2. *Hamiltonian formulation*

The dynamical system (2.1) can be written in the noncanonical Hamiltonian form (e.g., [6] or [7]; see also [8])

$$\mathbf{q}_t = [\mathbf{q}, H], \tag{2.5}$$

where

$$\mathbf{q} = (q_1, q_2)^T = (\Delta\eta + h, h)^T, \tag{2.6}$$

the Hamiltonian is given by

$$H(\mathbf{q}) \equiv \frac{1}{2}\int\int_\Omega \left\{\nabla\eta\cdot\nabla\eta + (h + h_B)^2\right\} dxdy - \eta_L \int_{\partial\Omega} \nabla\eta\cdot\mathbf{n}\, dS, \tag{2.7}$$

and the Poisson bracket is given by

$$[F,G] \equiv \int\int_\Omega \left\{ \frac{\delta F}{\delta q_1} J\left(\frac{\delta G}{\delta q_1}, q_1 + h_B\right) + \frac{\delta F}{\delta q_2} J\left(q_2, \frac{\delta G}{\delta q_2}\right)\right\} dxdy. \tag{2.8}$$

It is straightforward to show that the Poisson bracket satisfies all the required algebraic properties. (In the Appendix, we show that it satisfies the Jacobi identity.)

The Casimirs, the conserved quantities that lie in the kernel of the Poisson bracket, satisfy

$$[F,C] = 0,$$

for all sufficiently smooth functionals $F(\mathbf{q})$. Using the definition of the bracket (2.8) we obtain the general solution

$$C(\mathbf{q}) = \int\int_\Omega \Phi_1(q_1 + h_B)\, dxdy + \int\int_\Omega \Phi_2(q_2)\, dxdy, \tag{2.9}$$

where Φ_1 and Φ_2 are sufficiently smooth functions. The Casimirs are needed to construct a variational principle for steady solutions to the model.

3. Stability of steady solutions

3.1. *Variational principle*

To examine the stability of steady solutions, we first require a variational principle for arbitrary steady flows. General steady solutions $\eta = \eta_0(x, y)$ and $h = h_0(x, y)$ of (2.1a,b) satisfy

$$J(\eta_0, \Delta\eta_0 + h_0 + h_B) = 0,$$

$$J(\eta_0 + h_B, h_0) = 0,$$

which can be integrated to give

$$\eta_0 = F_1(\Delta\eta_0 + h_0 + h_B), \qquad (3.1a)$$

$$\eta_0 = F_2(h_0) - h_B, \qquad (3.1b)$$

where F_1 and F_2 are arbitrary functions of their arguments. It can be shown (e.g., [9]) that an analogue of Andrews' theorem [10] applies, which states that the only solutions that may satisfy the following stability conditions in the periodic channel domain (2.3) are those that are independent of x. Therefore, we henceforth assume that $\eta_0 = \eta_0(y)$ and $h_0 = h_0(y)$.

Consider the constrained Hamiltonian

$$\mathscr{H} = H + C, \qquad (3.2)$$

where H is the Hamiltonian given by (2.7) and C is the Casimir given by (2.9). It follows that

$$\delta\mathscr{H}(\eta, h) = \int\int_\Omega \{(\Phi_1' - \eta)\,\delta q_1 + (\Phi_2' + h + \eta + h_B)\,\delta h\}\,dx\,dy, \quad (3.3)$$

where $\Phi_1' \equiv d\Phi_1/d(q_1 + h_B)$ and $\Phi_2' \equiv d\Phi_2/dh$. Thus, we see that

$$\delta\mathscr{H}(\eta_0, h_0) = 0,$$

provided we choose the Casimir density functions Φ_1 and Φ_2 to be

$$\Phi_1(q_1 + h_B) = \int_0^{q_1 + h_B} F_1(\xi)\,d\xi, \qquad (3.4a)$$

$$\Phi_2(h) = -\int_0^h F_2(\xi)\,d\xi - \frac{h^2}{2}, \qquad (3.4b)$$

where F_1 and F_2 satisfy (3.1a) and (3.1b), respectively.

