


Hamiltonian Structure and a Variational Principle for Grounded Abyssal Flow on a Sloping Bottom in a Mid-Latitude β -Plane

By Gordon E. Swaters 

Observations, numerical simulations, and theoretical scaling arguments suggest that in mid-latitudes, away from the high-latitude source regions and the equator, the meridional transport of abyssal water masses along a continental slope correspond to geostrophic flows that are gravity or density driven and topographically steered. These dynamics are examined using a nonlinear reduced-gravity geostrophic model that describes grounded abyssal meridional flow over sloping topography that crosses the planetary vorticity gradient. It is shown that this model possesses a noncanonical Hamiltonian formulation. General nonlinear steady solutions to the model can be obtained for arbitrary bottom topography. These solutions correspond to nonparallel shear flows that flow across the planetary vorticity gradient. If the in-flow current along the poleward boundary is strictly equatorward, then no shock can form in the solution in the mid-latitude domain. It is also shown that the steady solutions satisfy the first-order necessary conditions for an extremal to a suitably constrained potential energy functional. Sufficient conditions for the definiteness of the second variation of the constrained energy functional are examined. The theory is illustrated with a nonlinear steady solution corresponding to an abyssal flow with upslope and down slope groundings in the height field.

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1. Introduction

The hemispheric scale meridional transport of abyssal water masses along the continental slope is a principal mechanism by which cold dense ocean water produced in high-latitude regions flows back toward the equator and beyond in the deep ocean. In the North Atlantic, the southward transport associated with deep western boundary currents (DWBC) along the North American continental slope is an example of such a flow as is, in the South Atlantic, the northward transport associated with Antarctic Bottom Water (AABW) along the South American continental slope. These equatorward flows are a significant component of the deep part of the global thermohaline overturning circulation. Their dynamics, accordingly, play an important role in climate evolution.

Observations (e.g., [1–3], among many others), theoretical considerations (e.g., [4–6], among others), and numerical simulations (e.g., [7–10], among others) suggest that in mid-latitudes away from the equator and from the polar source regions, these deep or abyssal flows are often grounded (i.e., where the abyssal current height intersects the bottom, which is sometimes also called an “incropping” in an obvious reference to outcroppings associated with surface currents) on the continental slope, in geostrophic balance, flow substantial distances coherently across the planetary vorticity gradient, are density or gravity driven, and are more or less topographically steered. These observations suggest that, to leading order, the dynamics of these flows can be described by a nonlinear time-dependent planetary-geostrophic reduced-gravity model that describes density or gravity-driven grounded abyssal meridional flow over sloping topography allowing for finite-amplitude dynamical deflections in the abyssal height or thickness, which will permit groundings in the height field all within the context of a mid-latitude β -plane. We hasten to add that, notwithstanding the broad range of dynamics the model incorporates, many physical processes are ignored that are important such as baroclinic, barotropic, and Kelvin–Helmholtz instability, vertical entrainment and mixing between the overlying water column and the abyssal current, and bottom friction. Some of these processes have been examined previously (e.g., [11–16]).

The hemispheric scale meridional transport of abyssal water masses is an example of a geophysical fluid dynamic flow that crosses the planetary vorticity gradient, i.e., the dependence of the Coriolis parameter on latitude. As is well known, there are very few exact analytical results known for such flows. Importantly, the model equation and solutions discussed in this paper (see also [6, 17, 18]) are among the very few known exact nonlinear results for oceanographic flows that cross the planetary vorticity gradient. The model examined here and its mathematical properties and solutions are of potential interest, therefore, to broader geophysical fluid dynamics in

addition to its specific importance and relevance in describing part of the global thermohaline overturning circulation.

It is emphasized that the principal purpose of this paper is *not* to apply the model to a specific oceanographic flow or to catalogue its various solutions (these can be found in [6, 17, 18]) but to describe some important *mathematical properties* of the model and its solutions. In particular, it is shown that this model has an infinite-dimensional noncanonical Hamiltonian structure and that nonlinear steady solutions to the model satisfy the first-order necessary conditions for an extremal to a suitably constrained potential energy functional. General conditions preventing the formation of a shock are described and their implication on the definiteness of the second variation of the constrained Hamiltonian are described. Finally, the theory is illustrated with a nontrivial example corresponding to a grounded equatorward abyssal flow on a sloping bottom with both an upslope and down slope grounding in the height field.

