<u>Shock formation example</u>

The pde and initial condition are given by

$$u_t + u\,u_x = 0,\ -\infty < x < \infty,\ t > 0, \text{ with } u\,(x,0) = 1/\left(1+x^2\right).$$

Using the method of characteristics, the solution may be written in the form

$$u = 1/\left(1+\tau^2\right) \ \text{ with } x = \tau + t/\left(1+\tau^2\right) \ \text{ where } \ -\infty < \tau < \infty.$$

Observing that the equation for the characteristics can be written in the form $\tau = x - u\,t$, allows the solution to be written in the form

$$u = 1/\left[1+(x-u\,t)^2\right] \Longleftrightarrow t^2u^3 - 2xt\,u^2 + \left(1+x^2\right)u - 1 = 0.$$

The only real solution to the cubic is given by

$$u\,(x,t) = 2x/\,(3t) + \sqrt[3]{-\beta + \sqrt{\beta^2+\alpha^3}} + \sqrt[3]{-\beta - \sqrt{\beta^2+\alpha^3}},$$

where

$$\alpha \equiv \left(3-x^2\right)/\left(9t^2\right) \ \text{ and } \beta \equiv \left[2x\left(x^2+9\right) - 27t\right]/\left(54t^3\right).$$

The space-time coordinates of first shock formation are $(x,t) = \left(\sqrt{3}, 8\sqrt{3}/9\right)$. Below, the solution is shown at $t = 0$, 0.5, 1.0 and $8\sqrt{3}/9$, respectively.