3.2. Linear stability conditions

The linear stability conditions are obtained by deriving conditions that ensure that the second variation of $\mathscr{H}$ evaluated at the steady solution is definite. The second variation of $\mathscr{H}$ is given by

$$\delta^2\mathscr{H}(\eta,h) = \iint_\Omega \left\{\nabla\delta\eta\cdot\nabla\delta\eta + \Phi_1''(\Delta\delta\eta+\delta h)^2 + (\Phi_2''+1)(\delta h)^2\right\} dx dy$$

$$+ \iint_\Omega \{(\Phi_1'-\eta)(\Delta\delta^2\eta+\delta^2 h) + (\Phi_2'+h+\eta+h_B)\delta^2 h\} dx dy.$$

It follows that

$$\delta^2\mathscr{H}(\eta_0,h_0) = \iint_\Omega \left\{\nabla\delta\eta\cdot\nabla\delta\eta + F_{10}'(\Delta\delta\eta+\delta h)^2 - F_{20}'(\delta h)^2\right\} dx dy, \quad (3.5)$$

where it is understood that $F_{10}' \equiv F_1'(\Delta\eta_0 + h_0 + h_B)$ and $F_{20}' \equiv F_2'(h_0)$. It is well known that $\delta^2\mathscr{H}(\eta_0,h_0)$ is an invariant of the linear stability problem associated with $\eta_0(y)$ and $h_0(y)$.

Our remaining analysis is facilitated by the introduction of the Poincaré inequality (e.g., [11])

$$\iint_\Omega \nabla(\delta\eta)\cdot\nabla(\delta\eta)\, dx dy \le \lambda \iint_\Omega [\Delta(\delta\eta)]^2 dx dy, \quad (3.6)$$

where $\lambda = (2L/\pi)^2$ and the perturbation boundary condition $\delta\eta = 0$ on $\partial\Omega$ has been used. Substitution of (3.6) into (3.5) leads to

$$\iint_\Omega \left\{F_{10}'(\Delta\delta\eta+\delta h)^2 - F_{20}'(\delta h)^2\right\} d dy \le \delta^2\mathscr{H}(\eta_0,h_0)$$

$$\le \iint_\Omega \left\{(\lambda+F_{10}')(\Delta\delta\eta+\gamma\delta h)^2 + (\lambda\gamma - F_{20}')(\delta h)^2\right\} dx dy, \quad (3.7)$$

where

$$\gamma \equiv \frac{F_{10}'}{\lambda + F_{10}'}. \quad (3.8)$$

Stability of the steady solutions η_0 and h_0 will be established if $\delta^2\mathscr{H}(\eta_0,h_0)$ is definite for all perturbations $\delta\eta$ and δh.

The steady solutions $h = h_0(y)$ and $\eta = \eta_0(y)$ are linearly stable in the sense of Liapunov with respect to the perturbation norm

$$\|\delta\mathbf{q}\|_1^2 = \iint_\Omega \left\{\nabla(\delta\eta)\cdot\nabla(\delta\eta) + (\Delta\delta\eta)^2 + (\delta h)^2\right\} dx dy, \quad (3.9)$$

if the steady solutions given by (3.1) satisfy

$$F'_{10} > 0, \tag{3.10a}$$

$$F'_{20} < 0, \tag{3.10b}$$

for all $(x, y) \in \Omega$. Alternatively, the steady solutions are linearly stable in the sense of Liapunov with respect to the perturbation norm

$$\|\delta \mathbf{q}\|_2^2 = \int\int_{\Omega} \{(\Delta \delta\eta)^2 + (\delta h)^2\}\, dx dy, \tag{3.11}$$

if the steady solutions given by (3.1) satisfy

$$F'_{10} < -\lambda, \tag{3.12a}$$

$$F'_{20} > \lambda\gamma, \tag{3.12b}$$

for all $(x, y) \in \Omega$, where γ is given by (3.8) and $\lambda > 0$ is given by (3.6).