The plan of the paper is as follows. In Section 2, the physical geometry and model is briefly described. Section 3 introduces the general nonlinear steady solution to the model, describes some important meridionally invariant cross-slope integrated flux properties associated with the solution, as well as a formula for the cross-slope position of the groundings for general bottom topography. Section 4 introduces the Hamiltonian structure including the Poisson bracket and Casimir invariants, and the variational principle for the general steady solution. The theory is illustrated with an example in Section 5. The paper is summarized in Section 6.

2. The model equation

Detailed derivations of the model equation as a distinguished asymptotic limit of the nonlinear shallow water equations appropriate for ocean basin length scales was described in [6], for a mid-latitude β -plane in [17], and in spherical coordinates in [18].

In standard notation [19], the *nondimensional* model is the nonlinear hyperbolic partial differential equation

$$h_t + \partial \left(h + h_B, \frac{h}{1 + \beta y} \right) = 0, \quad (1)$$

where the Jacobian $\partial(A, B) \equiv A_x B_y - A_y B_x$ (subscripts denote partial differentiation unless otherwise denoted), (x, y) are the eastward or zonal and northward or meridional Cartesian coordinates, respectively, $h(x, y, t) \geq 0$ is the thickness or height of the abyssal current above the bottom topography $h_B = h_B(x, y)$ (see Fig. 1), and the coefficient $1 + \beta y$ is the linearly varying Coriolis parameter associated with the β -plane approximation where for

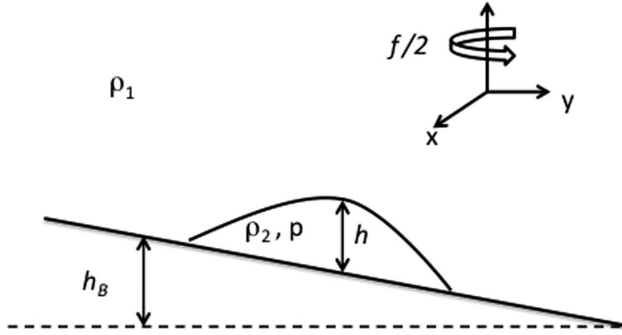


Figure 1. Geometry of the reduced gravity model used in this paper. The features are not shown to scale to facilitate their description.

typical length scales $\beta \simeq 0.02$ [6, 17] (but cannot be neglected over the meridional basin length scales of physical relevance). Formally, Eq. (1) is a small Rossby number limit of the shallow water equations for a differentially rotating fluid that permits *finite-amplitude* dynamical deflections in the thickness, or height, of the “grounded” current located immediately above the underlying variable topography, which is overlain by an infinitely deep but dynamically passive fluid layer [6, 17–19], i.e., the reduced gravity approximation [19].

The nondimensional eastward and northward Eulerian velocities, (u, v) , respectively, and the geostrophic pressure, denoted by p , are given by the geostrophic and the (integrated) hydrostatic relations, respectively,

$$u = -\frac{1}{1 + \beta y}(h + h_B)_y, \tag{2}$$

$$v = \frac{1}{1 + \beta y}(h + h_B)_x, \tag{3}$$

$$p = h + h_B. \tag{4}$$

Alternatively, (1) can be expanded into the quasi-linear form

$$h_t - \left[\frac{\beta h}{(1 + \beta y)^2} + \frac{\partial_y h_B}{1 + \beta y} \right] h_x + \frac{(\partial_x h_B) h_y}{1 + \beta y} = \frac{\beta (\partial_x h_B) h}{(1 + \beta y)^2}. \tag{5}$$

The potential vorticity (PV) equation associated this model is simply obtained by multiplying (1) with $(1 + \beta y)^{-1}$, yielding

$$\left(\frac{h}{1 + \beta y} \right)_t + \mathbf{u} \cdot \nabla \left(\frac{h}{1 + \beta y} \right) = 0, \tag{6}$$

where (2) and (3) has been used. Equation (5) may be interpreted as a variant of the so-called *planetary geostrophic wave equation* introduced by [20–23], generalized to allow for meridional flow on a mid-latitude β -plane with variable bottom topography.

One useful property of (1) is that it ensures that the appropriate kinematic condition associated with a grounding (i.e., a location where h intersects the bottom; as an example see Fig. 1) is automatically satisfied. That is, one does not need to apply the kinematic boundary condition as an additional auxiliary external constraint because the solution to (1) will necessarily automatically satisfy it [5, 6, 17, 18]. This is a direct consequence of the fact that a grounding must correspond to a streamline and (1) completely determines the evolution of the geostrophic pressure, which is the streamfunction.

