

Shock wave example

The pde and initial condition are given by

$$u_t + u u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad \text{with } u(x, 0) = \begin{cases} 1 - |x| & \text{when } -1 < x \leq 1, \\ 0 & \text{when } x \leq -1 \text{ or } x > 1. \end{cases}.$$

Using the method of characteristics, the solution may be written in the form, when $0 \leq t < 1$,

$$u(x, t) = \begin{cases} \frac{1+x}{1+t} & \text{when } -1 < x \leq t, \\ \frac{1-x}{1-t} & \text{when } t < x \leq 1, \\ 0 & \text{when } x \leq -1 \text{ or } x > 1. \end{cases}$$

The space-time coordinates of first shock formation are $(x, t) = (1, 1)$. The location of the shock wave ($x = h(t)$) is determined by the entropy condition

$$\frac{dh}{dt} = \frac{1+h}{2(1+t)}, \quad h(1) = 1 \implies h(t) = -1 + \sqrt{2(1+t)}.$$

Hence the solution, when $t \geq 1$, is given by

$$u(x, t) = \begin{cases} \frac{1+x}{1+t} & \text{when } -1 < x \leq -1 + \sqrt{2(1+t)}, \\ 0 & \text{when } x \leq -1 \text{ or } x > -1 + \sqrt{2(1+t)}. \end{cases}$$

Below, the solution is shown at $t = 0, 0.5, 1.0$ and 5 , respectively.