Conditions (3.10) and (3.12) guarantee that $\delta^2\mathscr{H}(\eta_0, h_0)$ is positive and negative definite, respectively. All that remains is to show the following a priori estimates. Assuming conditions (3.10) hold, it follows from (3.5) and (3.9) that

$$\begin{aligned}
\|\delta \mathbf{q}\|_1^2 &= \int\int_{\Omega} \{\nabla(\delta\eta)\cdot\nabla(\delta\eta) + (\Delta\delta\eta)^2 + (\delta h)^2\}\, dx dy \\
&\leq \Gamma_1^{-1}\delta^2\mathscr{H}(\eta_0, h_0) = \Gamma_1^{-1}[\delta^2\mathscr{H}(\eta_0, h_0)]_{t=0} \\
&\leq \Gamma_1^{-1}\Gamma_2\int\int_{\Omega} \{\nabla(\delta\tilde{\eta})\cdot\nabla(\delta\tilde{\eta}) + (\Delta\delta\tilde{\eta})^2 + (\delta\tilde{h})^2\}\, dx dy \\
&= \Gamma_1^{-1}\Gamma_2\|\delta\tilde{\mathbf{q}}\|_1^2,
\end{aligned}$$

where

$$\Gamma_1 = \min\left[1, \min_{\Omega} F'_{10}, \min_{\Omega}(-F'_{20})\right]/3 > 0,$$

$$\Gamma_2 = 3\max\left[1, \max_{\Omega} F'_{10}, \max_{\Omega}(-F'_{20})\right] > 0,$$

the inequality $(x + y)^2 \leq 2(x^2 + y^2)$ has been used, and $\delta\tilde{\mathbf{q}} = \delta\mathbf{q}|_{t=0}$ with corresponding definitions for $\delta\tilde{\eta}$ and $\delta\tilde{h}$. This establishes the linear stability of (η_0, h_0) provided conditions (3.10) hold.

Similarly, when conditions (3.12) hold it follows from (3.7) and (3.11) that

$$\begin{aligned}
\|\delta\mathbf{q}\|_2^2 &= \iint_\Omega \{(\Delta\delta\eta)^2 + (\delta h)^2\}\, dx dy \\
&\leq \max\left(2, 1 + 2\max_\Omega \gamma^2\right) \iint_\Omega \{(\Delta\delta\eta + \gamma\delta h)^2 + (\delta h)^2\}\, dx dy \\
&\leq \tilde{\Gamma}_1^{-1} \delta^2 \mathscr{H}(\eta_0, h_0) = \tilde{\Gamma}_1^{-1} [\delta^2 \mathscr{H}(\eta_0, h_0)]_{t=0} \\
&\leq \tilde{\Gamma}_1^{-1} \tilde{\Gamma}_2 \|\delta\tilde{\mathbf{q}}\|_2^2,
\end{aligned}$$

where

$$\tilde{\Gamma}_1 = \frac{\max[\lambda + \max_\Omega F'_{10}, \max_\Omega(-F'_{20} + \lambda\gamma)]}{\max(2, 1 + 2\max_\Omega \gamma^2)} < 0,$$

$$\tilde{\Gamma}_2 = 3\min\left[\min_\Omega F'_{10}, \min_\Omega(-F'_{20})\right] < 0,$$

which establishes the linear stability of (η_0, h_0) provided conditions (3.12) hold.

It should be noted that conditions (3.10a,b) can be reduced to the conditions

$$F'_{10} \geq 0, \tag{3.13a}$$

$$F'_{20} \leq 0, \tag{3.13b}$$

provided the stability norm considered is

$$\|\delta\mathbf{q}\|^2 = \iint_\Omega \nabla(\delta\eta) \cdot \nabla(\delta\eta)\, dx dy.$$

3.3. Nonlinear stability conditions

The nonlinear results proceed from examining the invariant functional

$$\begin{aligned}
\mathscr{L}(\mathbf{q}) &= H(\mathbf{q} + \mathbf{q}_0) - H(\mathbf{q}_0) + C(\mathbf{q} + \mathbf{q}_0) - C(\mathbf{q}_0) \\
&= \frac{1}{2} \iint_\Omega \nabla\eta \cdot \nabla\eta\, dx dy \\
&\quad + \iint_\Omega \left\{ \int_{q_{10} + h_B}^{q_1 + q_{10} + h_B} F_1(\xi)\, d\xi - F_1(q_{10} + h_B) q_1 \right\} dx dy \\
&\quad - \iint_\Omega \left\{ \int_{h_0}^{h + h_0} F_2(\xi)\, d\xi - F_2(h_0) h \right\} dx dy,
\end{aligned} \tag{3.14}$$