Swaters [6, 17, 18] has obtained and described a number of *exact nonlinear* steady boundary-value and time-dependent solutions to (1) for the Cauchy and time-dependent boundary-value problems, respectively. Our goal here is to focus on the Hamiltonian structure of the model, the establishment of a variational principle for arbitrary nonlinear steady solutions to the model, and to describe other related mathematical properties.

3. Nonlinear steady solutions

The steady or time-independent solutions of (5) satisfy the quasi-linear hyperbolic partial differential equation

$$(1 + \beta y)(\partial_x h_B)h_y - [\beta h + (1 + \beta y)(\partial_y h_B)]h_x = \beta(\partial_x h_B)h, \quad (7)$$

which can be solved exactly for *completely arbitrary bottom topography* $h_B(x, y)$ using the method of characteristics (for details see [6, 17, 18]). If we suppose that along $y = y_0$ that $h_B(x, y_0) = h_{B0}(x)$ and that $h(x, y_0) = h_0(x)$, then the nonlinear solution to (7) can be written in the form

$$h(x, y) = \frac{1 + \beta y}{1 + \beta y_0} h_0(\tau), \quad (8)$$

$$h_B(x, y) = \frac{\beta(y_0 - y)}{1 + \beta y_0} h_0(\tau) + h_{B0}(\tau), \quad (9)$$

where $\tau = x$ when $y = y_0$.

The characteristics, which are the isolines in the (x, y) -plane for constant τ , are also the geostrophic streamlines because (8) and (9) can be combined to yield

$$h(x, y) + h_B(x, y) = h_0(\tau) + h_{B0}(\tau), \quad (10)$$

because $h(x, y) + h_B(x, y)$ is the geostrophic pressure. In practice one determines $\tau(x, y)$ from (9) and substitutes into (8) to determine $h(x, y)$. Once h is known, the velocities are determined by the geostrophic relations (2) and (3). Thus, given knowledge of the cross-slope structure or shape of the abyssal current height h at the northern boundary of a region in the northern hemisphere (or southern boundary in the case of the southern hemisphere), (8) and (9) determines the steady equatorward flow equatorward of the location of the boundary condition.

The solutions (8) and (9) are *not a parallel shear flow* and are one of the *few known exact nonlinear solutions for oceanographically relevant steady flow that crosses the planetary vorticity gradient*. Qualitatively, the solution has the property that h decreases and the flow speeds up (while maintaining constant meridional volume transport [6]) and has a slight upslope trajectory as the flow moves equatorward (consistent with primitive equation simulations [7] or shallow water simulations [10]) and observations [2, 3, 24]. There is very good point wise agreement between this steady solution and the time-averaged mid-latitude height and velocity fields associated with the fully nonlinear shallow-water initial-value numerical simulations described by Kim *et al.* [10]. We will illustrate the solutions (8) and (9) with an explicit phenomenally relevant example in Section 5.

Because it is well known that the single layer planetary-geostrophic equations do not exhibit shear flow instability [25], the only “disorder” that can possibly arise in the solutions (8) and (9) is the possible emergence of a shock (as a consequence of the quasi-linearity in (12)). The shock will form at the first y -value equatorward of y_0 for which $|h_x| \rightarrow \infty$ (but h remains bounded). However, Swaters [6, 17, 18] has shown that if the meridional velocity $v(x, y_0)$ associated with the solutions (8) and (9) along the inflow poleward boundary, given by,

$$v(x, y_0) = \frac{h'_0(x) + h'_{B0}(x)}{1 + \beta y_0},$$

is equatorward for all x along y_0 within the abyssal current, then no shock forms in the solution (8) and (9) in the β -plane region equatorward of $y = y_0$. Of course, this applies only to a mid-latitude β -plane or on a sphere that does not extend to the equator. On an equatorial β -plane the analogue of these solutions for the geostrophic velocities ultimately become singular as the equator is approached regardless of the flow profile along the poleward boundary of the region [6, 17, 18] and new dynamics must prevail in the equatorial region [15, 16].