where H and C are given by (2.7) and (2.9) respectively, with the Casimir densities determined by (3.4a, b), and where $\mathbf{q}_0 = (q_{10}, q_{20})^T = (\Delta\eta_0 + h_0, h_0)^T$, where η_0 and h_0 are determined by (3.1a, b). The variable $\mathbf{q}$ represents the departure of the nonlinear time-dependent solution, $\mathbf{q}_T \equiv \mathbf{q} + \mathbf{q}_0$, from the steady solution, $\mathbf{q}_0$, and is therefore referred to as the (finite-amplitude) perturbation flow. Clearly, $\mathscr{L}(\mathbf{q})$ is conserved by the full nonlinear dynamics (2.1).

Because of the loss of compactness in Hilbert spaces, the definiteness of $\delta^2\mathscr{H}(\eta_0, h_0)$ is not sufficient to ensure that η_0 and h_0 correspond to a local extremum of the constrained Hamiltonian $\mathscr{H}(\eta, h)$. Nonlinear stability will require that the convexity conditions

$$\alpha_1 \le F_1'(\xi) \le \beta_1, \tag{3.15a}$$

$$\alpha_2 \le F_2'(\xi) \le \beta_2, \tag{3.15b}$$

hold for all arguments ξ where the prime indicates $d/d\xi$ and where α_1, β_1, α_2, and β_2 are real numbers. Integrating (3.15a, b) twice and substituting into (3.14) implies

$$\begin{aligned}\frac{1}{2}\int\int_\Omega \left\{\nabla\eta\cdot\nabla\eta + \alpha_1(\Delta\eta + h)^2 - \beta_2 h^2\right\} dx\,dy \\ \le \mathscr{L}(\mathbf{q}) \\ \le \frac{1}{2}\int\int_\Omega \left\{\nabla\eta\cdot\nabla\eta + \beta_1(\Delta\eta + h)^2 - \alpha_2 h^2\right\} dx\,dy,\end{aligned} \tag{3.16}$$

for all η and h. Substitution of the Poincaré inequality (3.6), written in terms η, into (3.16) leads to

$$\begin{aligned}\frac{1}{2}\int\int_\Omega \left\{\alpha_1(\Delta\eta + h)^2 - \beta_2 h^2\right\} dx\,dy \\ \le \mathscr{L}(\mathbf{q}) \\ \le \frac{1}{2}\int\int_\Omega \left\{(\lambda + \beta_1)(\Delta\eta + \gamma h)^2 + (\gamma\lambda - \alpha_2) h^2\right\} dx\,dy,\end{aligned} \tag{3.17}$$

where

$$\gamma \equiv \frac{\beta_1}{\lambda + \beta_1}. \tag{3.18}$$

Nonlinear stability of the steady solutions η_0 and h_0 will be established if $\mathscr{L}(\mathbf{q})$ is definite for all perturbations η and h.

The steady solutions $h = h_0(y)$ and $\eta = \eta_0(y)$ are nonlinearly stable in the sense of Liapunov with respect to the norm

$$\|\mathbf{q}\|_1^2 = \int\int_\Omega \{\nabla\eta\cdot\nabla\eta + (\Delta\eta)^2 + h^2\}\, dx dy, \tag{3.19}$$

if the Casimir densities functions $F_1(\xi)$ and $F_2(\xi)$, which determine the steady solution η_0 and h_0 through the relations (3.1a,b), satisfy the convexity estimates

$$0 < \alpha_1 \leq F_1'(\xi) \leq \beta_1 < \infty, \tag{3.20a}$$

$$-\infty < \alpha_2 \leq F_2'(\xi) \leq \beta_2 < 0, \tag{3.20b}$$

for all ξ and some real constants α_1, β_1, α_2, and β_2. Alternatively, the steady solutions are nonlinearly stable in the sense of Liapunov with respect to the norm

$$\|\mathbf{q}\|_2^2 = \int\int_\Omega \{(\Delta\eta)^2 + h^2\}\, dx dy, \tag{3.21}$$

if the Casimir densities functions $F_1(\xi)$ and $F_2(\xi)$ satisfy the convexity estimates

$$-\infty < \alpha_1 \leq F_1'(\xi) \leq \beta_1 < -\lambda < 0, \tag{3.22a}$$

$$\lambda\gamma < \alpha_2 \leq F_2'(\xi) \leq \beta_2 < \infty, \tag{3.22b}$$

for all ξ and some real constants α_1, β_1, α_2, β_2, where λ is the Poincaré constant given by (3.6) and γ is given by (3.18).

Clearly (3.20) and (3.22) establish the positive and negative definiteness of $\mathscr{L}(\mathbf{q})$, respectively. All that remains is to establish the following estimates.

From (3.16), (3.19), and (3.20) it follows that

$$
\begin{aligned}
\|\mathbf{q}\|_1^2 &= \int\int_\Omega \left\{\nabla\eta\cdot\nabla\eta + (\Delta\eta + h - h)^2 + h^2\right\} dx\,dy \\
&\leq \int\int_\Omega \left\{\nabla\eta\cdot\nabla\eta + 2(\Delta\eta + h)^2 + 3h^2\right\} dx\,dy \\
&\leq \Gamma\int\int_\Omega \left\{\nabla\eta\cdot\nabla\eta + \alpha_1(\Delta\eta + h)^2 - \beta_2 h^2\right\} dx\,dy \\
&\leq 2\Gamma\mathscr{L}(\mathbf{q}) \equiv 2\Gamma\mathscr{L}(\tilde{\mathbf{q}}) \\
&\leq \Gamma\int\int_\Omega \left\{\nabla\tilde{\eta}\cdot\nabla\tilde{\eta} + 2\beta_1(\Delta\tilde{\eta})^2 + (2\beta_1 - \alpha_2)\tilde{h}^2\right\} dx\,dy \\
&\leq \tilde{\Gamma}\|\tilde{\mathbf{q}}\|_1^2,
\end{aligned}
$$

where $\tilde{\mathbf{q}} = \mathbf{q}(x, y, t = 0)$, with corresponding definitions for $\tilde{\eta}$, $\tilde{h}$, $\Gamma^{-1} \equiv \min(1, \alpha_1, -\beta_2)/3 > 0$, $\tilde{\Gamma} \equiv \Gamma \max(1, 2\beta_1, 2\beta_1 - \alpha_2) > 0$ and we have used the invariance of $\mathscr{L}$ and the inequality $(x + y)^2 \leq 2(x^2 + y^2)$. It follows that

$$
\|\mathbf{q}\|_1 \leq (\tilde{\Gamma})^{1/2}\|\tilde{\mathbf{q}}\|_1,
$$

which thereby establishes nonlinear stability of (η_0, h_0) provided conditions (3.20) hold.

Similarly, for conditions (3.22), it follows from (3.17) and (3.21) that

$$
\|\mathbf{q}\|_2 \leq \hat{\Gamma}^{1/2}\|\tilde{\mathbf{q}}\|_2,
$$

where

$$
\hat{\Gamma} = \frac{\max(2, 1 + 2\gamma^2)}{\max(\lambda + \beta_1, \lambda\gamma - \alpha_2)}\min(2\alpha_1, 2\alpha_1 - \beta_2) > 0,
$$

establishing nonlinear stability of (η_0, h_0) provided conditions (3.22) hold.

As in the linear analysis, it is not necessary to bound α_1 and β_1 away from zero provided the stability norm is

$$
\|\mathbf{q}\|^2 = \int\int_\Omega \nabla\eta\cdot\nabla\eta\, dx\,dy.
$$

3.4. *Discussion of the stability results*

It is more interesting from a physical point of view to recast the stability conditions directly in terms of mean flow variables. It follows from (3.1a,b) that

$$F'_{10} = \frac{U_0}{U_{0_{yy}} - h_{0_y} - h_{B_y}}, \tag{3.23a}$$

$$F'_{20} = \frac{-U_0 + h_{B_y}}{h_{0_y}}, \tag{3.23b}$$

where $U_0(y) \equiv -d(\eta_0)/dy$ is the x-direction velocity in the upper layer. It then follows that (3.10) can be recast into the form