Suppose that the abyssal height $h_0(x)$ along $y = y_0$ has a grounding located at $x = a$, i.e., $h_0(a) = 0$. It follows that the cross-slope position of the grounding for $y \leq y_0$, denoted by $x = \tilde{a}(y)$, where $\tilde{a}(y_0) = a$, is located

on the isobath defined by $h_{B0}(a)$. To see this, observe that it follows from (13) that

$$h(\tilde{a}(y), y) = 0 \implies h_0(\tau(\tilde{a}(y), y)) = 0 \implies \tau(\tilde{a}(y), y) = a,$$

that is, a grounding must correspond, of course, to a streamline, which when substituted into (15) implies that

$$h_B(\tilde{a}(y), y) = h_{B0}(a),$$

i.e., the grounding is located along the isobath $h_{B0}(a)$. As a corollary, should the bottom topography be independent of y the cross-slope location of the grounding is constant with respect to y (see also [6, 17, 18]).

Additionally, the steady solution (8) and (9) has the property that the cross-slope integrated meridional energy flux is independent of y . To show this, suppose that the abyssal height $h_0(x)$ along $y = y_0$ has upslope and down slope groundings located at a_1 and a_2 , respectively, where $a_1 < a_2$, i.e., $h_0(a_{1,2}) = 0$. Further suppose that the cross-slope position of these groundings in the region $y \leq y_0$ is given by $x = \tilde{a}_1(y)$ and $x = \tilde{a}_2(y)$, respectively, as determined by, respectively, $h_B(\tilde{a}_{1,2}(y), y) = h_{B0}(a_{1,2})$ because $\tilde{a}_{1,2}(y_0) = a_{1,2}$.

The steady limit of the energy equation associated with (1), (2), and (3) can be written in the form

$$[uh(h + h_B)]_x + [vh(h + h_B)]_y = 0,$$

from which it follows that

$$\frac{d}{dy} \int_{\tilde{a}_1(y)}^{\tilde{a}_2(y)} vh(h + h_B) dx = 0,$$

where

$$M_{EF} \equiv \int_{\tilde{a}_1(y)}^{\tilde{a}_2(y)} vh(h + h_B) dx,$$

is, by definition, the cross-slope integrated meridional energy flux. In particular, it follows from (3), (8), and (10) that

$$M_{EF} = \frac{1}{1 + \beta y_0} \int_{a_1}^{a_2} h_0(\tau) [h_0(\tau) + h_{B0}(\tau)] [h'_0(\tau) + h'_{B0}(\tau)] d\tau.$$

There is another cross-slope integrated meridional flux that is constant with respect to y for the steady solutions (8) and (9). Suppose that F is an arbitrary differentiable function of the form

$$F = F\left(\frac{h}{1 + \beta y}\right),$$

with the property $F(0) = 0$. From the steady limit of the PV equation (6) we have

$$\begin{aligned} 0 &= (1 + \beta y) F' \left(\frac{h}{1 + \beta y} \right) \mathbf{u} \cdot \nabla \left(\frac{h}{1 + \beta y} \right) \\ &= \nabla \cdot \left[(1 + \beta y) \mathbf{u} F \left(\frac{h}{1 + \beta y} \right) \right], \end{aligned}$$

where F' means differentiation with respect to the argument, from which it follows that

$$\frac{d}{dy} \int_{\tilde{a}_1(y)}^{\tilde{a}_2(y)} (1 + \beta y) v F \left(\frac{h}{1 + \beta y} \right) dx = 0.$$

In particular, from (3), (8), and (10) we have

$$\begin{aligned} &\int_{\tilde{a}_1(y)}^{\tilde{a}_2(y)} (1 + \beta y) v F \left(\frac{h}{1 + \beta y} \right) dx \\ &= \int_{a_1}^{a_2} [h'_0(\tau) + h'_{B0}(\tau)] F \left(\frac{h_0(\tau)}{1 + \beta y_0} \right) d\tau. \end{aligned}$$

This property is important in consideration of the *Casimir* functionals that will be introduced in next section.

In the special case that $F(z) = z$, the invariance property associated with F reduces to the statement that the meridional volume transport is constant with respect to y , i.e.,

$$\frac{dT}{dy} = 0,$$

where

$$T \equiv \int_{\tilde{a}_1(y)}^{\tilde{a}_2(y)} v h dx,$$

is, by definition, the meridional volume transport. In particular, it follows that

$$T = \frac{1}{1 + \beta y_0} \int_{a_1}^{a_2} h_0(\tau) h'_B(\tau) d\tau.$$

These will be illustrated in the example presented in Section 5.