$$\frac{U_0}{U_{0_{yy}} - h_{0_y} - h_{B_y}} > 0, \tag{3.24a}$$

$$\frac{U_0 - h_{B_y}}{h_{0_y}} > 0, \tag{3.24b}$$

and (3.12) into the form

$$\frac{U_0}{U_{0_{yy}} - h_{0_y} - h_{B_y}} < -\lambda < 0, \tag{3.25a}$$

$$\frac{U_0 - h_{B_y}}{h_{0_y}} < -\frac{\lambda U_0}{\lambda\left[U_{0_{yy}} - h_{0_y} - h_{B_y}\right] + U_0}, \tag{3.25b}$$

for all $(x, y) \in \Omega$. It is clear in this notation that the stability conditions (3.24a,b) can be interpreted as the analogue of Fjørtoft's theorem (e.g. [12]) in a suitable reference frame.

To compare these results to those of Swaters [5], we assume $U_0 \equiv 0$. It follows that (3.13a,b) reduce to

$$\frac{h_{B_y}}{h_{0_y}} \leq 0. \tag{3.26}$$

This is identical to the sufficient condition for stability found in Ref. [5]. The corresponding nonlinear stability conditions reduce to the existence of real constants α and β such that

$$-\infty < \alpha \leq F'_2(\xi) \leq \beta \leq 0, \tag{3.27}$$

for all ξ, where $F_2(h_0) = h_B$ as defined by (3.1b).

The two conditions (3.26) and (3.27) appear similar, but in fact the nonlinear conditions are much stricter. This is best illustrated by an example. In Figure 2, two fronts are shown. The first has $h_B(y) = 1 - h_0(y)$ and the second $h_B(y) = (1 - h_0(y))^2$ for $-1 \le y \le 1$. Both fronts satisfy the linear stability condition (3.26), but only the first satisfies the nonlinear stability condition (3.27), since $F_2(\xi) = 1 - \xi$ implies $F_2'(\xi) = -1$, whereas

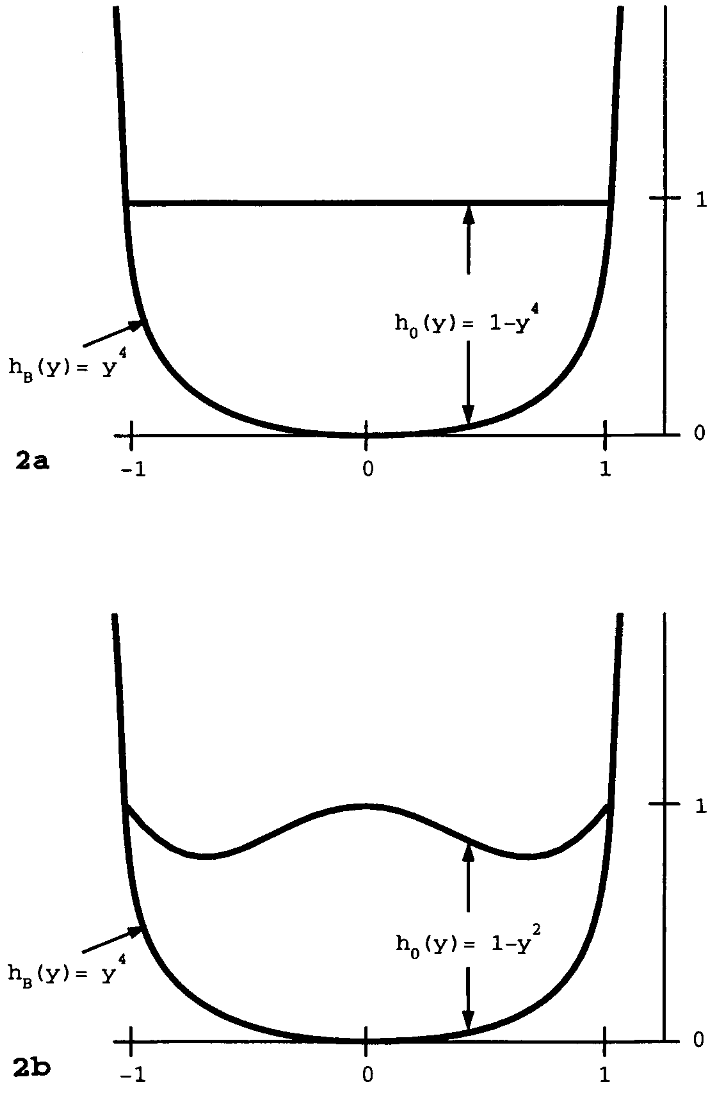

Figure 2. Two fronts where $h_B(y) = y^4$. In (a) $h_0(y) = 1 - y^4$, while in (b) $h_0(y) = 1 - y^2$. Both fronts satisfy the linear stability condition, but only the front in (a) satisfies the nonlinear stability condition.

in the second case $F_2(\xi)=(1-\xi)^2$ implies $F_2'(\xi)=-2(1-\xi)$, which is positive for $\xi>1$. It should be noted that the topographic and frontal variations illustrated in Figure 2 are with respect to nondimensional quantities. The $O(1)$ variations shown in these figures are consistent with the asymptotic requirements in the derivation of (2.1).

Physically, it is obvious that Figure 2a corresponds to a nonlinearly stable configuration since there can be no release of any potential energy from the perfectly horizontal front. In Figure 2b, the flow is linearly stable according to our model but destabilization for sufficiently large perturbations cannot be ruled out. Indeed, we expect that if sufficiently perturbed, the configuration in Figure 2b would evolve toward the one shown in Figure 2a.

These examples serve to illustrate that there is a necessary coupling between the ambient topography and frontal configuration to generate instability. Not every baroclinic front over sloping topography will be linearly unstable according to this theory. Configurations that are linearly stable may still be susceptible to nonlinear instability, as shown above. This raises the issue of attempting to apply the model to real oceanographic flows. We have recently applied the model to study the stability characteristics of deep water renewal in the Strait of Georgia and the interested reader is referred there [3].

Appendix: Jacobi identity for the Poisson bracket

Jacobi's identity is given by

$$[[F,G],K] + \text{cyc.} = 0, \tag{A.1}$$

for all sufficiently smooth functionals $F(\mathbf{q})$, $G(\mathbf{q})$, and $K(\mathbf{q})$ and where cyc. indicates all cyclic perturbations.

To prove that the Poisson bracket given by (2.8) satisfies this condition we follow the method of Scinocca and Shepherd [13]. We introduce the notation

$$[F,G] = \left\langle \frac{\delta F}{\delta \mathbf{q}}, \Gamma \frac{\delta G}{\delta \mathbf{q}} \right\rangle,$$

where Γ is given by

$$\Gamma_{ij} = -\delta_{i1}\delta_{j1}J(q_1+h_B,*) + \delta_{i2}\delta_{j2}J(q_2,*),$$

and where $\langle \mathbf{a},\mathbf{b}\rangle$ is the inner product

$$\langle \mathbf{a},\mathbf{b}\rangle = \int_\Omega \mathbf{a}\mathbf{b}^T\,dx\,dy.$$

Equation (A.1) reduces to

$$\left\langle \frac{\delta \langle \mathbf{f}, \Gamma \mathbf{g} \rangle}{\delta \mathbf{q}}, \Gamma \mathbf{k} \right\rangle + \text{cyc.} = 0, \tag{A.2}$$

where we have introduced

$$\mathbf{f} = (f_1, f_2) = \frac{\delta F}{\delta \mathbf{q}}, \qquad \mathbf{g} = (g_1, g_2) = \frac{\delta G}{\delta \mathbf{q}}, \qquad \mathbf{k} = (k_1, k_2) = \frac{\delta K}{\delta \mathbf{q}}.$$