4. Hamiltonian structure and a variational principle

The time-dependent energy equation associated with (1), (2), and (3), and hence (5), can be written in the form

$$\partial_t [(h + h_B)^2 - h_B^2] + \nabla \cdot [2\mathbf{u}h(h + h_B)] = 0.$$

The area-integrated positive-definite potential energy functional, denoted by H , given by

$$H = \frac{1}{2} \iint_{\Omega} (h + h_B)^2 - h_B^2 dx dy, \quad (11)$$

where Ω is the “periodic” meridionally aligned channel domain

$$\Omega = \{(x, y) | b_1(y) < x < b_2(y), y_- < y < y_0\},$$

where $x = b_{1,2}(y)$ are the (fixed, but potentially y dependent) cross-slope positions of the upslope and down slope channel walls, respectively, and $-\beta^{-1} \ll y_- < y < y_0$, will be invariant in time, i.e.,

$$\frac{dH}{dt} = - \iint_{\Omega} \nabla \cdot [\mathbf{u}h(h + h_B)] dx dy = 0,$$

where it has been assumed that $\mathbf{u} \cdot \mathbf{n} = 0$ on $x = b_{1,2}(y)$, where \mathbf{n} is the unit outward normal on the upslope and down slope channel walls, and where the cross-slope integrated meridional energy flux along $y = y_-$ is assumed to be the same as along the inflow poleward boundary $y = y_0$ (as the steady solution satisfies). The time-invariance of H still holds even when time-dependent groundings are present provided the integration in the above is handled properly.

Equation (1) (and hence (5)) has the noncanonical Hamiltonian formulation [26,27]

$$h_t = J \frac{\delta H}{\delta h}, \quad (12)$$

where the cosymplectic operator J is given by

$$J(*) = \partial \left(\frac{h}{1 + \beta y}, * \right), \quad (13)$$

and where $\delta H / \delta h$ is the variational derivative

$$\frac{\delta H}{\delta h} = h + h_B.$$

The Poisson bracket associated with the Hamiltonian formulation is given by

$$[F, G] = \iint_{\Omega} \frac{\delta F}{\delta h} \partial \left(\frac{h}{1 + \beta y}, \frac{\delta G}{\delta h} \right) dx dy. \tag{14}$$

Tedious calculation (not shown here) will confirm that (14) satisfies the *Jacobi Identity*.

The *Casimir* functionals, denoted by C , span the Kernel of the cosymplectic operator J and will be determined by

$$\begin{aligned} J \frac{\delta C}{\delta h} = \partial \left(\frac{h}{1 + \beta y}, \frac{\delta C}{\delta h} \right) = 0 &\implies \frac{\delta C}{\delta h} = \widehat{F} \left(\frac{h}{1 + \beta y} \right) \\ \implies C = \iint_{\Omega} (1 + \beta y) F \left(\frac{h}{1 + \beta y} \right) dx dy, &\tag{15} \end{aligned}$$

where F is an arbitrary function of its argument, which we will assume satisfies $F(0) = 0$. The Casimirs are, of course, invariant in time as can be seen from

$$\begin{aligned} \frac{dC}{dt} &= \iint_{\Omega} (1 + \beta y) F' \left(\frac{h}{1 + \beta y} \right) \left(\frac{h}{1 + \beta y} \right)_t dx dy \\ &= - \iint_{\Omega} (1 + \beta y) F' \left(\frac{h}{1 + \beta y} \right) \mathbf{u} \cdot \nabla \left(\frac{h}{1 + \beta y} \right) dx dy \\ &= - \iint_{\Omega} \nabla \cdot [\mathbf{u} (1 + \beta y) F] dx dy = 0, \end{aligned}$$

where F' means differentiation with respect to the argument, and where the PV equation (6), and the geostrophic relations (2) and (3) have been used, and it is assumed that the cross-slope integrated meridional flux $(1 + \beta y)vF$ along $y = y_-$ is assumed to be the same as along the inflow poleward boundary $y = y_0$ (as the steady solution satisfies). Again, the time-invariance of C still follows if there are time-dependent groundings in h , provided the integration in the above derivation is handled properly. The Casimir functionals are required in constructing a variational principle for general steady solutions to the model.

4.1. Variational principle

There is an alternate characterization of the steady solutions to (1) or equivalently (5) that lends itself to a variational principle. The variational

principle described here is the barotropic limit of the variational principle described in [5]. From (1) we see that the steady solutions can be written in the form

$$h_B(x, y) + h(x, y) = F\left(\frac{h(x, y)}{1 + \beta y}\right), \quad (16)$$

for some function F .