The first term in (A.2) can be expanded to give

$$\begin{aligned} \left\langle \left\langle \frac{\delta \mathbf{f}}{\delta \mathbf{q}}, \Gamma \mathbf{g} \right\rangle, \Gamma \mathbf{k} \right\rangle &= \left\langle \left\langle \mathbf{f}, \frac{\delta \Gamma}{\delta \mathbf{q}} \mathbf{g} \right\rangle, \Gamma \mathbf{k} \right\rangle + \left\langle \left\langle \mathbf{f}, \Gamma \frac{\delta \mathbf{g}}{\delta \mathbf{q}} \right\rangle, \Gamma \mathbf{k} \right\rangle \\ &= \left\langle \left\langle \frac{\delta \mathbf{f}}{\delta \mathbf{q}}, \Gamma \mathbf{g} \right\rangle, \Gamma \mathbf{k} \right\rangle - \left\langle \left\langle \frac{\delta \mathbf{g}}{\delta \mathbf{q}}, \Gamma \mathbf{f} \right\rangle, \Gamma \mathbf{k} \right\rangle + \left\langle \left\langle \mathbf{f}, \frac{\delta \Gamma}{\delta \mathbf{q}} \mathbf{g} \right\rangle, \Gamma \mathbf{k} \right\rangle, \end{aligned}$$

where we have used the skew symmetry of Γ. It follows that

$$\left\langle \left\langle \frac{\delta \mathbf{f}}{\delta \mathbf{q}}, \Gamma \mathbf{g} \right\rangle, \Gamma \mathbf{k} \right\rangle - \left\langle \left\langle \frac{\delta \mathbf{g}}{\delta \mathbf{q}}, \Gamma \mathbf{f} \right\rangle, \Gamma \mathbf{k} \right\rangle + \text{cyc.} = 0,$$

so that the Jacobi identity is reduced to establishing

$$\left\langle \left\langle \mathbf{f}, \frac{\delta \Gamma}{\delta \mathbf{q}}, \mathbf{g} \right\rangle, \Gamma \mathbf{k} \right\rangle + \text{cyc.} = 0. \tag{A.3}$$

Since we have eliminated all terms in $\delta \mathbf{f}/\delta \mathbf{q}$, $\delta \mathbf{q}/\delta \mathbf{q}$, and $\delta \mathbf{k}/\delta \mathbf{q}$, we can, without loss of generality, write

$$\begin{aligned} \left\langle \frac{\delta \mathbf{f}}{\delta \mathbf{q}}, \Gamma \mathbf{g} \right\rangle &= \frac{\delta}{\delta \mathbf{q}} \langle \mathbf{f}, \Gamma \mathbf{g} \rangle \\ &= \frac{\delta}{\delta \mathbf{q}} \int\int_{\Omega} \{ -f_1 J(q_1 + h_B, g_1) + f_2 J(q_2, g_2) \} \, dx\,dy \\ &= \frac{\delta}{\delta \mathbf{q}} \int\int_{\Omega} \{ (q_1 + h_B) J(f_1, g_1) + q_2 J(g_2, f_2) \} \, dx\,dy \\ &\equiv \frac{\delta(*)}{\delta \mathbf{q}} = \langle J(f_1, g_1), J(g_2, f_2) \rangle, \end{aligned} \tag{A.4}$$

where we have used $f_1 = f_2 = 0$ on $\partial\Omega$. It follows that (A.3) can be written as

$$\int\int_\Omega \left\{ -\frac{\delta(*)}{\delta q_1} J(q_1 + h_B, k_1) + \frac{\delta(*)}{\delta q_2} J(q_2, k_2) \right\} dxdy + \text{cyc.}$$

$$= \int\int_\Omega \left\{ (q_1 + h_B) J\left(\frac{\delta(*)}{\delta q_1}, k_1 \right) + q_2 J\left(k_2, \frac{\delta(*)}{\delta q_2} \right) \right\} dxdy + \text{cyc.}$$

$$= \int\int_\Omega \{ (q_1 + h_B) [J(J(f_1, g_1), k_1) + \text{cyc.}]$$

$$+ q_2 [J(k_2, J(g_2, f_2)) + \text{cyc.}] \} dxdy$$

$$= 0,$$

where we have used (A.4) and that

$$J(J(A,B),C) + \text{cyc.} = 0,$$

for sufficiently smooth functions A, B, C. Therefore, the Poisson bracket (2.8) satisfies the Jacobi identity.

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