The nonlinear steady solutions (8) and (9) has this form, of course, because (8) formally implies the relationship

$$\tau = \tilde{\tau}\left(\frac{h(x, y)}{1 + \beta y}\right),$$

for some function $\tilde{\tau}$, which when substituted into (10) implies

$$\begin{aligned} & h(x, y) + h_B(x, y) \\ &= h_0\left(\tilde{\tau}\left(\frac{h(x, y)}{1 + \beta y}\right)\right) + h_{B0}\left(\tilde{\tau}\left(\frac{h(x, y)}{1 + \beta y}\right)\right) = F\left(\frac{h(x, y)}{1 + \beta y}\right), \end{aligned}$$

for some function F . We will illustrate this construction with an example presented in Section 5.

Written in the form (16), the steady solution (8) and (9) can be seen to satisfy the first-order conditions for an extremum to the time-invariant Casimir-constrained Hamiltonian

$$\mathcal{H} = \iint_{\Omega} \left\{ \frac{(h + h_B)^2 - h_B^2}{2} - (1 + \beta y) \int_0^{h/(1+\beta y)} F(\xi) d\xi \right\} dx dy, \quad (17)$$

because it follows that

$$\delta\mathcal{H} = \iint_{\Omega} \left[h + h_B - F\left(\frac{h}{1 + \beta y}\right) \right] \delta h \, dx dy, \quad (18)$$

so that (16) implies

$$\delta\mathcal{H}|_{\text{evaluated for (16)}} = 0.$$

Equation (18) holds whether or not time-dependent groundings exist. The fact that the variations in any groundings do not appear in (18) is a reflection of the fact that the evolution of the groundings is solely determined by the dynamical system (1) itself and no additional constraints are required.

4.2. The second variation of \mathcal{H}

The second variation of \mathcal{H} , i.e., $\delta^2\mathcal{H}$, evaluated at the steady solution (16), is given by

$$\delta^2\mathcal{H}|_{\text{evaluated for (16)}} = \iint_{\Omega} \left[1 - \frac{1}{1 + \beta y} F' \left(\frac{h}{1 + \beta y} \right) \right] (\delta h)^2 dx dy.$$

However, from (16) we have

$$F' \left(\frac{h(x, y)}{1 + \beta y} \right) = \frac{(1 + \beta y)(h + h_B)_x}{h_x},$$

and from (8) and (9) we have that

$$\frac{(1 + \beta y)(h + h_B)_x}{h_x} = \frac{(1 + \beta y_0) [h'_{B0}(\tau) + h'_0(\tau)]}{h'_0(\tau)},$$

so that

$$1 - \frac{1}{1 + \beta y} F' \left(\frac{h}{1 + \beta y} \right) = 1 - \frac{(1 + \beta y_0) [h'_{B0}(\tau) + h'_0(\tau)]}{(1 + \beta y) h'_0(\tau)}. \quad (19)$$

If the no shock condition $v(x, y_0) < 0$ holds, then it follows that

$$1 - \frac{1}{1 + \beta y} F' \left(\frac{h}{1 + \beta y} \right) \neq 0,$$

for any $-\beta^{-1} \ll y \leq y_0$ (the mid-latitude β -plane domain with northern boundary located at $y = y_0$). If the right-hand side of (19) is a continuous function of τ , then the no-shock condition would be sufficient to ensure that

$$\delta^2\mathcal{H}|_{\text{evaluated for (16)}},$$

would be definite for all δh . For example, if $h'_0(\tau) > 0$ (within the context of satisfying $v(x, y_0) < 0$), it would follow that

$$\delta^2\mathcal{H}|_{\text{evaluated for (16)}} > 0.$$

Or, if for example, if $h'_0(\tau) < 0$ (within the context of satisfying $v(x, y_0) < 0$), it would follow that

$$\delta^2\mathcal{H}|_{\text{evaluated for (16)}} < 0.$$

However, the right-hand side of (19) will *not* be a continuous function of τ for an abyssal height profile with upslope and down slope groundings such as that shown in Fig. 1 because there clearly exists a τ such that $h'_0(\tau) = 0$. In this case one cannot conclude the definiteness of the second variation when evaluated for the steady solution.

We hasten to add that the definiteness of the second variation evaluated for the steady solution is *not* a proof of “linear stability” or is the indefiniteness of the second variation evaluated for the steady solution indicative of “linear instability” in the sense of classical hydrodynamic instability *with respect to the model equation* (1) because the underlying model (1) is completely incapable of describing any such instability in any event [25]. The only “disorder” that can arise in this model is the formation of a shock. The *baroclinic instability* of these solutions has been examined in [5, 11, 28, 29]. Within the context of model that can describe baroclinic instability, an abyssal height profile such as that shown in Fig. 1 is baroclinically unstable. However, the baroclinic instability saturates at finite amplitude [29].

5. An example with upslope and down slope groundings

As a nontrivial example for which it is possible to explicitly solve for $h(x, y)$ let us consider the linearly sloping bottom

$$h_B(x) = -x, \quad (20)$$

with the abyssal height profile along $y = y_0$ given by

$$h_0(x) = \begin{cases} \tilde{h}_0(1 - x^2/a^2) & \text{if } |x| \leq a, \\ 0 & \text{if } |x| > a, \end{cases} \quad (21)$$

with $0 < \tilde{h}_0 < a/2$ (ensuring that the meridional velocity is strictly equatorward along $y = y_0$, which in turn ensures that $v(x, y_0) < 0$). The abyssal height (21) corresponds to a “parabolic” cross-slope boundary profile with half-width $a > 0$ and maximum height $\tilde{h}_0 > 0$ located at $x = 0$, which possesses upslope and down slope groundings located at $x = \pm a$, respectively. Figure 2 is a graph of $h_0(x)$ on the sloping bottom with $a = 1$ and $\tilde{h}_0 = 0.45$ (corresponding, dimensionally, to a current half-width of 100 km and a maximum height of about 250 m). There is no loss of generality in assuming a bottom slope $h'_B(x) = -1$ on account of the nondimensionalization scheme used to derive (1).

The nondimensional meridional volume transport along $y = y_0$ is given by

$$T = \frac{1}{1 + \beta y_0} \int_{-a}^a [h_B(x) + h_0(x)]_x h_0(x) dx = -\frac{4a\tilde{h}_0}{3(1 + \beta y_0)}, \quad (22)$$

where (20) and (21) have been used and, as has been shown, the meridional volume transport is invariant in y . Assuming $a = 1$, $\tilde{h}_0 = 0.45$, and $y_0 = 5$, $T_0 \simeq -0.55$ (or dimensionally about -1.22 Sv; 1 Sv = 10^6 m³/s). These transport values are reasonably consistent with observations of the “deep”

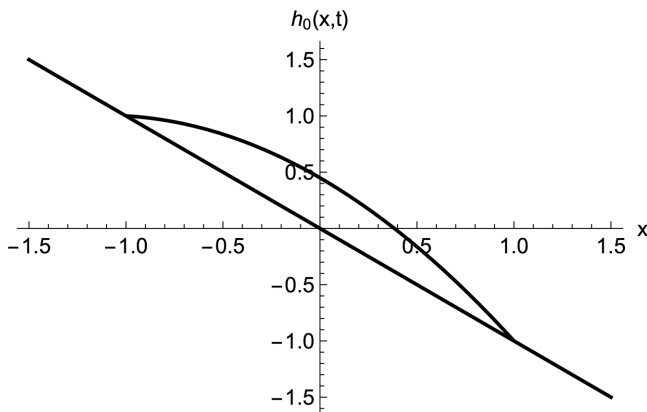


Figure 2. Graph of the abyssal height profile $h_0(x)$ along the poleward boundary $y = y_0$ on the sloping bottom $h_B = -x$.

transport associated with “overflow/lower deep water” associated with the DWBC near Cape Cod [3].

Substitution of (20) and (21) into (9) implies, after a little algebra, that

$$\tau(x, y) = \begin{cases} \Psi(x, y) & \text{if } |x| \leq a, \\ x & \text{if } |x| > a, \end{cases} \quad (23)$$

where

$$\Psi(x, y) \equiv \frac{2a(\Gamma + x)}{a + \sqrt{4\Gamma^2 + 4x\Gamma + a^2}}, \quad (24)$$

with

$$\Gamma(y) \equiv \frac{\beta(y_0 - y)\tilde{h}_0}{1 + \beta y_0} \geq 0, \quad (25)$$

for $-\beta^{-1} \ll y \leq y_0$. We note that $\Psi(x, y_0) = x$ and $\Gamma(y_0) = 0$. Substitution of (21) and (23) through to (35) into (8) explicitly determines $h(x, y)$. The nonlinear solution so obtained corresponds to an equatorward flowing nonparallel shear flow that crosses the planetary vorticity gradient, which possesses an upslope and down slope grounding. As commented on previously, there are very few known exact nonlinear solutions for meridionally flowing oceanographic flows.

It follows from (2), (3), and (8) that the flow speeds up and the height decreases as y decreases so that the meridional volume transport is invariant with respect to y (as is the cross-slope integrated meridional energy flux). In addition, because $h_B = h_B(x)$ it *necessarily* follows that the location of the groundings are invariant with respect to y , that is, their cross-slope location

is fixed by the location of the groundings along $y = y_0$. Graphical depictions of these properties can be seen in the many images contained in [6, 17, 18].

With respect to the variational principle, it follows from (16) that Casimir density for this example is given by

$$F(\xi) = (1 + \beta y_0)\xi - a \operatorname{sgn}(\tau) \sqrt{1 - (1 + \beta y_0)\xi/\tilde{h}_0}, \quad (26)$$

which implies that

$$\int_0^{h/(1+\beta y)} F(\xi) d\xi = \frac{(1 + \beta y_0)h^2}{2(1 + \beta y)^2} - \frac{2a\tilde{h}_0 \operatorname{sgn}(\tau)}{3(1 + \beta y_0)} \left[\left(1 - \frac{(1 + \beta y_0)h}{(1 + \beta y)\tilde{h}_0}\right)^{3/2} - 1 \right], \quad (27)$$

which is to be substituted into (17) to finally obtain \mathcal{H} .

From (19) we find that

$$1 - \frac{1}{1 + \beta y} F' \left(\frac{h}{1 + \beta y} \right) = 1 - \frac{(1 + \beta y_0)(1 + 2\tau\tilde{h}_0/a^2)}{2(1 + \beta y)\tau\tilde{h}_0/a^2}, \quad (28)$$

where $-a \leq \tau \leq a$. The no-shock condition $0 < \tilde{h}_0 < a/2$ implies that $1 + 2\tau\tilde{h}_0/a^2 > 0$ and that the right-hand side of (28) is *never* zero for any $-\beta^{-1} \ll y \leq y_0$ for all $\tau \in [-a, a]$. When $-a \leq \tau < 0$ (where $h'_0 > 0$), the right-hand side of (28) is positive definite. However, when $0 < \tau \leq a$ (where $h'_0 < 0$), the right-hand side of (28) is negative definite because

$$\frac{(1 + \beta y_0)(1 + 2\tau\tilde{h}_0/a^2)}{2(1 + \beta y)\tau\tilde{h}_0/a^2} > 1,$$

for $-\beta^{-1} \ll y \leq y_0$. The right-hand side of (28) is discontinuous at $\tau = 0$.

6. Conclusions

Observations, theoretical considerations, and numerical simulations suggest that in mid-latitudes away from the equator and from the polar source regions, hemispheric scale abyssal flows, such as the DWBC and AABW, are often grounded on the continental slope, in geostrophic balance, flow substantial distances coherently across the planetary vorticity gradient, are density or gravity driven, and are more or less topographically steered. These dynamics result in a surprisingly simple but nevertheless illuminating nonlinear time-dependent planetary-geostrophic reduced-gravity model that can describe density or gravity-driven grounded abyssal meridional flow over sloping topography while permitting groundings in the abyssal water height in a mid-latitude β -plane. This model is an extension of the so-called *planetary geostrophic wave equation*, generalized to allow for meridional flow on a mid-latitude β -plane with variable bottom topography.

Our goal here was not to apply the model to a specific oceanographic setting, but rather to describe a number of mathematical properties associated

with the solutions and model itself. It was shown that this model possessed an infinite-dimensional noncanonical Hamiltonian structure and that nonlinear steady solutions to the model satisfy the first-order necessary conditions for an extremal to a suitably constrained potential energy functional. Conditions preventing the formation of a shock were discussed and their implication on the definiteness of the second variation of the constrained Hamiltonian was described. Finally, the theory was illustrated with a nontrivial example corresponding to a grounded equatorward abyssal flow on a sloping bottom with both an upslope and down slope grounding in the height field.

As mentioned in the Introduction, the model examined here, while encompassing a wide range dynamics, nevertheless does not include several physical processes that play a role in large-scale dynamics of abyssal ocean currents such as baroclinic, barotropic, and Kelvin–Helmholtz instability, vertical entrainment and mixing between the overlying water column and the abyssal current, and bottom friction. It would, of course, be quite interesting to examine the role played by these other physical process in modulating the dynamics described here.

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